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ON GENERAL LATTICE REPLETENESS AND COMPLETENESS

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Introduction

In this paper we wish to initiate a systematic study of various concepts pertaining to repleteness or completeness of a lattice. Special cases include such notions as realcompactness, α-completeness, Borel completeness, N-compactness, almost-realcompactness, and so on.

Specifically we consider an arbitrary set X and an arbitrary lattice $\mathscr{L}$ of subsets of X. We denote the algebra of subsets of X generated by $\mathscr{L}$ by $\mathscr{A}(\mathscr{L})$ and the set of all (finitely additive) two-valued measures on $\mathscr{A}(\mathscr{L})$ by $I(\mathscr{L})$. We then consider various subsets of $I(\mathscr{L})$ and, denoting the general element of $I(\mathscr{L})$ by μ, we demand that the support of μ, $S(\mu)$ be non-empty for μ in these subsets. Particular choices of these subsets, in the case where X is a topological space and $\mathscr{L}$ a particular lattice of subsets of X, give all the special cases referred to above as well as many others.

We proceed to analyze in the abstract setting of $\langle X, \mathscr{L}\rangle$ interrelations between these various concepts of repleteness-completeness, and then consider the important situation of two lattices $\mathscr{L}_1$, $\mathscr{L}_2$, with $\mathscr{L}_1 \subset \mathscr{L}_2$, and investigate when $\mathscr{L}_1$-(repleteness-completeness) implies $\mathscr{L}_2$-(repleteness-completeness), and conversely. Our results subsume all the known relationships in the special cases referred to above and they also yield new applications. We give a few of the applications, but it should be clear from these that many more such applications are available by appropriate choice of the lattices.

We then investigate particular lattices in subsets of $I(\mathscr{L})$ which are replete-complete and show how these can be utilized in constructing repletions-completions in particular cases.

The important point throughout the paper is that we can systematically treat all cases of repleteness-completeness, uniformly, by general measure-theoretic techniques. This was done to a limited extent in [2], just for repleteness, and in [10] using filter arguments with just one lattice and with just certain completeness notions. The advantage of the measure approach is that it is particularly simple with respect to extension-restriction matters and that much of it can be extended to the case of arbitrary measures—not necessarily two-valued; in this paper we will just pursue the case of

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the two-valued measures for the sake of the topological applications and in order to keep the definitions to a minimum.

We take pleasure in acknowledging our indebtedness to the referee for improving throughout the presentation of this paper and for considerably strengthening a number of results. In particular, we just cite the improved presentation of Lemma 5.2 which is due to the referee, as well as Lemma 4.1, which greatly strengthened and shortened the proof of Theorem 4.4.

Section 1

For convenience, we review some terminology which is consistent with that used in [4], [16], and elsewhere.

(a) Consider any set X and any lattice $\mathscr{L}$ of subsets of X. We shall always assume, without loss of generality for our purposes, that $\emptyset, X \in \mathscr{L}$. $\mathscr{L}$ is said to be δ iff, for every subset $\{L_\alpha; \alpha \in A\}$ of $\mathscr{L}$, if A is countable then $\cap \{L_\alpha; \alpha \in A\} \in \mathscr{L}$. $\mathscr{L}$ is said to be *complemented* iff, for every element L of $\mathscr{L}$, $L' \in \mathscr{L}$. The set whose general element is the complement of an element of $\mathscr{L}$ is denoted by $\mathscr{L}'$. $\mathscr{L}$ is said to be *complement generated* iff, for every element L of $\mathscr{L}$, there exists a subset $\{L_\alpha; \alpha \in A\}$ of $\mathscr{L}$ such that $L = \cap\{L'_\alpha; \alpha \in A\}$ and A is countable. $\mathscr{L}$ is said to be *separating* iff, for any two elements a, b of X, if $a \neq b$ then there exists an element A of $\mathscr{L}$ such that $a \in A$ and $b \notin A$. $\mathscr{L}$ is said to be *disjunctive* iff, for every element a of X and every element B of $\mathscr{L}$, if $a \notin B$ then there exists an element A of $\mathscr{L}$ such that $a \in A$ and $A \cap B = \emptyset$. $\mathscr{L}$ is said to be *regular* iff, for every element a of X and every element B of $\mathscr{L}$, if $a \notin B$ then there exist two elements C, D of $\mathscr{L}$ such that $a \in C'$ and $B \subset D'$ and $C' \cap D' = \emptyset$. $\mathscr{L}$ is said to be *normal* iff, for any two elements A, B of $\mathscr{L}$, if $A \cap B = \emptyset$ then there exist two elements C, D of $\mathscr{L}$ such that $A \subset C'$ and $B \subset D'$ and $C' \cap D' = \emptyset$. $\mathscr{L}$ is said to be *Lindelöf* iff, for every subset $\{L_\alpha; \alpha \in A\}$ of $\mathscr{L}$, if

$$\cap\{L_\alpha; \alpha \in A\} = \emptyset$$

then there exists a subset A^* of A such that $\cap\{L_\alpha; \alpha \in A^*\} = \emptyset$ and A^* is countable. $\mathscr{L}$ is said to be *compact* iff, for every subset $\{L_\alpha; \alpha \in A\}$ of $\mathscr{L}$, if $\cap\{L_\alpha; \alpha \in A\} = \emptyset$ then there exists a subset A^* of A such that $\cap\{L_\alpha; \alpha \in A^*\} = \emptyset$ and A^* is finite. A subset S of X is said to be *$\mathscr{L}$-compact* iff the lattice

$$S \cap \mathscr{L} \quad (S \cap \mathscr{L} = \{S \cap L | L \in \mathscr{L}\})$$

is compact. $\mathscr{L}$ is said to be *countably compact* iff, for every subset $\{L_\alpha; \alpha \in A\}$ of $\mathscr{L}$, if $\cap\{L_\alpha; \alpha \in A\} = \emptyset$ and A is countable then there exists a subset A^* of A such that $\cap\{L_\alpha; \alpha \in A^*\} = \emptyset$ and A^* is finite. $\mathscr{L}$ is said to be *countably paracompact* iff, for every sequence $\langle A_n \rangle$ in $\mathscr{L}$, if $\langle A_n \rangle$ is decreasing and $\lim_n A_n = \emptyset$ then there exists a sequence $\langle B_n \rangle$ in $\mathscr{L}$ such that for every n, $A_n \subset B'_n$ and $\langle B'_n \rangle$ is decreasing and $\lim_n B'_n = \emptyset$.

Next, consider any two lattices $\mathcal{L}_1$, $\mathcal{L}_2$ of subsets of X. $\mathcal{L}_1$ is said to *semiseparate* L_2 iff, for every element L_1 of $\mathcal{L}_1$ and every element L_2 of $\mathcal{L}_2$, if $L_1 \cap L_2 = \emptyset$ then there exists an element $\tilde{L}_1$ of $\mathcal{L}_1$ such that $L_2 \subset \tilde{L}_1$ and $L_1 \cap \tilde{L}_1 = \emptyset$. $\mathcal{L}_1$ is said to *separate* $\mathcal{L}_2$ iff for any two elements L_2, $\tilde{L}_2$ of $\mathcal{L}_2$, if $L_2 \cap \tilde{L}_2 = \emptyset$ then there exist two elements L_1, $\tilde{L}_1$ of $\mathcal{L}_1$ such that $L_2 \subset L_1$, $\tilde{L}_2 \subset \tilde{L}_1$ and $L_1 \cap \tilde{L}_1 = \emptyset$. $\mathcal{L}_2$ is said to be $\mathcal{L}_1$-*countably bounded* iff, for every sequence $\langle B_n \rangle$ in $\mathcal{L}_2$, if $\langle B_n \rangle$ is decreasing and $\lim_n B_n = \emptyset$ then there exists a sequence $\langle A_n \rangle$ in $\mathcal{L}_1$ such that, for every n, $B_n \subset A_n$, $\langle A_n \rangle$ is decreasing and $\lim_n A_n = \emptyset$. $\mathcal{L}_2$ is said to be $\mathcal{L}_1$-*countably paracompact* iff, for every sequence $\langle B_n \rangle$ in $\mathcal{L}_2$, if $\langle B_n \rangle$ is decreasing and $\lim_n B_n = \emptyset$ then there exists a sequence $\langle A_n \rangle$ in $\mathcal{L}_1$ such that for every n, $B_n \subset A_n'$, $\langle A_n' \rangle$ is decreasing and $\lim_n A_n' = \emptyset$.

(b) The set of natural numbers is denoted by N. For an arbitrary function f, the domain of f is denoted by D_f. The set whose general element is the intersection of an arbitrary subset of $\mathcal{L}$ which is countable is denoted by $\delta\mathcal{L}$. The set whose general element is the intersection of an arbitrary subset of $\mathcal{L}$ is denoted by $t\mathcal{L}$. The set whose general element is the union of an arbitrary subset of $\mathcal{L}$ which is countable is denoted by $\Sigma\mathcal{L}$. The algebra of subsets of X generated by $\mathcal{L}$ is denoted by $\mathcal{A}(\mathcal{L})$. The σ-algebra of subsets of X generated by $\mathcal{L}$ is denoted by $\sigma(L)$. The family of subsets of X which is closed under countable unions and intersections and contains $\mathcal{L}$, and is minimal is denoted by $\rho(\mathcal{L})$. The family of subsets of X obtainable from X by the lattice Souslin operations is denoted by $s(\mathcal{L})$. Next, consider any algebra $\mathcal{A}$ of subsets of X. A *measure* on $\mathcal{A}$ is defined to be a function μ from $\mathcal{A}$ to R such that μ is bounded and finitely additive. The set whose general element is a measure on $\mathcal{A}(\mathcal{L})$ is denoted by $M(\mathcal{L})$. For an element μ of $M(\mathcal{L})$, the support of μ is defined to be

$$\cap\{L \in \mathcal{L}; |\mu|(L) = |\mu|(X)\}$$

and is denoted by $S(\mu)$. An element μ of $M(\mathcal{L})$ is said to be $\mathcal{L}$-*regular* iff, for every element E of $\mathcal{A}(\mathcal{L})$ and every positive number ε, there exists an element L of $\mathcal{L}$ such that $L \subset E$ and $|\mu(E) - \mu(L)| < \varepsilon$. The set whose general element is an element of $M(\mathcal{L})$ which is $\mathcal{L}$-regular is denoted by $MR(\mathcal{L})$. An element μ of $M(\mathcal{L})$ is said to be σ-smooth iff, for every sequence $\langle A_n \rangle$ in $\mathcal{A}(\mathcal{L})$, if $\langle A_n \rangle$ is decreasing and $\lim_n A_n = \emptyset$ then $\lim_n \mu(A_n) = 0$. The set whose general element is an element of $M(\mathcal{L})$ which is σ-smooth is denoted by $M(\sigma, \mathcal{L})$. The set whose general element is an element of $M(\mathcal{L})$ which is σ-smooth just for $\langle A_n \rangle$ in $\mathcal{L}$ is denoted by $M(\sigma^*, \mathcal{L})$. Note that if $\mu \in MR(\mathcal{L})$, then $\mu \in MR(\sigma, \mathcal{L})$ iff $\mu \in M(\sigma^*, \mathcal{L})$. The set whose general element is an element μ of $M(\mathcal{L})$ such that $\mu(\mathcal{A}(\mathcal{L})) = \{0, 1\}$ is denoted by $I(\mathcal{L})$. $\mathcal{L}$ is said to be *replete* iff, whenever an element μ of $I(\mathcal{L})$ belongs to $IR(\sigma, \mathcal{L})$ then $S(\mu) \neq \emptyset$. Next, consider any topological space X, and denote its collection of zero sets by $\mathcal{Z}$, its collection of closed sets by $\mathcal{F}$, its collection of clopen sets by $\mathcal{C}$, and its collection of Borel sets by $\mathscr{B}$. If X is 3 1/2, X is said to be realcompact iff $\mathscr{Z}$ is replete [11]. X is said to be α-complete iff $\mathscr{F}$ is replete [8]. X is said to be N-compact iff $\mathscr{C}$ is replete [13]. X is said to be Borel-complete iff $\mathscr{B}$ is replete [12].

Since every element of $M(\mathscr{L})$ is equal to the difference of nonnegative elements of $M(\mathscr{L})$, without loss of generality we may work exclusively with nonnegative elements of $M(\mathscr{L})$.

Note. Occasionally we shall use variants of the notation introduced above either for simplicity or for clarity.

(c) We note for convenience that there exists a one-to-one correspondence between $I(\mathscr{L})$ and the set of all prime $\mathscr{L}$-filters and a one-to-one correspondence between $IR(\mathscr{L})$ and the set of all $\mathscr{L}$-ultrafilters. (Details can be found in [1] and [7].) It follows, therefore, that for every element μ of $I(\mathscr{L})$, there exists an element ν of $IR(\mathscr{L})$ such that $\mu \leqslant \nu$ on $\mathscr{L}$. (The proof involves a filter-ultrafilter argument.) It is interesting to note that this fact can be extended to nonnegative measures in $M(\mathscr{L})$; i.e., for every element μ of $M(\mathscr{L})$, if $\mu \geqslant 0$ then there exists an element ν of $MR(\mathscr{L})$ such that $\mu \leqslant \nu$ on $\mathscr{L}$ and $\mu(X) = \nu(X)$. (Details can be found in [18].)

In addition, we observe that for any two lattices $\mathscr{L}_1$, $\mathscr{L}_2$ of subsets of X, if $\mathscr{L}_1 \subset \mathscr{L}_2$, then for every element μ of $IR(\mathscr{L}_1)$, there exists an element ν of $IR(\mathscr{L}_2)$ such that $\nu|_{\mathscr{A}(\mathscr{L}_1)} = \mu$. (The proof involves a filter-ultrafilter argument.) Again, this fact can be extended to the more general case: for every element μ of $MR(\mathscr{L}_1)$, there exists an element ν of $MR(\mathscr{L}_2)$ such that $\nu|_{\mathscr{A}(\mathscr{L}_1)} = \mu$; moreover, such a ν is unique if $\mathscr{L}_1$ separates $\mathscr{L}_2$. (Details can be found in [3] and [4].) Also, if $\mu \in MR(\sigma, \mathscr{L}_1)$ and $\mathscr{L}_2$ is $\mathscr{L}_1$-countably paracompact, or $\mathscr{L}_1$-countably bounded, then $\nu \in MR(\sigma, \mathscr{L}_2)$.

Section 2

In this section we define repleteness, fully-repleteness, prime completeness, Cauchy completeness, and almost-repleteness. Then, we present various relationships among these concepts (additional such relationships are presented in Section 3). Finally, we make some relevant observations.

Consider any set X and any lattice $\mathscr{L}$ of subsets of X.

DEFINITION 2.1. Denote the general element of $M(\mathscr{L})$ by μ.

(α) $\mathscr{L}$ is *replete* iff $S(\mu) \neq \emptyset$ whenever $\mu \in IR(\sigma, \mathscr{L})$.
(β) $\mathscr{L}$ is *fully-replete* iff $S(\mu) \neq \emptyset$ whenever $\mu \in I(\sigma, \mathscr{L})$.
(γ) $\mathscr{L}$ is *prime complete* iff $S(\mu) \neq \emptyset$ whenever $\mu \in I(\sigma^*, \mathscr{L})$.
(δ) $\mathscr{L}$ is *Cauchy complete,* iff $S(\mu) \neq \emptyset$ whenever $\mu \in I(\sigma^*, \mathscr{L}')$.
(ε) $\mathscr{L}$ is *almost-replete* iff $S(\mu) \neq \emptyset$ whenever $\mu \in IR(\mathscr{L}') \cap I(\sigma^*, \mathscr{L})$.

Observation. Prime completeness implies fully-repleteness implies repleteness.

THEOREM 2.1. *Assume $\mathscr{L}$ is complement generated. Then:*

1. *Repleteness is equivalent to Cauchy completeness.*
2. *Repleteness is equivalent to fully-repleteness.*

Proof. 1. Since $\mathscr{L}$ is complement generated, $I(\sigma^*, \mathscr{L}') \subset IR(\mathscr{L})$. Also, since $\mathscr{L}$ is complement generated, $\mathscr{L}$ is countably paracompact; hence $I(\sigma^*, \mathscr{L}') \subset I(\sigma^*, \mathscr{L})$. Consequently $I(\sigma^*, \mathscr{L}') \subset IR(\sigma, \mathscr{L})$. Also, since $IR(\sigma, \mathscr{L}) \subset I(\sigma^*, \mathscr{L}')$, $IR(\sigma, \mathscr{L}) = I(\sigma^*, \mathscr{L}')$. Hence repleteness is equivalent to Cauchy completeness.

2. Since fully-repleteness implies repleteness, it suffices to show the converse. Assume $\mathscr{L}$ is replete. Consider any element μ of $I(\sigma, \mathscr{L})$. Note that $\mu \in I(\sigma^*, \mathscr{L}')$. Since $\mathscr{L}$ is replete and repleteness implies Cauchy completeness, $\mathscr{L}$ is Cauchy complete. Consequently $S(\mu) \neq \emptyset$. Hence $\mathscr{L}$ is fully-replete.

THEOREM 2.2. *Assume $\mathscr{L}$ is normal and countably paracompact. Then:*

1. *Repleteness is equivalent to fully-repleteness.*
2. *Fully-repleteness is equivalent to prime completeness.*
3. *Prime completeness is equivalent to Cauchy completeness.*

Proof. First, establish 1 and 2 by showing that repleteness is equivalent to prime completeness. Since prime completeness implies repleteness, it suffices to show the converse. Assume $\mathscr{L}$ is replete. Consider any element μ of $I(\sigma^*, \mathscr{L})$. Next, consider any element ν of $IR(\mathscr{L})$ such that $\mu \leq \nu$ on $\mathscr{L}$. Then $S(\mu) \supset S(\nu)$. Show $\nu \in IR(\sigma, \mathscr{L})$. Consider any sequence $\langle A_n \rangle$ in $\mathscr{L}$ such that $\langle A_n \rangle$ is decreasing and $\lim_n A_n = \emptyset$. Since $\mathscr{L}$ is countably paracompact, there exists a sequence $\langle B_n \rangle$ in $\mathscr{L}$ such that for every n, $A_n \subset B_n'$ and $\langle B_n' \rangle$ is decreasing and $\lim_n B_n' = \emptyset$. For every n, since $A_n \subset B_n'$ and $\mathscr{L}$ is normal, there exist two elements C_n, D_n of $\mathscr{L}$ such that $A_n \subset C_n'$, $B_n \subset D_n'$ and $C_n' \cap D_n' = \emptyset$; then $A_n \subset C_n' \subset D_n \subset B_n'$. (Assume, without loss of generality, that these inclusions hold with $\langle D_n \rangle$ decreasing.) Then $\nu(A_n) \leq \nu(C_n') \leq \mu(C_n') \leq \mu(D_n)$. Since $\langle B_n' \rangle$ is decreasing and $\lim_n B_n' = \emptyset$, $\lim_n D_n = \emptyset$. Hence, since $\mu \in I(\sigma^*, \mathscr{L})$, $\lim_n \mu(D_n) = 0$. Consequently $\lim_n \nu(A_n) = 0$. Hence $\nu \in I(\sigma^*, \mathscr{L})$. Consequently $\nu \in IR(\sigma, \mathscr{L})$, and, since $\mathscr{L}$ is replete, $S(\nu) \neq \emptyset$. Thus $S(\mu) \neq \emptyset$, and $\mathscr{L}$ is prime complete.

The proof of 3 is similar and will be omitted.

Applications. (1) Consider any topological space X such that S is $T_{3\,1/2}$. Then "$\mathscr{Z}$ is replete" (or X is realcompact [11]), "$\mathscr{Z}$ is fully-replete", "$\mathscr{Z}$ is prime complete" and "$\mathscr{Z}$ is Cauchy complete" are equivalent.

(2) Consider any topological space X such that X is normal and countably paracompact. Then "$\mathscr{F}$ is replete" (or X is α-complete [8]), "$\mathscr{F}$ is fully-replete", "$\mathscr{F}$ is prime complete" and "$\mathscr{F}$ is Cauchy complete" are equivalent.

(3) Consider any topological space X such that X is 0-dimensional and

T_1. Then "$\mathscr{C}$ is replete" (or X is N-compact [13]), "$\mathscr{C}$ is fully-replete", etc. are equivalent.

THEOREM 2.3. *Assume $\mathscr{L}$ is regular. Then prime completeness is equivalent to almost-repleteness.*

Proof. We omit the proof that prime completeness implies almost-repleteness.

To show that almost-repleteness implies prime completeness, assume $\mathscr{L}$ is almost-replete. Consider any element μ of $I(\sigma^*, \mathscr{L})$. Note that $\mu \in I(\mathscr{L}')$. Consider any element ρ of $IR(\mathscr{L}')$ such that $\mu \leqslant \rho$ on L'. Then $\rho \leqslant \mu$ on $\mathscr{L}$. Hence, since $\mathscr{L}$ is regular, $S(\mu) = S(\rho)$. To show that $\rho \in I(\sigma^*, \mathscr{L})$, consider any sequence $\langle A_n \rangle$ in $\mathscr{L}$ such that $\langle A_n \rangle$ is decreasing and $\lim_n A_n = \emptyset$. Note that for every n, $\rho(A_n) \leqslant \mu(A_n)$. Hence

$$\lim_n \rho(A_n) \leqslant \lim_n \mu(A_n).$$

Since $\mu \in I(\sigma^*, \mathscr{L})$, $\lim_n \mu(A_n) = 0$. Consequently $\lim_n \rho(A_n) = 0$. Hence $\rho \in I(\sigma^*, \mathscr{L})$. Consequently $\rho \in IR(\mathscr{L}') \cap I(\sigma^*, \mathscr{L})$, and since $\mathscr{L}$ is almost-replete, $S(\rho) \neq \emptyset$. Thus $S(\mu) \neq \emptyset$, and $\mathscr{L}$ is prime complete.

Applications. (1) Consider any topological space X such that X is $T_{3\,1/2}$. Then "$\mathscr{Z}$ is prime complete" is equivalent to "$\mathscr{Z}$ is almost-replete". Moreover, since in this case "$\mathscr{Z}$ is replete" is equivalent to "$\mathscr{Z}$ is prime complete", "$\mathscr{Z}$ is replete" (or X is realcompact) is equivalent to "$\mathscr{Z}$ is almost-replete".

(2) Consider any topological space X such that X is regular. Then "$\mathscr{F}$ is prime complete" is equivalent to "$\mathscr{F}$ is almost-replete".

The purpose of the following two observations is to present situations of non-repleteness.

Observation 2.1. (α). Assume X is uncountable. Note that the set whose general element is a subset E of X such that E or E' is countable is a σ-algebra. Denote this σ-algebra by $\mathscr{A}$. Now, consider the measure μ on $\mathscr{A}$ determined by $\mu(E) = 0$ if E is countable, and $\mu(E) = 1$ if E' is countable. Note that $\mu \in I(\sigma, \mathscr{A}) = IR(\sigma, \mathscr{A})$, but $S(\mu) = \emptyset$. Hence $\mathscr{A}$ is non-replete.

(β) Assume $\mathscr{L}$ is disjunctive. Consider any element μ of $IR(\mathscr{L}')$. Assume $S_{\mathscr{L}'}(\mu) \neq \emptyset$. Consider any element y of $S_{\mathscr{L}'}(\mu)$. Then $\mu \leqslant \mu_y$ on $\mathscr{L}'$. Hence $\mu_y \leqslant \mu$ on $\mathscr{L}$. Hence, since $\mathscr{L}$ is disjunctive, $\mu_y = \mu$. Consequently $\mu_y \in IR(\mathscr{L}')$.

Special Case. Consider any topological space X such that X is T_1 and let $\mathscr{L} = \mathscr{F}$. Further, assume for every element x of X, $\{x\}$ is not open. Then for every element μ of $IR(\mathscr{L}')$, $S_{\mathscr{L}'}(\mu) = \emptyset$. Hence, in case $IR(\sigma, \mathscr{L}') \neq \emptyset$, $\mathscr{L}'$ is non-replete.

The purpose of the following observation is to show that prime completeness does not imply "normality and countable paracompactness."

Observation 2.2. Let $X = R$. Denote the usual topology on R by $\mathscr{O}$.

(α) We will show that $\mathscr{O}$ is prime complete. Since $\langle R, \mathscr{O}\rangle$ is metrizable, $\mathscr{F} = \mathscr{Z}$. Hence $I(\sigma^*, \mathscr{O}) = I(\sigma^*, \mathscr{Z}')$. Since $\mathscr{Z}$ is complement generated, $I(\sigma^*, \mathscr{Z}') \subset IR(\sigma, \mathscr{Z})$. (See Theorem 2.1, Part 1.) Since $\mathscr{Z}$ is Lindelöf, $\mathscr{Z}$ is replete. Hence $IR(\sigma, \mathscr{Z}) = \{\mu_x; x \in X\}$. Consequently $I(\sigma^*, \mathscr{O}) \subset \{\mu_x; x \in X\}$. Hence $\mathscr{O}$ is prime complete.

(β) $\mathscr{O}$ is neither normal nor countably paracompact.

(γ) Consequently prime completeness does not imply "normality and countable paracompactness."

The purpose of the following observation is to show that prime completeness does not imply normality.

Observation 2.3. Assume X is uncountable. Denote the co-countable topology on X by $\mathscr{O}$.

(α) $\mathscr{F}$ is Lindelöf. Hence $\mathscr{F}$ is prime complete.

(β) $\mathscr{F}$ is not normal.

(γ) Consequently prime completeness does not imply normality.

The purpose of the following observation is to show that normality and countable paracompactness does not imply repleteness.

Observation 2.4. Assume X is uncountable. Denote the co-countable topology on X by $\mathscr{O}$.

(α) $\mathscr{O}$ is normal (vacuously) and countably paracompact (vacuously).

(β) We will show that $\mathscr{O}$ is not replete. Since $\mathscr{O}$ is normal and countably paracompact, $IR(\sigma, \mathscr{O}) \neq \emptyset$. Consider any element μ of $IR(\sigma, \mathscr{O})$. Note that

$$S(\mu) = \cap \{0 \in \mathscr{O} | \mu(0) = 1\}.$$

Next, note for every element x of X, $\{x\} \in \mathscr{A}(\mathscr{O})$ and $\mu(\{x\}) = 0$. Hence for every element F of $\mathscr{F}$, if $F \neq X$, then $\mu(F) = 0$. Hence for every element O of $\mathscr{O}$, if $O \neq \emptyset$, then $\mu(O) = 1$. Further, note for element x of X, $X - \{x\} \in \mathscr{O}$ and $X - \{x\} \neq \emptyset$; thus $\mu(X - \{x\}) = 1$. Consequently

$$S(\mu) \subset \cap \{X - \{x\}; x \in X\} = \emptyset,$$

and $\mathscr{O}$ is not replete.

(γ) Consequently "normality and countable paracompactness" does not imply repleteness.

The purpose of the following observation is to present a situation, under which the cardinal of a set is measurable. (See [11] for the definition of measurable cardinal.)

Observation 2.5. Assume that $\langle X, \mathscr{L}\rangle$ is a topological space and replace $\mathscr{L}$ by $\mathscr{F}$. Further, assume there exists an element μ of $I(\mathscr{F})$ such that $\mu \in IR(\sigma, \mathscr{O})$ and $S_{\mathscr{F}}(\mu) = \emptyset$. Next, consider any subset $\{U_\alpha; \alpha \in \Lambda\}$ of $\mathscr{O}$ such that for every α, $\mu(U_\alpha) = 0$ and $\{U_\alpha;\ \alpha \in \Lambda\}$ is disjoint, and $\{U_\alpha;\ \alpha \in \Lambda\}$ is maximal. Note that $\cup\ \{U_\alpha;\ \alpha \in \Lambda\} \in \mathcal{O}$. Denote $\cup\ \{U_\alpha;\ \alpha \in \Lambda\}$ by U.

(α) We will show that $\overline{U} = X$. Consider any element x of X. Note that either there exists an α such that $x \in U_\alpha$ (Case 1), or for every α, $x \notin U_\alpha$ (Case 2). For Case 1, note that $x \in \overline{U}$. For Case 2, consider any element O of $\mathcal{O}$ such that $x \in O$. Since $S_{\mathcal{F}}(\mu) = \emptyset$, $\cap\ \{F \in \mathcal{F} | \mu(F) = 1\} = \emptyset$. Hence

$$\cup\ \{F' \in \mathcal{F}' | \mu(F') = 0\} = X.$$

Consequently there exists an element $\tilde{O}$ of $\mathcal{O}$ such that $\mu(\tilde{O}) = 0$ and $x \in \tilde{O}$. Then $O \cap \tilde{O} \in \mathcal{O}$ and $\mu(O \cap \tilde{O}) = 0$ and $x \in O \cap \tilde{O}$. Consequently for every α, $O \cap \tilde{O} \neq U_\alpha$. Hence, since $\{U_\alpha;\ \alpha \in \Lambda\}$ is maximal, there exists an α such that $(O \cap \tilde{O}) \cap U_\alpha \neq \emptyset$. Hence $O \cap U \neq \emptyset$. Hence $x \in \overline{U}$. Consequently $\overline{U} = X$. This argument is due to Frolik [10].

(β) We omit the proof that $\mu(U) = 1$.

Next, consider the function ρ which is such that $D_\rho = \mathcal{P}(\Lambda)$ and for every element M of $\mathcal{P}(\Lambda)$, $\rho(M) = \mu(\cup\ \{U_\alpha;\ \alpha \in M\})$. Note that

$$\rho \in I(\mathcal{P}(\Lambda)) = IR(\mathcal{P}(\Lambda)).$$

Next, note that $\rho \in I(\sigma, \mathcal{P}(\Lambda))$. Consequently $\rho \in IR(\sigma, \mathcal{P}(\Lambda))$. Now, show $S(\rho) = \emptyset$. Assume $S(\rho) \neq \emptyset$. Consider any element α_0 of $S(\rho)$. Then $\rho \leqslant \mu_{\alpha_0}$ on $\mathcal{P}(\Lambda)$. Hence, since $\rho \in IR(\mathcal{P}(\Lambda))$, $\rho = \mu_{\alpha_0}$. Consequently

$$1 = \mu_{\alpha_0}(\{\alpha_0\}) = \rho(\{\alpha_0\}) = \mu(\cup\ \{U_\alpha; \alpha \in \{\alpha_0\}\}) = \mu(U_{\alpha_0}) = 0,$$

a contradiction. Hence $S(\rho) = \emptyset$.

Hence card Λ is measurable, so card X is measurable.

Section 3

In this section we consider an arbitrary set X and two arbitrary lattices $\mathcal{L}_1$, $\mathcal{L}_2$ of subsets of X such that $\mathcal{L}_1 \subset \mathcal{L}_2$. We investigate criteria under which various repleteness or completeness properties of $\mathcal{L}_1$ will hold for $\mathcal{L}_2$, and conversely.

Consider any set X and any two lattices $\mathcal{L}_1$, $\mathcal{L}_2$ of subsets of X such that $\mathcal{L}_1 \subset \mathcal{L}_2$.

THEOREM 3.1. *Assume $\mathcal{L}_1$ is replete. Further, assume one of the following conditions is satisfied:*

1. *$\mathcal{L}_1$ is complement generated.*
2. *$\mathcal{L}_1$ is δ and $\sigma(\mathcal{L}_1) \subset s(\mathcal{L}_1)$.*
3. *$\mathcal{L}_1$ semiseparates $\mathcal{L}_2$.*
4. *$\mathcal{L}_1$ is normal and countably paracompact.*

Finally, assume $\mathcal{L}_1 \subset \mathcal{L}_2 \subset t\mathcal{L}_1$. Then $\mathcal{L}_2$ is replete.

Proof. Consider any element ν of $IR(\sigma, \mathscr{L}_2)$. Denote $\nu|_{\mathscr{A}(\mathscr{L}_1)}$ by μ. Since $\mathscr{L}_1 \subset \mathscr{L}_2 \subset t\mathscr{L}_1$, $S(\nu) = S(\mu)$. Moreover, note in cases 1, 2, and 3, $\mu \in IR(\sigma, \mathscr{L}_1)$. Hence, since $\mathscr{L}_1$ is replete, $S(\mu) \neq \emptyset$. Consequently $S(\nu) \neq \emptyset$. As for case 4, note that $\mu \in I(\sigma^*, \mathscr{L}_1)$, and, using the normality and countable paracompactness of $\mathscr{L}_1$, consider any element ρ of $IR(\sigma, \mathscr{L}_1)$ such that $\mu \leq \rho$ on $\mathscr{L}_1$. Then $S(\mu) \supset S(\rho)$. Moreover, since $\mathscr{L}_1$ is replete, $S(\rho) \neq \emptyset$. Consequently $S(\nu) \neq \emptyset$, and $\mathscr{L}_2$ is replete.

Applications. (1) Consider any topological space X such that X is $T_{3\,1/2}$ and let $\mathscr{L}_1 = \mathscr{Z}$ and $\mathscr{L}_2 = \mathscr{F}$. Then $\mathscr{Z}$ is replete implies $\mathscr{F}$ is replete; i.e., X is realcompact implies X is α-complete.

(2) Consider any topological space X such that X is 0-dimensional and T_1, and let $\mathscr{L}_1 = \mathscr{C}$ and $\mathscr{L}_2 = \mathscr{Z}$. Then $\mathscr{C}$ is replete implies $\mathscr{Z}$ is replete; i.e., X is N-compact implies X is realcompact. (Consequently X is N-compact implies X is α-complete.)

THEOREM 3.2. *Assume $\mathscr{L}_1$ is fully-replete (resp. prime complete, Cauchy complete). Further, assume $\mathscr{L}_1 \subset \mathscr{L}_2 \subset t\mathscr{L}_1$. Then $\mathscr{L}_2$ is fully-replete (resp. prime complete, Cauchy complete).*

(Proof omitted.)

THEOREM 3.3. *Assume $\mathscr{L}_1$ is replete. Further, assume $\mathscr{L}_1$ is normal and countably paracompact, and $\mathscr{L}_1 \subset \mathscr{L}_2 \subset t\mathscr{L}_1$. Then $\mathscr{L}_2$ is almost-replete.*

(The proof involves essentially the same argument as the proof of Theorem 3.1, Part 4, and will be omitted.)

Applications. (1) In Theorem 3.3, let $\mathscr{L}_2 = \mathscr{L}_1$. Then $\mathscr{L}_1$ is replete implies $\mathscr{L}_1$ is almost-replete.

(2) Consider any topological space X such that X is $T_{3\,1/2}$ and let $\mathscr{L}_1 = \mathscr{Z}$ and $\mathscr{L}_2 = \mathscr{F}$. Then $\mathscr{Z}$ is replete implies $\mathscr{F}$ is almost-replete; i.e., X is realcompact implies $\mathscr{F}$ is almost-realcompact. The terminology "almost-realcompact" is due to Frolik [9].

THEOREM 3.4. *Assume $\mathscr{L}_2$ is replete. Further, assume $\mathscr{L}_2$ is $\mathscr{L}_1$-countably paracompact or $\mathscr{L}_1$-countably bounded. Then $\mathscr{L}_1$ is replete.*

Proof. Consider any element μ of $IR(\sigma, \mathscr{L}_1)$. Using the condition "$\mathscr{L}_2$ is $\mathscr{L}_1$-countably paracompact or $\mathscr{L}_1$-countably bounded", consider any element ν of $IR(\sigma, \mathscr{L}_2)$ such that $\nu|_{\mathscr{A}(\mathscr{L}_1)} = \mu$. (This is a special case of a general measure-extension theorem for lattice-regular measures [3], [4]; in the present situation it can also be achieved in a more elementary fashion, using the correspondence between $IR(\mathscr{L})$ and the set of all $\mathscr{L}$-ultrafilters [1], [7]. Also, see Section 1(c).) Then $S(\mu) \supset S(\nu)$. Moreover, since $\mathscr{L}_2$ is replete, $S(\nu) \neq \emptyset$. Consequently $S(\mu) \neq \emptyset$. Hence $\mathscr{L}_1$ is replete.

Remark. Note the condition "$\mathscr{L}_2$ is $\mathscr{L}_1$-countably paracompact or $\mathscr{L}_1$-countably bounded" is satisfied for example if $\mathscr{L}_2$ is countably paracompact and $\mathscr{L}_1$ separates $\mathscr{L}_2$.

Applications. (1) Consider any topological space X such that X is $T_{3\,1/2}$ and let $\mathscr{L}_1 = \mathscr{Z}$ and $\mathscr{L}_2 = \mathscr{F}$. Further, assume X is countably paracompact and normal. Then $\mathscr{F}$ is replete implies $\mathscr{Z}$ is replete; i.e., X is α-complete implies X is realcompact.

(2) Consider any topological space X such that X is 0-dimensional and T_1 and let $\mathscr{L}_1 = \mathscr{C}$ and $\mathscr{L}_2 = \mathscr{Z}$. Further, assume X is strongly 0-dimensional (i.e., $\mathscr{C}$ separates $\mathscr{Z}$). Then $\mathscr{Z}$ is replete implies $\mathscr{C}$ is replete; i.e., X is realcompact implies X is N-compact.

(3) Consider any topological space X such that X is 0-dimensional, T_1, and countably paracompact, and let $\mathscr{L}_1 = \mathscr{C}$ and $\mathscr{L}_2 = \mathscr{F}$. Further, assume X is ultranormal (i.e., $\mathscr{C}$ separates $\mathscr{F}$). Then X is α-complete implies X is N-compact.

THEOREM 3.5. *Assume $\mathscr{L}_2$ is replete. Further, assume $\mathscr{L}_2$ is $\mathscr{L}_1$-countably paracompact. Then $\mathscr{L}_1$ is fully-replete.*

Proof. Consider any element μ of $I(\sigma, \mathscr{L}_1)$. Using Tarski's Extension Theorem [6], consider any element ν of $I(\mathscr{L}_2)$ such that $\nu|_{\mathscr{A}(\mathscr{L}_1)} = \mu$. Then $S(\mu) \supset S(\nu)$. Now, consider any element ρ of $IR(\mathscr{L}_2)$ such that $\nu \leqslant \rho$ on $\mathscr{L}_2$. Then $S(\nu) \supset S(\rho)$. To show that $\rho \in IR(\sigma, \mathscr{L}_2)$, consider any sequence $\langle A_n \rangle$ in $\mathscr{L}_2$ such that $\langle A_n \rangle$ is decreasing and $\lim_n A_n = \emptyset$. Since $\mathscr{L}_2$ is $\mathscr{L}_1$-countably paracompact, there exists a sequence $\langle B_n \rangle$ in $\mathscr{L}_1$ such that for every n, $A_n \subset B_n'$, $\langle B_n' \rangle$ is decreasing and $\lim_n B_n' = \emptyset$. Then for every n,

$$\rho(A_n) \leqslant \rho(B_n') \leqslant \nu(B_n') = \mu(B_n').$$

Hence $\lim_n \rho(A_n) \leqslant \lim_n \mu(B_n')$. Since $\mu \in I(\sigma, \mathscr{L}_1)$, $\lim_n \mu(B_n') = 0$. Consequently $\lim_n \rho(A_n) = 0$. Hence $\rho \in I(\sigma^*, \mathscr{L}_2)$, so $\rho \in IR(\sigma, \mathscr{L}_2)$. Hence, since $\mathscr{L}_2$ is replete, $S(\rho) \neq \emptyset$. Consequently $S(\mu) \neq \emptyset$, and $\mathscr{L}_1$ is fully-replete.

THEOREM 3.6. *Assume $\mathscr{L}_2$ is prime complete. Further, assume $\mathscr{L}_2$ is $\mathscr{L}_1$-countably bounded. Then $\mathscr{L}_1$ is prime complete.*

Proof. Consider any element μ of $I(\sigma^*, \mathscr{L}_1)$. Using Tarski's Extension Theorem, consider any element ν of $I(\mathscr{L}_2)$ such that $\nu|_{\mathscr{A}(\mathscr{L}_1)} = \mu$. Then $S(\mu) \supset S(\nu)$. Moreover, since $\mathscr{L}_2$ is $\mathscr{L}_1$-countably bounded, $\nu \in I(\sigma^*, \mathscr{L}_2)$. Hence, since $\mathscr{L}_2$ is prime complete, $S(\nu) \neq \emptyset$. Consequently $S(\mu) \neq \emptyset$, so $\mathscr{L}_1$ is prime complete.

THEOREM 3.7. *Assume $\mathscr{L}_2$ is Cauchy complete. Further, assume $\mathscr{L}_2$ is $\mathscr{L}_1$-countably paracompact. Then $\mathscr{L}_1$ is Cauchy complete.*

Proof. Consider any element μ of $I(\sigma^*, \mathscr{L}_1')$. Next, consider any element ν of $I(\mathscr{L}_2)$ such that $\nu|_{\mathscr{A}(\mathscr{L}_1)} = \mu$. Then $S(\mu) \supset S(\nu)$. Now, consider any element ρ of $IR(\mathscr{L}_2)$ such that $\nu \leqslant \rho$ on $\mathscr{L}_2$. Then $S(\nu) \supset S(\rho)$. To show that $\rho \in I(\sigma^*, \mathscr{L}_2')$, consider any sequence $\langle A_n' \rangle$ in $\mathscr{L}_2'$ such that $\langle A_n' \rangle$ is decreasing and $\lim_n A_n' = \emptyset$. Then for every n, since $\rho \in IR(\mathscr{L}_2)$, there exists an element B_n of $\mathscr{L}_2$ such that $B_n \subset A_n'$ and $\rho(A_n') = \rho(B_n)$. Assume $\langle B_n \rangle$ is decreasing. Then $\langle B_n \rangle$ is in $\mathscr{L}_2$ and $\langle B_n \rangle$ is decreasing and $\lim_n B_n = \emptyset$. Hence, since $\mathscr{L}_2$ is $\mathscr{L}_1$-countably paracompact, there exists a sequence $\langle C_n \rangle$ in $\mathscr{L}_1$ such that for every n, $B_n \subset C_n'$, $\langle C_n' \rangle$ is decreasing and $\lim_n C_n' = \emptyset$. Then for every n,

$$\rho(A_n') = \rho(B_n) \leqslant \rho(C_n') \leqslant \nu(C_n') = \mu(C_n').$$

Hence $\lim_n \rho(A_n') \leqslant \lim_n \mu(C_n')$. Hence, since $\mu \in I(\sigma^*, \mathscr{L}_1')$, $\lim_n \mu(C_n') = 0$. Consequently $\lim_n \rho(A_n') = 0$. Hence $\rho \in I(\sigma^*, \mathscr{L}_2')$. Hence, since $\mathscr{L}_2$ is Cauchy complete, $S(\rho) \neq \emptyset$. Consequently $S(\mu) \neq \emptyset$, and $\mathscr{L}_1$ is Cauchy complete.

THEOREM 3.8. *Assume $\mathscr{L}_2$ is almost-replete. Further, assume $\mathscr{L}_2$ is regular and $\mathscr{L}_2$ is $\mathscr{L}_1$-countably paracompact or $\mathscr{L}_1$-countably bounded. Then $\mathscr{L}_1$ is replete.*

(The result follows from Theorems 3.4 and 2.3.)

Remark. The general extension and restriction theorems we have just established can be readily applied, in a systematic fashion, to mappings T described by $T\colon \langle X, \mathscr{L}_1 \rangle \to \langle X, \mathscr{L}_2 \rangle$, to obtain conditions for $\mathscr{L}_1$-repleteness-completeness to imply $\mathscr{L}_2$-repleteness-completeness, and conversely.

Section 4

In this section we consider some special cases of the setting considered in Section 3.

Consider any set X and any lattice $\mathscr{L}$ of subsets of X.

Part I. $\mathscr{L}_1 = \mathscr{L}$ and $\mathscr{L}_2 = \sigma(\mathscr{L})$.

THEOREM 4.1. 1. *If $\sigma(\mathscr{L})$ is replete, then $\mathscr{L}$ is fully-replete.*
2. *If $\mathscr{L}$ is replete, δ and $\sigma(\mathscr{L}) \subset s(\mathscr{L})$, then $\sigma(\mathscr{L})$ is replete.*

Proof. 1. Assume $\sigma(\mathscr{L})$ is replete. Consider any element μ of $I(\sigma, \mathscr{L})$. Denote the extension of μ to $I(\sigma, \sigma(\mathscr{L}))$ by ν. Note $S(\mu) \supset S(\nu)$. Since $\sigma(\mathscr{L})$ is an algebra, $I(\sigma(\mathscr{L})) = IR(\sigma(\mathscr{L}))$. Consequently $\nu \in IR(\sigma, \sigma(\mathscr{L}))$. Hence, since $\sigma(\mathscr{L})$ is replete, $S(\nu) \neq \emptyset$. Consequently $S(\mu) \neq \emptyset$, and $\mathscr{L}$ is fully-replete.

2. Assume $\mathscr{L}$ is replete, δ and $\sigma(\mathscr{L}) \subset s(\mathscr{L})$. Consider any element ν of $IR(\sigma, \sigma(\mathscr{L}))$. Consider $\nu|_{\mathscr{A}(\mathscr{L})}$ and denote it by μ. Since $\nu \in IR(\sigma, \sigma(\mathscr{L}))$,

$\mu \in I(\sigma, \mathscr{L})$. Since $\mathscr{L}$ is δ and $\sigma(\mathscr{L}) \subset s(\mathscr{L})$, $\mu \in IR(\sigma, \mathscr{L})$, and μ is $\mathscr{L}$-regular. Hence, since $\mathscr{L}$ is δ, ν is also $\mathscr{L}$-regular. Hence for every element B of $\sigma(\mathscr{L})$, if $\nu(B) = 1$ then there exists an element L of $\mathscr{L}$ such that $L \subset B$ and $\mu(L) = \nu(L) = 1$. Then

$$\cap \{B \in \sigma(\mathscr{L}) | \nu(B) = 1\} \supset \cap \{L \in \mathscr{L} | \mu(L) = 1\}.$$

Consequently $S(\nu) \supset S(\mu)$. Moreover, since $\mu \in IR(\sigma, \mathscr{L})$ and $\mathscr{L}$ is replete, $S(\mu) \neq \emptyset$. Consequently $S(\nu) \neq \emptyset$, and $\sigma(\mathscr{L})$ is replete.

Observation. If $\mathscr{L}$ is replete and $\mathscr{L}$ is complement generated, then $\sigma(\mathscr{L})$ is replete.

Applications. (1) Consider any topological space X such that X is $T_{3\,1/2}$ and let $\mathscr{L} = \mathscr{Z}$. Then, by the preceding theorem, $\mathscr{Z}$ is replete iff $\sigma(\mathscr{Z})$ is replete; i.e., X is realcompact iff X is Baire-replete. This result is due to Hewitt [14].

(2) Consider any topological space X such that X is analytic [15] and let $\mathscr{L} = \mathscr{F}$. Then, by the preceding theorem, $\mathscr{F}$ is replete iff $\sigma(\mathscr{F})$ is replete, i.e., X is α-complete iff X is Borel-complete.

Part II. $\mathscr{L}_1 = \mathscr{L}$ and $\mathscr{L}_2 = \delta(\mathscr{L})$.

THEOREM 4.2. *Assume $\mathscr{L}$ is replete. Further, assume any one of the four conditions of Theorem 3.1 is satisfied. Finally, note that $\mathscr{L} \subset \delta(\mathscr{L}) \subset t\mathscr{L}$. Then, by Theorem 3.1, $\delta(\mathscr{L})$ is replete.*

Application. Consider any topological space X such that X is 0-dimensional and T_1 and let $\mathscr{L} = \mathscr{C}$. Then, by the preceding theorem, $\mathscr{C}$ is replete implies $\delta(\mathscr{C})$ is replete.

THEOREM 4.3. *Assume $\mathscr{L}$ is fully-replete, or prime complete, or Cauchy complete. Further, note that $\mathscr{L} \subset \delta(\mathscr{L}) \subset t\mathscr{L}$. Then, by Theorem 3.2, $\delta(\mathscr{L})$ is respectively fully-replete, prime complete, Cauchy complete.*

The following lemmas will prove exceedingly useful in the applications.

LEMMA 4.1. *If $\mathscr{L}$ is countably paracompact, then $\delta(\mathscr{L})$ is $\mathscr{L}$-countably paracompact.*

Proof. Assume $\mathscr{L}$ is countably paracompact. Consider any sequence $\langle A_n \rangle$ in $\delta(\mathscr{L})$ such that $\langle A_n \rangle$ is decreasing and $\lim_n A_n = \emptyset$. Then for every n, $A_n = \cap_{i=1}^{\infty} L_{n,i}$ where $L_{n,i} \in \mathscr{L}$. Now, for every n, let $M_n = \cap_{i,j \leq n} L_{i,j}$. Then $\langle M_n \rangle$ is in $\mathscr{L}$ and $\langle M_n \rangle$ is decreasing and $\lim_n M_n = \emptyset$. Hence, since $\mathscr{L}$ is countably paracompact, there exists a sequence $\langle B_n \rangle$ in $\mathscr{L}$ such that for every n, $M_n \subset B_n'$ and $\langle B_n' \rangle$ is decreasing and $\lim_n B_n' = \emptyset$. Now, note for every n, $A_n \subset A_i \subset L_{i,j}$ where $i, j \leq n$; consequently $A_n \subset M_n \subset B'_n$. Consequently $\delta(\mathscr{L})$ is $\mathscr{L}$-countably paracompact.

COROLLARY TO THEOREM 3.4. *If $\mathscr{L}$ is countably paracompact, then $\delta(\mathscr{L})$ is replete implies $\mathscr{L}$ is replete.*

(The result follows from Theorem 3.4 and the above lemma.)

The last result gives the following special case:

THEOREM 4.4. *If $\mathscr{L}$ is complemented, then $\delta(\mathscr{L})$ is replete implies $\mathscr{L}$ is replete.*

Proof. Assume $\mathscr{L}$ is complemented. Then $\mathscr{L}$ is countably paracompact, and the result follows from the preceding corollary.

We will also need the following result for applications:

LEMMA 4.2. *If $\Sigma(\mathscr{L}) \cap \delta(\mathscr{L}) \subset \mathscr{L}$ and $\mathscr{L}$ is complemented then $\mathscr{L}$ separates $\delta(\mathscr{L})$.*

(The proof is straightforward, though tedious, and will be omitted.)

The following lemma has importance in itself. It will also be used in some applications of Theorem 4.4.

LEMMA 4.3. *Consider any two lattices $\mathscr{L}_1$, $\mathscr{L}_2$ of subsets of X such that $\mathscr{L}_1 \subset \mathscr{L}_2$. Further, assume $\mathscr{L}_1$ is δ, $\mathscr{L}_1$ separates $\mathscr{L}_2$, and $\mathscr{L}_2$ is complement generated. Then $\mathscr{L}_1 = \mathscr{L}_2$.*

Proof. Consider any element B of $\mathscr{L}_2$. Since $\mathscr{L}_2$ is complement generated, there exists a sequence $\langle B_n \rangle$ in $\mathscr{L}_2$ such that $B = \bigcap_n B'_n$. Note for every n, $B \cap B_n = \emptyset$; hence, since $\mathscr{L}_1$ separates $\mathscr{L}_2$, there exist two elements A_n, C_n of $\mathscr{L}_1$ such that $B \subset A_n$, $B_n \subset C_n$ and $A_n \cap C_n = \emptyset$; then

$$B \subset A_n \subset C'_n \subset B'_n.$$

Hence $B \subset \bigcap_n A_n \subset \bigcap_n B'_n = B$. Hence $B = \bigcap_n A_n$. Since $\mathscr{L}_1$ is δ, $\bigcap_n A_n \in \mathscr{L}_1$. Consequently $B \in \mathscr{L}_1$. Hence $\mathscr{L}_2 \subset \mathscr{L}_1$. Consequently $\mathscr{L}_1 = \mathscr{L}_2$.

Application. Consider any topological space X such that X is 0-dimensional and T_1. Then X is strongly 0-dimensional (i.e., $\mathscr{C}$ separates $\mathscr{Z}$) iff $\delta(\mathscr{C}) = \mathscr{Z}$.

Proof. (a) Assume X is strongly 0-dimensional. Then $\mathscr{C}$ separates $\mathscr{Z}$. Hence, since $\mathscr{C} \subset \delta(\mathscr{C}) \subset \mathscr{Z}$, $\delta(\mathscr{C})$ separates $\mathscr{Z}$. Hence, by Lemma 4.3, $\delta(\mathscr{C}) = \mathscr{Z}$.

(b) Assume $\delta(\mathscr{C}) = \mathscr{Z}$. Since $\Sigma(\mathscr{C}) \cap \delta(\mathscr{C}) \subset \mathscr{C}$ and $\mathscr{C}$ is complemented, by Lemma 4.2, $\mathscr{C}$ separates $\delta(\mathscr{C})$. Consequently $\mathscr{C}$ separates $\mathscr{Z}$. Hence X is strongly 0-dimensional.

An application of Theorem 4.4. Consider any topological space X such that X is 0-dimensional and T_1 and let $\mathscr{L} = \mathscr{C}$. Then, by Theorem 4.4, $\delta(\mathscr{C})$ is replete implies $\mathscr{C}$ is replete.

In particular, if X is strongly 0-dimensional, then $\delta(\mathscr{C}) = \mathscr{Z}$ (see the preceding application), and, consequently, $\mathscr{Z}$ is replete implies $\mathscr{C}$ is replete; i.e., X is realcompact implies X is N-compact.

Part III. In this part we construct other lattices $\tilde{\mathscr{L}}$ from $\mathscr{L}$, by adjoining certain collections of subsets of X to $\mathscr{L}$, such as the collection of finite subsets, or the collection of countable subsets, or the collection of $\mathscr{L}$-compact sets, etc. Then we give conditions on $\mathscr{L}$ under which $\tilde{\mathscr{L}}$ is replete, and vice versa.

For any two collections $\mathscr{S}_1$, $\mathscr{S}_2$ of subsets of X, let

$$\mathscr{S}_1 \cup \mathscr{S}_2 = \{S_1 \cup S_2 | S_1 \in \mathscr{S}_1 \text{ and } S_2 \in \mathscr{S}_2\}$$

and

$$\mathscr{S}_1 \cap \mathscr{S}_2 = \{S_1 \cap S_2 | S_1 \in S_1 \text{ and } S_2 \in S_2\}.$$

THEOREM 4.5. *Denote the collection of all finite subsets of X by $\mathscr{F}$. Then $\mathscr{L} \cup \mathscr{F}$ is a lattice.* (Proof omitted.) *Denote $\mathscr{L} \cup \mathscr{F}$ by $\tilde{\mathscr{L}}$. Note that $\mathscr{L} \subset \tilde{\mathscr{L}}$.*

1. *If $\mathscr{L}$ is replete then $\tilde{\mathscr{L}}$ is replete.*
2. *If $\tilde{\mathscr{L}}$ is replete and $\tilde{\mathscr{L}}$ is $\mathscr{L}$-countably paracompact or $\mathscr{L}$-countably bounded then $\mathscr{L}$ is replete.*

Proof. To prove 1, assume $\mathscr{L}$ is replete. Consider any element ν of $IR(\sigma, \tilde{\mathscr{L}})$. By the definition of support of a measure,

$$S(\nu) = \cap \{A \in \tilde{\mathscr{L}} | \nu(A) = 1\}.$$

Note that either there exists an element F of $\mathscr{F}$ such that $\nu(F) = 1$ (Case 1) or for every element F of $\mathscr{F}$, $\nu(F) = 0$ (Case 2).

Case 1. Consider any element F_0 of $\mathscr{F}$ such that $\nu(F_0) = 1$. Then, since $\nu \in I(\tilde{\mathscr{L}})$, there exists an element x_0 of F_0 such that $\nu(\{x_0\}) = 1$ and x_0 is unique. Hence for every element A of $\tilde{\mathscr{L}}$, if $\nu(A) = 1$, then $x_0 \in A$. Consequently $x_0 \in S(\nu)$. Hence $S(\nu) \neq \emptyset$.

Case 2. Consider $\nu|_{\mathscr{A}(\mathscr{L})}$ and denote it by μ. Note that

$$S(\mu) = \{L \in \mathscr{L} | \mu(L) = 1\}.$$

Show $S(\nu) \supset S(\mu)$. Denote the general element of $\tilde{\mathscr{L}}$ by A. Then there exist an element L of $\mathscr{L}$ and an element F of $\mathscr{F}$ such that $A = L \cup F$. Then $\nu(A) = \nu(L \cup F) = \nu(L)$, since $\nu(F) = 0$ by the assumption. Consequently if $\nu(A) = 1$, then there exists an element L of $\mathscr{L}$ such that $L \subset A$ and $\mu(L) = \nu(L) = 1$. Hence

$$\cap \{A \in \tilde{\mathscr{L}} | \nu(A) = 1\} \supset \cap \{L \in \mathscr{L} | \mu(L) = 1\}.$$

Consequently $S(\nu) \supset S(\mu)$. Next, show $\mu \in IR(\sigma, \mathscr{L})$. Since $\nu \in IR(\sigma, \tilde{\mathscr{L}})$, $\mu \in I(\sigma, \mathscr{L})$. Also, show $\mu \in IR(\mathscr{L})$. Consider any element L of $\mathscr{L}$ such that $\mu(L') = 1$. Then $\nu(L') = 1$. Hence, since $\nu \in IR(\tilde{\mathscr{L}})$, there exists an element A of $\tilde{\mathscr{L}}$ such that $A \subset L'$ and $\nu(A) = 1$. Then there exists an element $\hat{L}$ of $\mathscr{L}$ such that $\hat{L} \subset A$ and $\mu(\hat{L}) = 1$. Then $\hat{L} \in \mathscr{L}$ and $\hat{L} \subset L'$ and $\mu(\hat{L}) = 1$. Hence μ is $\mathscr{L}$-regular on $\mathscr{L}'$, and $\mu \in IR(\mathscr{L})$. Consequently $\mu \in IR(\sigma, \mathscr{L})$. Hence, since $\mathscr{L}$ is replete, $S(\mu) \neq \emptyset$. Consequently $S(\nu) \neq \emptyset$, and $\tilde{\mathscr{L}}$ is replete.

For a proof of 2, use Theorem 3.4.

Remark. Denote the collection of all countable subsets of X by $\mathscr{C}$. If in the formulation of Theorem 4.5, $\mathscr{F}$ is replaced by $\mathscr{C}$, then the resulting statement is true.

More generally, consider any subset S of X and denote the collection of all countable subsets of S by $\mathscr{C}$. If in the formulation of Theorem 4.5, $\mathscr{F}$ is replaced by $\mathscr{C}$, then the resulting statement is true.

Application. Let $X = R$. Denote the usual topology on R by $\mathscr{O}_1$ and Smirnov's deleted sequence topology on R by $\mathscr{O}_2$. (Smirnov's deleted sequence topology is defined as follows: Let

$$A = \{1/n; n \in N\}.$$

For any subset V of X, $V \in \mathscr{O}_2$ iff there exist an element U of $\mathscr{O}_1$ and a subset B of A such that $V = U - B$.) Since $\langle R, \mathscr{O}_1\rangle$ is metrizable, $\mathscr{F}_1 = \mathscr{Z}_1$. Since $\mathscr{Z}_1$ is Lindelöf, it is replete. Consequently $\langle R, \mathscr{F}_1\rangle$ is α-complete. Further, $\mathscr{F}_1 \subset \mathscr{F}_2$ and $\mathscr{F}_2$ is obtained from $\mathscr{F}_1$ by adjoining the collection of all subsets of A. Hence, by the above remark, $\mathscr{F}_2$ is replete. Hence $\langle R, \mathscr{O}_2\rangle$ is α-complete.

THEOREM 4.6. *Denote the collection of $\mathscr{L}$-compact sets by $\mathscr{K}$ and assume $\mathscr{K} \subset t\mathscr{L}$. Then $\mathscr{L} \cup \mathscr{K}$ is a lattice.* (Proof omitted.) *Denote $\mathscr{L} \cup \mathscr{K}$ by $\tilde{\mathscr{L}}$.* Note that $\mathscr{L} \subset \tilde{\mathscr{L}}$.

1. *If $\mathscr{L}$ is replete then $\tilde{\mathscr{L}}$* is replete.
2. *If $\mathscr{L}$ is replete and $\tilde{\mathscr{L}}$ is $\mathscr{L}$-countably paracompact or $\mathscr{L}$-countably bounded then $\mathscr{L}$ is replete.*

Proof. For a proof of 1, assume $\mathscr{L}$ is replete. Consider any element ν of $IR(\sigma, \tilde{\mathscr{L}})$. Note there exists an element K of $\mathscr{K}$ such that $\nu(K) = 1$ (Case 1), or for every element K of $\mathscr{K}$, $\nu(K) = 0$ (Case 2).

Case 1. Consider any element K_0 of $\mathcal{K}$ such that $\nu(K_0) = 1$. Recall that

$$S(\nu) = \cap \{A \in \tilde{\mathcal{L}} | \nu(A) = 1\}.$$

Consider any element A of $\tilde{\mathcal{L}}$ such that $\nu(A) = 1$. Then $\nu(A \cap K_0) = 1$. Hence $A \cap K_0 \neq \emptyset$. Consequently for every natural number n, for every n elements $A_1, \ldots, A_n$ of $\mathcal{L}$, if $\nu(A_i) = 1$, $i = 1, \ldots, n$, then $\cap_{i=1}^{n} A_i \cap K_0 \neq \emptyset$. Further, note since K_0 is $\mathcal{L}$-compact, K_0 is $t\mathcal{L}$-compact. Also, since $\tilde{\mathcal{L}} = \mathcal{L} \cup \mathcal{K}$ and $\mathcal{K} \subset t\mathcal{L}$, $\tilde{\mathcal{L}} \subset t\mathcal{L}$. Consequently

$$\cap \{A \in \tilde{\mathcal{L}} | \nu(A) = 1\} \cap K_0 \neq \emptyset.$$

Hence $\cap \{A \in \tilde{\mathcal{L}} \mid \nu(A) = 1\} \neq \emptyset$, so $S(\nu) \neq \emptyset$. Note the condition $\mathcal{K} \subset t\mathcal{L}$ is satisfied, if $\mathcal{L}$ is separating, disjunctive, and normal, or if $\mathcal{L}$ is T_2.

Case 2. To show $S(\nu) \neq \emptyset$, use the same argument as for the case of $\mathcal{F}$.

Consequently $\tilde{\mathcal{L}}$ is replete.

For a proof of 2, use Theorem 3.4.

Remark. If $\mathcal{L}$ is replete and for every element ν of $IR(\sigma, \tilde{\mathcal{L}})$, $\nu|_{\mathcal{K}}$ is $\mathcal{L}$-regular, then $\tilde{\mathcal{L}}$ is replete. (Proof omitted.)

Applications. (1) Consider any topological space X such that X is $T_{3\,1/2}$ and realcompact and denote its collection of $\mathcal{Z}$-compact sets by $\mathcal{K}$. (Note that since $\mathcal{F} = t\mathcal{Z}$, $\mathcal{K}$ is the collection of compact sets of X.) Then $\mathcal{Z} \cup \mathcal{K}$ is replete.

Proof. Since $\mathcal{Z}$ is replete and $\mathcal{K} \subset t\mathcal{Z}$, by Theorem 4.6, part 1, $\mathcal{Z} \cup \mathcal{K}$ is replete.

(2) Consider any topological space X such that X is T_1, normal and countably paracompact (and denote its collection of compact sets by $\mathcal{K}$). Then if $\mathcal{Z} \cup \mathcal{K}$ is replete, X is realcompact.

Proof. Assume $\mathcal{Z} \cup \mathcal{K}$ is replete. Note that $\mathcal{K} \subset t\mathcal{Z}$. Also, since $t\mathcal{Z}$ is countably paracompact and $\mathcal{Z}$ separates $t\mathcal{Z}$, $t\mathcal{Z}$ is $\mathcal{Z}$-countably paracompact. Consequently $\mathcal{Z} \cup \mathcal{K}$ is $\mathcal{Z}$-countably paracompact. Hence, by Theorem 4.6, part 2, $\mathcal{Z}$ is replete. Hence X is realcompact.

DEFINITION 4.1. A collection $\mathcal{D}$ of subsets of X is an $\mathcal{L}$-ideal in X iff $\mathcal{D}$ is a lattice and for every element D of $\mathcal{D}$ and every element L of $\mathcal{L}$, $L \cap D \in \mathcal{D}$.

Note that $\mathcal{F}$, $\mathcal{C}$, $\mathcal{K}$ are $\mathcal{L}$-ideals in X, where in the case of $\mathcal{K}$ we assume $\mathcal{K} \subset t\mathcal{L}$. Also, the collection of all $\mathcal{L}$-countably compact sets is an $\mathcal{L}$-ideal in X.

THEOREM 4.7. *Consider any $\mathcal{L}$-ideal $\mathcal{J}$ in X. Then $\mathcal{L} \cup \mathcal{J}$ is a lattice.* (Proof omitted.) *Denote $\mathcal{L} \cup \mathcal{J}$ by $\tilde{\mathcal{L}}$. Note that $\mathcal{L} \subset \tilde{\mathcal{L}}$.*

1. *If $\mathscr{L}$ is replete, any one of the four conditions of Theorem 3.1 is satisfied and $\tilde{\mathscr{L}} \subset t\mathscr{L}$ then $\tilde{\mathscr{L}}$ is replete.*

2. *If $\tilde{\mathscr{L}}$ is replete and $\tilde{\mathscr{L}}$ is $\mathscr{L}$-countably paracompact or $\mathscr{L}$-countably bounded then $\mathscr{L}$ is replete.*

Proof. Use Theorem 3.1 to prove part 1, and Theorem 3.4 for part 2.

Remarks. (1) The condition $\tilde{\mathscr{L}} \subset t\mathscr{L}$ is equivalent to $\mathscr{I} \subset t\mathscr{L}$.

(2) If $\mathscr{L}$ is replete and for every element ν of $IR(\sigma, \tilde{\mathscr{L}})$, $\nu|_{\mathscr{I}}$ is $\mathscr{L}$-regular and $\mathscr{L}$ is Lindelöf, then $\tilde{\mathscr{L}}$ is replete. (Proof omitted.)

The following lemma is related to Theorem 4.7, part 1, in that it gives an answer to the question of when $\mathscr{L}$ semiseparates $\tilde{\mathscr{L}}$; it also gives an answer to the question of when $\mathscr{L}$ separates $\tilde{\mathscr{L}}$.

LEMMA 4.4. *Consider any $\mathscr{L}$-ideal $\mathscr{I}$ in X.*

1. *If $\mathscr{L}$ semiseparates $\mathscr{I}$ then $\mathscr{L}$ semiseparates $\tilde{\mathscr{L}}$.*
2. *If $\mathscr{L}$ semiseparates $\mathscr{I}$ and $\mathscr{L}$ separates $\mathscr{I}$ then $\mathscr{L}$ separates $\tilde{\mathscr{L}}$.*

Proof. To prove part 1, assume $\mathscr{L}$ semiseparates $\mathscr{I}$. Consider any element L of $\mathscr{L}$ and any element A of $\tilde{\mathscr{L}}$ such that $L \cap A = \emptyset$. Since $A \in \tilde{\mathscr{L}}$, there exist an element $\hat{L}$ of $\mathscr{L}$ and an element I of $\mathscr{I}$ such that $A = \hat{L} \cup I$. Then

$$\emptyset = L \cap A = L \cap (\hat{L} \cup I) = (L \cap \hat{L}) \cup (L \cap I).$$

Hence $L \cap \hat{L} = \emptyset$ and $L \cap I = \emptyset$. Since $L \cap I = \emptyset$ and $\mathscr{L}$ semiseparates $\mathscr{I}$, there exists an element $\check{L}$ of $\mathscr{L}$ such that $I \subset \check{L}$ and $L \cap \check{L} = \emptyset$. Then $\hat{L} \cup \check{L} \in \mathscr{L}$, $A \subset \hat{L} \cup \check{L}$ and

$$L \cap (\hat{L} \cup \check{L}) = (L \cap \hat{L}) \cup (L \cap \check{L}) = \emptyset \cup \emptyset = \emptyset.$$

Hence $\mathscr{L}$ semiseparates $\tilde{\mathscr{L}}$.

We omit the proof of part 2.

Section 5

In this section we consider an arbitrary set X and an arbitrary lattice $\mathscr{L}$ of subsets of X, and we define compactification of X, repletion of X, fully-repletion of X, prime completion of X, Cauchy completion of X, and almost-repletion of X. Then, we introduce a model for the representation of the pairs

$$\langle I(\mathscr{L}), V(\mathscr{L})\rangle, \quad \langle IR(\mathscr{L}), W(\mathscr{L})\rangle, \quad \langle IR(\sigma, \mathscr{L}), W(\sigma, \mathscr{L})\rangle,$$

$$\langle I(\sigma, \mathscr{L}), V(\sigma, \mathscr{L})\rangle, \quad \langle I(\sigma^*, \mathscr{L}), V(\sigma^*, \mathscr{L})\rangle$$

and others. (For the definitions of $V(\mathscr{L})$, $W(\mathscr{L})$, $W(\sigma, \mathscr{L})$, etc., see special cases below.) By introducing suitable conditions, we obtain a compactification of X, a repletion of X, a fully-repletion of X, a prime completion of X, a Cauchy completion of X, and an almost-repletion of X.

DEFINITION 5.1. A topological space $\langle Y, \mathcal{F}\rangle$ is a compactification of X, a repletion of X, a fully-repletion of X, a prime completion of X, a Cauchy completion of X, or an almost-repletion of X iff $\mathcal{F}$ is compact, replete, fully-replete, prime complete, Cauchy complete, or almost-replete, respectively, and there exists a function ϕ from X to Y such that ϕ is a $\langle t\mathcal{L}, \mathcal{F} \cap \phi(X)\rangle$-homeomorphism and $\phi(X)$ is dense in Y.

Next, we introduce the model mentioned above.

Preliminaries. Consider any subset I of $I(\mathcal{L})$ and any lattice $\mathcal{S}$ contained in $\mathcal{A}(\mathcal{L})$. Denote the general element of $\mathcal{A}(\mathcal{S})$ by A and let $H(A) = \{\mu \in I | \mu(A) = 1\}$.

1. If $A \in \mathcal{A}(\mathcal{S})$, then $I - H(A) = H(A')$.
2. Let $A, B \in \mathcal{A}(\mathcal{S})$.

(α) $H(A \cup B) = H(A) \cup H(B)$;
(β) $H(A \cap B) = H(A) \cap H(B)$;
(γ) If $A \supset B$ then $H(A) \supset H(B)$;
(δ) If $\{\mu_x; x \in X\} \subset I$, then $H(A) \supset H(B)$ implies $A \supset B$;
(ε) If $\{\mu_x; x \in X\} \subset I$, then $A = B$ iff $H(A) = H(B)$.

We shall prove only (δ). Assume $\{\mu_x; x \in X\} \subset I$ and consider any two elements A, B of $\mathcal{A}(\mathcal{S})$ such that $H(A) \supset H(B)$. Consider the case $B \neq \emptyset$. Consider any element x of B. Note $\mu_x \in I$ and $\mu_x(B) = 1$. Hence $\mu_x \in H(B)$. Consequently $\mu_x \in H(A)$, and $x \in A$. Consequently $A \supset B$. Hence (δ) is true.

We summarize these results as follows.

LEMMA 5.1. *Consider the function H such that $D_H = \mathcal{A}(\mathcal{S})$, and for every element A of $\mathcal{A}(\mathcal{S})$, $H(A) = \{\mu \in I \mid \mu(A) = 1\}$. Then H is a Boolean homomorphism from $\mathcal{A}(\mathcal{S})$ to the algebra of subsets of I. If $\{\mu_x; x \in X\} \subset I$, then H is one-to-one.*

Also $\mathcal{A}(H(\mathcal{S})) = H(\mathcal{A}(\mathcal{S}))$.
(Proof omitted.)

LEMMA 5.2. *Given $\nu \in I(H(\mathcal{S}))$, define $\bar{\nu}$ by $\bar{\nu}(A) = \nu(H(A))$ for $A \in \mathcal{A}(\mathcal{S})$. Then $\bar{\nu} \in I(\mathcal{S})$. The map $^-$: $I(H(\mathcal{S})) \to I(\mathcal{S})$ is one-to-one and is also onto when (and only when) H is one-to-one. If H is one-to-one, then $\nu \in IR(H(\mathcal{S}))$ iff $\bar{\nu} \in IR(\mathcal{S})$.*

Proof. For any Boolean algebra $\mathcal{B}$, $\mu \in I(\mathcal{B})$ can be identified with the ultrafilter $\{B \in \mathcal{B} \mid \mu(B) = 1\}$. Thus the interrelationship of the maps H

and $^-$ is an instance of the well-known (see Chapter IV, Section 3 of [5]) contravariant coequivalence of the categories of Boolean algebras and Boolean spaces. The final assertion is obvious.

If H is one-to-one, then for each element λ of $I(\mathcal{S})$, we designate by $\hat{\lambda}$ the unique element of $I(H(\mathcal{S}))$ such that $\bar{\hat{\lambda}} = \lambda$.

Observation. If $\mu \in I(H(\mathcal{S}))$ satisfies $\bar{\mu} \in I$, then $S(\mu) \neq \emptyset$.

Proof. Consider any element μ of $I(H(\mathcal{S}))$ such that $\bar{\mu} \in I$. Note that

$$S(\mu) = \cap \{H(S) | S \in \mathcal{S} \text{ and } \mu(H(S)) = 1\}.$$

Further, note that for every element S of $\mathcal{S}$, if $\mu(H(S)) = 1$ then $\bar{\mu}(S) = 1$, and, since $\bar{\mu} \in I$, $\bar{\mu} \in H(S)$. Consequently $\bar{\mu} \in S(\mu)$, and $S(\mu) \neq \emptyset$.

The following lemma will be used in the study of the various special cases represented by the model.

LEMMA 5.3. *Assume $\mathcal{L}$ is separating and consider any subset I of $I(\mathcal{L})$. Consider the function ϕ such that $D_\phi = X$ and, for every element x of X, $\phi(x) = \mu_x$, and assume $\phi(X) \subset I$, if necessary. Since $\mathcal{L}$ is separating, ϕ is one-to-one. Identify $\phi(X)$ with X. Then for every element L of $\mathcal{L}$, the $tH(\mathcal{L})$-closure of L (denoted by $\bar{L}$) is equal to $H(L)$.*

Proof. Consider any element L of $\mathcal{L}$. Since $\phi(X) \subset I$, $L \subset H(L)$. Hence $\bar{L} \subset H(L)$. To show that $H(L) \subset \bar{L}$, consider the case $H(L) \neq \emptyset$. Consider any element μ of $H(L)$ and any basic open set $H(\tilde{L})'$ (complement with respect to I) such that $\mu \in H(\tilde{L})'$ and show $H(\tilde{L})' \cap L \neq \emptyset$. Assume $H(\tilde{L})' \cap L = \emptyset$. Then, since $\phi(X) \subset I$, $\tilde{L}' \cap L = \emptyset$. Hence $L \subset \tilde{L}$. Since $\mu \in H(L)$, $\mu(L) = 1$. Consequently $\mu(\tilde{L}) = 1$. Moreover, since $\mu \in H(\tilde{L})'$, $\mu(\tilde{L}') = 1$, a contradiction. Hence $H(\tilde{L})' \cap L \neq \emptyset$, and $\mu \in \bar{L}$. Consequently $\bar{L} = H(L)$.

Observation. $\bar{X} = H(X) = I$; i.e., X is dense in I.

Special case (1). Let $I = I(\mathcal{L})$, $\mathcal{S} = \mathcal{L}$, and denote $H(\mathcal{L})$ by $V(\mathcal{L})$. If $\mu \in I(V(\mathcal{L}))$, then $\bar{\mu} \in I$, from which it follows that $V(\mathcal{L})$ is compact.

THEOREM 5.1. *Assume that $\mathcal{L}$ is separating and consider the topological space $\langle I(\mathcal{L}), tV(\mathcal{L})\rangle$.*

1. *$\langle I(\mathcal{L}), tV(\mathcal{L})\rangle$ is T_0.*
2. *$\langle I(\mathcal{L}), tV(\mathcal{L})\rangle$ is a compactification of X.*

Proof. 1. Consider any two elements μ_1, μ_2 of $I(\mathcal{L})$ such that $\mu_1 \neq \mu_2$. Then there exists an element L of $\mathcal{L}$ such that $\mu_1(L) \neq \mu_2(L)$. Assume for example that $\mu_1(L) = 1$. Then $\mu_2(L) = 0$. Consequently $\mu_2 \in V(L)'$ and $\mu_1 \notin V(L)'$. Hence $\langle I(\mathcal{L}), tV(\mathcal{L})\rangle$ is T_0.

2. Since $V(\mathcal{L})$ is compact, $tV(\mathcal{L})$ is compact. Next, consider the function

ϕ introduced in Lemma 5.3. Since $tV(\mathscr{L}) \cap \phi(X) = t\phi(\mathscr{L})$ and ϕ is one-to-one, ϕ is a $\langle t\mathscr{L}, tV(\mathscr{L}) \cap \phi(X)\rangle$-homeomorphism. Moreover, X is dense in $I(\mathscr{L})$ (see Lemma 5.3). Consequently $\langle I(\mathscr{L}), tV(\mathscr{L})\rangle$ is a compactification of X.

Special case (2). Let $I = IR(\mathscr{L})$, $\mathscr{S} = \mathscr{L}$, and denote $H(\mathscr{L})$ by $W(\mathscr{L})$. If H is one-to-one, then, by Lemma 5.2, $\mu \in IR(W(\mathscr{L}))$ implies $\bar{\mu} \in I$, from which it follows that $W(\mathscr{L})$ is compact. In particular, note that

$$\{\mu_x; x \in X\} \subset IR(\mathscr{L})$$

if $\mathscr{L}$ is disjunctive (and conversely), and so the foregoing statement is true in this case. However, it is easy to show that $W(\mathscr{L})$ is always compact.

THEOREM 5.2. *Assume $\mathscr{L}$ is separating and disjunctive and consider the topological space $\langle IR(\mathscr{L}), tW(\mathscr{L})\rangle$.*

1. *$\langle IR(\mathscr{L}), tW(\mathscr{L})\rangle$ is T_1.*
2. *$\langle IR(\mathscr{L}), tW(\mathscr{L})\rangle$ is a compactification of X.*

This theorem is well known (e.g., see [17]).

Special case (3). Let $I = IR(\sigma, \mathscr{L})$, $\mathscr{S} = \mathscr{L}$ with $\mathscr{L}$ disjunctive, and denote $H(\mathscr{L})$ by $W(\sigma, \mathscr{L})$. If $\mu \in IR(\sigma, W(\sigma, \mathscr{L}))$, then $\bar{\mu} \in I$, from which it follows that $W(\sigma, \mathscr{L})$ is replete.

Proof. Assume $\mu \in IR(\sigma, W(\sigma, \mathscr{L}))$. Since $\mathscr{L}$ is disjunctive,

$$\{\mu_x; x \in X\} \subset I.$$

Hence H is one-to-one, and, by Lemma 5.2, $\bar{\mu} \in IR(\mathscr{L})$. Now, to show that $\bar{\mu} \in IR(\sigma, \mathscr{L}) = I$, consider any sequence $\langle L_n\rangle$ in $\mathscr{L}$ such that $\langle L_n\rangle$ is decreasing and $\lim_n L_n = \emptyset$. Note that

$$lim_n\, \bar{\mu}(L_n) = \lim_n \mu(W(\sigma, L_n)).$$

Since $\langle L_n\rangle$ is decreasing and $\lim_n L_n = \emptyset$, $\langle W(\sigma, L_n)\rangle$ is decreasing and $\lim_n W(\sigma, L_n) = \emptyset$. Hence, since $\mu \in IR(\sigma, W(\sigma, \mathscr{L}))$,

$$lim_n\, \mu(W(\sigma, L_n)) = 0.$$

Consequently $\lim_n \bar{\mu}(L_n) = 0$, and $\bar{\mu} \in IR(\sigma, \mathscr{L}) = I$.

THEOREM 5.3. *Assume that $\mathscr{L}$ is separating and disjunctive and consider the topological space $\langle IR(\sigma, \mathscr{L}), tW(\sigma, \mathscr{L})\rangle$.*

(α) *Since $\langle IR(\sigma, \mathscr{L}), tW(\sigma, \mathscr{L})\rangle$ is a subspace of $\langle IR(\mathscr{L}), tW(\mathscr{L})\rangle$ and the latter space is T_1, the former space is T_1.*

(β) *Further, assume any one of the four conditions of Theorem 3.1 is satisfied by $W(\sigma, \mathscr{L})$. Then, since $W(\sigma, \mathscr{L})$ is replete, by Theorem 3.1,*

$tW(\sigma, \mathscr{L})$ is replete. Hence $\langle IR(\sigma, \mathscr{L}), tW(\sigma, \mathscr{L})\rangle$ is α-complete. Consequently

$$\langle IR(\sigma, \mathscr{L}), tW(\sigma, \mathscr{L})\rangle$$

is a repletion of X.

Special case (4). Let $I = I(\sigma, \mathscr{L})$, $\mathscr{S} = \mathscr{L}$, and denote $H(\mathscr{L})$ by $V(\sigma, \mathscr{L})$. If

$$\mu \in I(\sigma, V(\sigma, \mathscr{L}))$$

then $\bar{\mu} \in I$, from which it follows that $V(\sigma, \mathscr{L})$ is fully-replete.

THEOREM 5.4. *Assume $\mathscr{L}$ is separating and consider the topological space $\langle I(\sigma, \mathscr{L}), tV(\sigma, \mathscr{L})\rangle$.*

1. *$\langle I(\sigma, \mathscr{L}), tV(\sigma, \mathscr{L})\rangle$ is T_0.*
2. *$\langle I(\sigma, \mathscr{L}), tV(\sigma, \mathscr{L})\rangle$ is a fully-repletion of X.*

(Proof omitted.)

Special case (5). Let $I = I(\sigma^*, \mathscr{L})$, $\mathscr{S} = \mathscr{L}$, and denote $H(\mathscr{L})$ by $V(\sigma^*, \mathscr{L})$. If

$$\mu \in I(\sigma^*, V(\sigma^*, \mathscr{L}))$$

then $\bar{\mu} \in I$, from which it follows that $V(\sigma^*, \mathscr{L})$ is prime complete.

THEOREM 5.5. *Assume that $\mathscr{L}$ is separating and consider the topological space $\langle I(\sigma^*, \mathscr{L}), tV(\sigma^*, \mathscr{L})\rangle$.*

1. *$\langle I(\sigma^*, \mathscr{L}), tV(\sigma^*, \mathscr{L})\rangle$ is T_0.*
2. *$\langle I(\sigma^*, \mathscr{L}), tV(\sigma^*, \mathscr{L})\rangle$ is a prime completion of X.*

(Proof omitted.)

Special case (6). Let $I = I(\sigma^*, \mathscr{L}')$, $\mathscr{S} = \mathscr{L}$, and denote $H(\mathscr{L})$ by $U(\mathscr{L})$. If

$$\mu \in I(\sigma^*, U(\mathscr{L})')$$

then $\bar{\mu} \in I$, from which it follows that $U(\mathscr{L})$ is Cauchy complete.

THEOREM 5.6. *Assume that $\mathscr{L}$ is separating and consider the topological space $\langle I(\sigma^*, \mathscr{L}'), tU(\mathscr{L})\rangle$.*

1. *$\langle I(\sigma^*, \mathscr{L}'), tU(\mathscr{L})\rangle$ is T_0.*
2. *$\langle I(\sigma^*, \mathscr{L}'), tU(\mathscr{L})\rangle$ is a Cauchy completion of X.*

(Proof omitted.)

Special case (7). Let $I = \{\mu_x; x \in X\} \cup (IR(\mathscr{L}') \cap I(\sigma^*, \mathscr{L}))$, $\mathscr{S} = \mathscr{L}$, and denote $H(\mathscr{L})$ by $T(\mathscr{L})$. If $\mu \in IR(T(\mathscr{L})') \cap I(\sigma^*, T(\mathscr{L}))$, then $\bar{\mu} \in I$, from which it follows that $T(\mathscr{L})$ is almost-replete.

Proof. Assume

$$\mu \in IR(T(\mathscr{L})') \cap I(\sigma^*, T(\mathscr{L})).$$

Since $\{\mu_x; x \in X\} \subset I$, H is one-to-one, and, by Lemma 5.2, $\bar{\mu} \in IR(\mathscr{L}')$. The argument that $\bar{\mu} \in I(\sigma^*, \mathscr{L})$ is similar to (3), and, hence, also to (4), (5), and (6).

THEOREM 5.7. *Assume that $\mathscr{L}$ is separating and consider the topological space $\langle I, tT(\mathscr{L})\rangle$.*

1. *$\langle I, tT(\mathscr{L})\rangle$ is T_0; it is T_1 iff $\mathscr{L}'$ is disjunctive.*
2. *$\langle I, tT(\mathscr{L})\rangle$ is an almost-repletion of X if $T(\mathscr{L})'$ semiseparates $(tT(\mathscr{L}))'$.*

Remark 1. We will show that $\langle I, tT(\mathscr{L})\rangle$ is T_1 iff $\mathscr{L}'$ is disjunctive.

(α) Assume $\mathscr{L}'$ is disjunctive. Consider any two elements ν_1, ν_2 of I. Note that

$$\nu_1, \nu_2 \in IR(\mathscr{L}') \cap I(\sigma^*, \mathscr{L}) \quad \text{(Case 1)}$$

or

$$\nu_1, \nu_2 \in \{\mu_x; x \in X\} \quad \text{(Case 2)}$$

or

$$\nu_1 \in \{\mu_x; x \in X\} \text{ and } \nu_2 \in IR(\mathscr{L}') \cap I(\sigma^*, \mathscr{L}) \quad \text{(Case 3).}$$

In Case 1, to obtain the desired separation of ν_1 and ν_2, use the $\mathscr{L}'$-regularity of ν_1 and ν_2. As for Cases 2 and 3, first use the disjunctiveness of $\mathscr{L}'$ to establish the $\mathscr{L}'$-regularity of ν_1 and ν_2 in Case 2, or of ν_1 in Case 3, and then proceed as in Case 1.

(β) Assume $\mathscr{L}'$ is not disjunctive. Then there exists an element x of X such that μ_x is not $\mathscr{L}'$-regular. Now, consider any element ν of $IR(\mathscr{L}')$ such that $\mu_x \leq \nu$ on $\mathscr{L}'$. Then for every basic open set $V(L)' \cap I$, if $\mu_x \in V(L)' \cap I$, then $\nu \in V(L)' \cap I$, so that the desired separation of μ_x and ν is not possible. Consequently $\langle I, tT(\mathscr{L})\rangle$ is not T_1.

Remark 2. $T(\mathscr{L})'$ semiseparates $(tT(\mathscr{L}))'$ if $\mathscr{L}$ is an algebra.

Section 6

In this section we consider an arbitrary set X and an arbitrary lattice $\mathscr{L}$ of subsets of X, and we construct a prime completion of X which is T_1. We note that it is possible to construct a fully-repletion of X which is T_1, a Cauchy completion of X which is T_1, and an almost-repletion of X which is T_1, by using the same method.

LEMMA 6.1. *Denote any element of the set*

$$\{IR(\sigma, \mathscr{L}), I(\sigma, \mathscr{L}), I(\sigma^*, \mathscr{L}), I(\sigma^*, \mathscr{L}'), \{\mu_x; x \in X\} \cup (IR(\mathscr{L}') \cap I(\sigma^*, \mathscr{L}))\}$$

by I. Then each of the sets $W(\sigma, \mathscr{L})$, $V(\sigma, \mathscr{L})$, $V(\sigma^*, \mathscr{L})$, $U(\mathscr{L})$, $T(\mathscr{L})$, is denoted by $H(\mathscr{L})$. (See Section 5, special cases (3), (4), (5), (6), (7).) *Next, denote the collection of all finite subsets of I by $\mathscr{F}$, and denote $H(\mathscr{L}) \cup \mathscr{F}$ by $\tilde{H}(\mathscr{L})$* (cf. Theorem 4.5). *Then $\tilde{H}(\mathscr{L})$ is respectively replete (if $\mathscr{L}$ is disjunctive), fully-replete, prime complete, Cauchy complete, almost-replete (if $H(\mathscr{L})'$ semiseparates $\tilde{H}(\mathscr{L})'$).*

Proof. Consider any element ν of

$$IR(\sigma, \tilde{H}(\mathscr{L})), I(\sigma, \tilde{H}(\mathscr{L})), I(\sigma^*, \tilde{H}(\mathscr{L})), I(\sigma^*, \tilde{H}(\mathscr{L}')), IR(\tilde{H}(\mathscr{L})') \cap I(\sigma^*, \tilde{H}(\mathscr{L})),$$

and show (imposing suitable conditions, if necessary) that $S(\nu) \neq \emptyset$. Note that either there exists an element F of $\mathscr{F}$ such that $\nu(F) = 1$ (Case 1) or for every element F of $\mathscr{F}$, $\nu(F) = 0$ (Case 2).

Case 1. Consider any element F_0 of $\mathscr{F}$ such that $\nu(F_0) = 1$. Then, since $\nu \in I(\tilde{H}(\mathscr{L}))$, there exists an element μ_0 of F_0 such that $\nu(\{\mu_0\}) = 1$ and μ_0 is unique. Hence for every element A of $\tilde{H}(\mathscr{L})$, if $\nu(A) = 1$, then $\mu_0 \in A$. Consequently $\mu_0 \in S(\nu)$. Hence $S(\nu) \neq \emptyset$.

Case 2. Consider $\nu|_{\mathscr{A}(H(\mathscr{L}))}$ and denote it by μ. Show $S(\nu) \supset S(\mu)$. Consider any element A of $\tilde{H}(\mathscr{L})$. Then there exist an element L of $\mathscr{L}$ and an element F of $\mathscr{F}$ such that $A = H(L) \cup F$. Then

$$\nu(A) = \nu(H(L) \cup F) = \nu(H(L)),$$

since $\nu(F) = 0$ by the assumption. Consequently if $\nu(A) = 1$, then there exists an element L of $\mathscr{L}$ such that

$$H(L) \subset A \quad \text{and} \quad \mu(H(L)) = \nu(H(L)) = 1.$$

Hence

$$\cap \{A \in \tilde{H}(\mathscr{L}) | \nu(A) = 1\} \supset \cap \{H(L) | L \in \mathscr{L} \text{ and } \mu(H(L)) = 1\}.$$

Consequently $S(\nu) \supset S(\mu)$. Next, we will show that $S(\mu) \neq \emptyset$.

(α) Assume $\mathscr{L}$ is disjunctive. Then $W(\sigma, \mathscr{L})$ is replete. (See special case (3).) Hence, by Theorem 4.5, Case 1, $\tilde{W}(\sigma, \mathscr{L})$ is replete.

(β) Since $V(\sigma, \mathscr{L})$ is fully-replete (see special case (4)) and $\mu \in I(\sigma, V(\sigma, \mathscr{L}))$, $S(\mu) \neq \emptyset$.

(γ) Since $V(\sigma^*, \mathscr{L})$ is prime complete (see special case (5)) and $\mu \in I(\sigma^*, V(\sigma^*, \mathscr{L}))$, $S(\mu) \neq \emptyset$.

(δ) Since $U(\mathscr{L})$ is Cauchy complete (see special case (6)) and $\mu \in I(\sigma^*, U(\mathscr{L})')$, $S(\mu) \neq \emptyset$.

(ε) Consider special case (7). Note that $T(\mathscr{L})$ is almost-replete. Further, assume that $T(\mathscr{L})'$ semiseparates $\tilde{T}(\mathscr{L})'$. Then, since

$$\nu \in IR(\tilde{T}(\mathscr{L})') \cap I(\sigma^*, \tilde{T}(\mathscr{L})),$$

we have

$$\mu \in IR(T(\mathscr{L})') \cap I(\sigma^*, T(\mathscr{L})).$$

Consequently $S(\mu) \neq \emptyset$ (except possibly in case (α) where this condition turned out to be irrelevant).

Observation. $t\tilde{H}(\mathscr{L})$ is respectively replete (if any one of the four conditions of Theorem 3.1 is satisfied), fully-replete, prime complete, and so on.

LEMMA 6.2. *Assume $\mathscr{L}$ is separating and $t\mathscr{L}$ is T_1. Consider the topological space $\langle I(\sigma^*, \mathscr{L}), t\tilde{V}(\sigma^*, \mathscr{L})\rangle$.*

If (1) $t\mathscr{L} = \mathscr{L}$, *or* (2) *for every finite nonempty subset S of X, $S \notin (t\mathscr{L})'$, (i.e., no finite nonempty subset of X is clopen) then $\overline{X} = I(\sigma^*, \mathscr{L})$ (i.e., X is dense in $I(\sigma^*, \mathscr{L})$).*

Proof. Case (1). Consider any basic open set, i.e., any element A' of $\tilde{V}(\sigma^*, \mathscr{L})'$ such that $A' \neq \emptyset$, and show $A' \cap X \neq \emptyset$. Since $A \in \tilde{V}(\sigma^*, \mathscr{L})$, there exist an element L of $\mathscr{L}$ and an element F of $\mathscr{F}$ such that $A = V(\sigma^*, L) \cup F$. Since $I(\sigma^*, \mathscr{L}) = X \cup (I(\sigma^*, \mathscr{L}) - X)$,

$$F = (X \cap F) \cup ((I(\sigma^*, \mathscr{L}) - X) \cap F).$$

Note that

$$X \cap F \in \mathscr{F} \quad \text{and} \quad (I(\sigma^*, \mathscr{L}) - X) \cap F \in \mathscr{F}.$$

Denote $X \cap F$ by F_1 and $(I(\sigma^*, \mathscr{L}) - X) \cap F$ by F_2. Then

$$F = F_1 \cup F_2, \quad F_1 \subset X \quad \text{and} \quad F_2 \subset I(\sigma^*, \mathscr{L}) - X.$$

Since F_1 is finite and $t\mathscr{L}$ is T_1, $F_1 \in t\mathscr{L}$. Hence, since $t\mathscr{L} = \mathscr{L}$, $F_1 \in \mathscr{L}$. Next, we show that $F_1 = V(\sigma^*, F_1)$.

(α) Note that $F_1 \subset V(\sigma^*, F_1)$.

(β) We will show that $V(\sigma^*, F_1) \subset F_1$. Consider the case $V(\sigma^*, F_1) \neq \emptyset$. Consider any element ν of $V(\sigma^*, F_1)$. Then $\nu(F_1) = 1$. Hence, since $\nu \in I(\mathscr{L})$, there exists an element x of F_1 such that $\nu(\{x\}) = 1$ and x is unique. Then $\nu = \mu_x$. Hence $V(\sigma^*, F_1) \subset F_1$.

(γ) So $F_1 = V(\sigma^*, F_1)$. Consequently

$$\begin{aligned} A &= V(\sigma^*, L) \cup F \\ &= V(\sigma^*, L) \cup (F_1 \cup F_2) \\ &= (V(\sigma^*, L) \cup F_1) \cup F_2 \\ &= (V(\sigma^*, L) \cup V(\sigma^*, F_1)) \cup F_2 \\ &= V(\sigma^*, L \cup F_1) \cup F_2. \end{aligned}$$

Since $F_1 \in \mathscr{L}$, $L \cup F_1 \in \mathscr{L}$. Denote $L \cup F_1$ by $\tilde{L}$. Then

$$A = V(\sigma^*, \tilde{L}) \cup F_2.$$

Hence

$$A' = V(\sigma^*, \tilde{L})' \cap F_2' \supset \tilde{L}' \cap F_2'.$$

Since $F_2 \subset I(\sigma^*, \mathscr{L}) - X$, $F_2' \supset X$. Consequently $A' \supset \tilde{L}' \cap X = \tilde{L}'$. Since $A' \neq \emptyset$ and $A' = V(\sigma^*, \tilde{L}') \cap F_2'$, $\tilde{L}' \neq \emptyset$. Consequently $A' \cap X \neq \emptyset$.

Case (2). Consider any basic open set A' such that $A' \neq \emptyset$, and show $A' \cap X \neq \emptyset$. Since $A = V(\sigma^*, L) \cup F$ (see Case 1), $A' \cap X \neq \emptyset$ iff $A \not\supset X$ iff $V(\sigma^*, L) \cup F \not\supset X$ iff $L \cup F \not\supset X$ iff $L \cup F_1 \neq X$, where $F_1 = X \cap F$. Since $L \in t\mathscr{L}$, and $F_1 \notin (t\mathscr{L})'$ by the assumption, $L \cup F_1 \neq X$. Consequently $A' \cap X \neq \emptyset$.

Thus for every element G of $(t\tilde{V}(\sigma^*, L))'$, if $G \neq \emptyset$, then $G \cap X \neq \emptyset$. Hence $\overline{X} = I(\sigma^*, \mathscr{L})$.

THEOREM 6.1. *Assume that $\mathscr{L}$ is separating and $t\mathscr{L}$ is T_1. Then if $t\mathscr{L} = \mathscr{L}$, or for every finite nonempty subset S of X, $S \notin (t\mathscr{L})'$ (i.e., no finite nonempty subset of X is clopen) then*

1. *$\langle I(\sigma^*, \mathscr{L}), t\tilde{V}(\sigma^*, \mathscr{L})\rangle$ is T_1 and*
2. *$\langle I(\sigma^*, \mathscr{L}), t\tilde{V}(\sigma^*, \mathscr{L})\rangle$ is a prime completion (hence, a repletion) of X.*

Proof. 1. Since $t\tilde{V}(\sigma^*, \mathscr{L})$ contains $\mathscr{F}$, $\langle I(\sigma^*, \mathscr{L}), t\tilde{V}(\sigma^*, \mathscr{L})\rangle$ is T_1.

2. Recall that $t\tilde{V}(\sigma^*, \mathscr{L})$ is prime complete. (See Lemma 6.1.) Next, consider the function ϕ introduced in Lemma 5.3. We will show that ϕ is a $\langle t\mathscr{L}, t\tilde{V}(\sigma^*, \mathscr{L}) \cap \phi(X)\rangle$-homeomorphism. Since ϕ is one-to-one, it suffices to show that $t\tilde{V}(\sigma^*, \mathscr{L}) \cap X = t\mathscr{L}$. To show that

$$t\tilde{V}(\sigma^*, \mathscr{L}) \cap X \subset t\mathscr{L},$$

consider any element S of $t\tilde{V}(\sigma^*, \mathscr{L}) \cap X$. Then there exists a subset $\{A_\lambda; \lambda \in \Lambda\}$ of $\tilde{V}(\sigma^*, \mathscr{L})$ such that

$$S = (\cap \{A_\lambda; \lambda \in \Lambda\}) \cap X.$$

Note that for every element λ of Λ, since $A_\lambda \in \tilde{V}(\sigma^*, \mathscr{L})$, there exist an element L_λ of $\mathscr{L}$ and an element F of $\mathscr{F}$ such that

$$A_\lambda = V(\sigma^*, L_\lambda) \cup F_\lambda.$$

Then $S = \cap \{L_\lambda \cup (F_\lambda \cap X); \lambda \in \Lambda\}$. Note that for every λ, $F_\lambda \cap X$ is a finite subset of X; hence, since $t\mathscr{L}$ is T_1, $F_\lambda \cap X \in t\mathscr{L}$; hence

$$L_\lambda \cup (F_\lambda \cap X) \in t\mathscr{L}.$$

Consequently $S \in t\mathscr{L}$. Hence $t\tilde{V}(\sigma^*, \mathscr{L}) \cap X \subset t\mathscr{L}$. We omit the proof that

$$t\mathscr{L} \subset t\tilde{V}(\sigma^*, \mathscr{L}) \cap X.$$

Consequently

$$t\tilde{V}(\sigma^*, \mathscr{L}) \cap X = t\mathscr{L},$$

and ϕ is a $\langle t\mathscr{L}, t\tilde{V}(\sigma^*, \mathscr{L}) \cap \phi(X)\rangle$-homeomorphism.

Assume also that $t\mathscr{L} = \mathscr{L}$, or for every finite nonempty subset S of X, $S \notin (t\mathscr{L})'$. Then $\bar{X} = I(\sigma^*, \mathscr{L})$ (Lemma 6.2).

Consequently $\langle I(\sigma^*, \mathscr{L}), t\tilde{V}(\sigma^*, \mathscr{L})\rangle$ is a prime completion of X.

Application. Consider any topological space X such that X is T_1 and denote its collection of closed sets by $\mathscr{F}$. Then

$$\langle I(\sigma^*, \mathscr{F}), t\tilde{V}(\sigma^*, \mathscr{F})\rangle$$

is T_1 and is a prime completion (hence, a repletion) of X.

Remark. It is possible to construct a fully-repletion of X which is T_1, a Cauchy completion of X which is T_1, and an almost-repletion of X which is T_1, by using the same method.

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