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SPHERES AND CYLINDERS: A LOCAL GEOMETRIC CHARACTERIZATION

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This note characterizes the smooth hypersurfaces in Euclidean space which satisfy locally the following geometric condition.

(*) For each two points of the surface, the chord joining them meets the normal to the surface in equal angles at the two points.

This condition arose in the study of the Bochner-Martinelli integral formula, which is a higher-dimensional analogue of Cauchy's integral formula in the complex plane. In [1] the author proved that the Bochner-Martinelli operator, viewed as a bounded singular integral operator acting on the Hilbert space of square-integrable functions on the boundary of a smooth bounded domain, is self-adjoint if and only if the domain is a ball. The proof hinged on the following geometrical result, which has nothing to do with complex analysis.

GLOBAL CHARACTERIZATION THEOREM. *Let G be a bounded C^1 smooth domain in $R^k, k \geq 2$. Then the boundary of G satisfies* (*) *if and only if G is a ball.*

The proof of the Global Characterization Theorem in [1] uses a compactness argument, and therefore does not address the question, asked by N. Kerzman, of which surfaces satisfy (*) locally. It is easy to see that, in addition to subsets of spheres S^n, subsets of planes R^n and subsets of spherical cylinders $S^n \times R^m$ do satisfy (*). Our result in this article is that there are no other possibilities.

LOCAL CHARACTERIZATION THEOREM. *Let M be a connected smooth C^1 local hypersurface in R^k, $k \geq 2$. Then M satisfies* (*) *if and only if M lies in a surface of the form*

$$S^{k-1-j} \times R^j, \quad 0 \leq j \leq k-1.$$

Our hypotheses are, more explicitly, that there is an open set $U \subset R^k$ and a continuously differentiable normalized defining function

$$r: U \to R$$

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such that

$$M = \{x \in U \colon r(x) = 0\}$$

and the gradient ∇r *has unit length on* M, M *is connected, and for all points* x *and* y *of* M

$$(x - y) \cdot \nabla r(x) = (y - x) \cdot \nabla r(y), \tag{1}$$

where the dot denotes the usual Euclidean scalar product.

Proof. It is enough to show that M has the required form in some neighborhood of an arbitrary point $p \in M$, for the theorem then follows by an obvious connectedness argument. Therefore we may shrink U without loss of generality, and since condition (*) is preserved by conformal coordinate transformations, we are also free to carry out translations, rotations, and dilations. Making a rigid motion we may assume that $p = 0$ and that the unit normal $\nabla r(p)$ is the k-th basis vector $e_k = (0, \ldots, 0, 1)$, and shrinking U we may assume that U is a cube centered at the origin such that

$$\nabla r(x) \cdot e_k > 1/\sqrt{2}$$

for all $x \in U$. The latter choice, which invokes the continuity of the derivatives of r, in particular ensures by the implicit function theorem that M may be realized as the graph of a function of the first $k - 1$ variables.

For reference we list two direct consequences of (*): If x and y are two points of M with corresponding unit normals ν and μ, then

$$\nu \cdot x = -e_k \cdot x, \tag{2}$$

$$\mu \cdot x - \nu \cdot y = e_k \cdot (x - y). \tag{3}$$

Equation (2) results from (1) by setting $y = 0$. Equation (3) follows by subtracting (2) from (1) and then adding equation (2) with x replaced by y.

The theorem is proved by induction on the dimension k. The case $k = 2$ is in essence known [2], with a geometric demonstration. We give here an analytic proof. Let (u, v) denote any point of M other than the origin, and let $(m, \sqrt{1 - m^2})$ denote the corresponding unit normal. Then

$$mu + \sqrt{1 - m^2}\; v = -v$$

by equation (2). Hence $m = 0$ if and only if $v = 0$. In any case it follows that $u^2 - v^2 > 0$ and that

$$m = -2uv/(u^2 + v^2).$$

Since the slope of the curve M at (u, v) is $-m/\sqrt{1 - m^2}$, M satisfies the differential equation

$$\frac{dv}{du} = \frac{2uv}{u^2 - v^2} \tag{4}$$

in the domain $\{(u, v) \in U: u^2 > v^2\}$. The substitution $w = v/u$ transforms (4) into

$$u \frac{dw}{du} = \frac{w(1 + w^2)}{1 - w^2}$$

which has the integrals $cu = w/(1 + w^2)$, $c \in R$ arbitrary, that is, $v = c(u^2 + v^2)$. Thus both to the right and to the left of the origin M has the form of a circle (if we regard the line $v = 0$ as a degenerate circle). Since the hypothesis (*) holds at all points, it is clear that M is the same circle on both sides of the origin.

The basis step of the induction being established, we now assume the result for dimension $k - 1$, where $k > 2$. The proof divides into two cases, depending on whether or not the tangent plane intersects M at points other than the origin.

Case 1. $M \setminus \{0\}$ intersects the tangent plane $\{x: x \cdot e_k = 0\}$. After a rotation about the e_k-axis and a uniform dilation of the coordinates, we may assume that a point of intersection is the first basis vector $e_1 = (1, 0, \ldots, 0)$. Let ν denote the corresponding unit normal. The planar slice

$$M \cap \{x: x \cdot e_j = 0, j \neq 1, k\}$$

is a differentiable curve passing through the origin and the point e_1. By Rolle's theorem it follows that there is a point y on the curve, $0 < y \cdot e_1 < 1$, with unit normal $\mu = \nabla r(y)$ satisfying $\mu \cdot e_1 = 0$. By (2) we have

$$- e_k \cdot y = \mu \cdot y = (\mu \cdot e_k)(e_k \cdot y).$$

By assumption $\mu \cdot e_k > 1/\sqrt{2}$, so this forces $y \cdot e_k = 0$. Thus y lies on the e_1-axis, say $y = ce_1$, $0 < c < 1$.

Let x be an arbitrary point of M, with unit normal ω. By (3),

$$\omega \cdot e_1 - \nu \cdot x = e_k \cdot (e_1 - x) = - e_k \cdot x,$$

$$\omega \cdot ce_1 - \mu \cdot x = e_k \cdot (ce_1 - x) = - e_k \cdot x,$$

whence

$$(1 - c)\omega \cdot e_1 - (\nu - \mu) \cdot x = 0, \tag{5}$$

$$(\mu - c\nu) \cdot x = (1 - c)e_k \cdot x. \tag{6}$$

If M lies in a $(k - 1)$-dimensional hyperplane there is nothing to prove. Otherwise, there are k linearly independent points in M, and since (6) holds for all $x \in M$, it follows that

$$\mu - c\nu = (1 - c)e_k.$$

Then $\mu = \nu = e_k$, since all three are unit vectors. Equation (5) then reduces to $\omega \cdot e_1 = 0$. Thus the component of the gradient of r in the e_1-direction vanishes everywhere on M; M is independent of the e_1-direction. Hence M is the cartesian product of an interval I of the e_1-axis with the slice

$$M_1 = M \cap \{x : x \cdot e_1 = 0\}.$$

The slice M_1 inherits condition (*), so by the induction hypothesis M_1 has the required form. Then so does $M = I \times M_1$. This completes the proof of Case 1.

Case 2. M intersects the tangent plane $\{x : x \cdot e_k = 0\}$ only at the origin. In this case, $M \setminus \{0\}$ lies entirely on one side of the tangent plane, say below. It follows from (2) that if $x \in M \setminus \{0\}$ then $\nu \cdot e_k \neq 1$; since the k-th component of the normal ν by assumption is also different from -1, this means that the other components of ν do not all vanish. Therefore relatively open subsets of horizontal slices

$$M \cap \{x : x \cdot e_k = -a\}$$

are smooth. For $0 < \epsilon < \inf\{-x \cdot e_k : x \in M \cap \partial(\frac{1}{2} U)\}$, the slice

$$M_\epsilon = M \cap \{x : x \cdot e_k = -\epsilon\} \cap \frac{1}{2} U$$

is a compact set which we view as the boundary of a smooth C^1 bounded domain in the affine subspace

$$R^{k-1} \cong \{x : x \cdot e_k = -\epsilon\}.$$

There is a chord through M_ϵ of maximal length, and this chord necessarily meets M_ϵ orthogonally at both endpoints q_1 and q_2. Equation (1) then implies that the k-th components of the unit normals to M are equal at q_1 and q_2, and the remaining components are equal in magnitude and opposite in sign. Since $e_k \cdot q_1 = -\epsilon = e_k \cdot q_2$, equation (3) implies

$$(q_2 - q_1) \cdot (q_1 + q_2) = 0.$$

In other words, the hyperplane $H = \{x : x \cdot (q_2 - q_1) = 0\}$ through the origin orthogonal to the chord $q_2 - q_1$ also bisects this chord.

Let y be any point of the normal slice $M \cap H$. By symmetry, the chords $q_1 - y$ and $q_2 - y$ make equal angles with the normals to M at q_1 and q_2. Hence the chords $y - q_1$ and $y - q_2$ make equal angles with the normal to M at y. This means that the normal at y lies in H, so the slice $M \cap H$, viewed as a surface in $H \cong R^{k-1}$, inherits condition (*). By the induction hypothesis $M \cap H$ lies in a $(k-1)$-sphere, which has the e_k-direction as an axis of symmetry. (This slice is neither planar nor cylindrical because it contacts the tangent plane $\{x : x \cdot e_k = 0\}$ only at the origin.)

Via a rotation about the e_k-axis and a uniform dilation, we may assume that $H = \{x : x \cdot e_1 = 0\}$, $q_1 \cdot e_1 = 1$, and $q_2 \cdot e_1 = -1$. Let

$$me_1 + \sqrt{1 - m^2}\, e_k$$

denote the unit normal at q_1, where

$$m - \epsilon \sqrt{1 - m^2} = \epsilon \tag{7}$$

by equation (2), so that $m = 2\epsilon/(1 + \epsilon^2)$. Since $M \cap H$ is spherical, there is $a > 0$ such that for all $j \neq 1, k$, the points $\pm ae_j - \epsilon e_k$ lie in $M_\epsilon \cap H$ and have unit normals

$$\pm ne_j + \sqrt{1 - n^2}\, e_k,$$

where

$$an - \epsilon\sqrt{1 - n^2} = \epsilon. \tag{8}$$

By (3), we have

$$(\pm ne_j + \sqrt{1 - n^2}\, e_k) \cdot q_1 = (me_1 + \sqrt{1 - m^2}\, e_k) \cdot (\pm ae_j - \epsilon e_k), \quad j \neq 1, k.$$

Therefore $e_j \cdot q_1 = 0, j \neq 1, k$, and $m = n$. Hence $a = 1$ by (7) and (8). In sum, the points $\pm e_j - \epsilon e_k, j \neq k$, lie in M, and the corresponding unit normals are

$$\pm me_j + \sqrt{1 - m^2}\, e_k.$$

Now let x be any point of M, with corresponding unit normal ν. By (3)

$$\nu \cdot (\pm e_j - \epsilon e_k) - (\pm me_j + \sqrt{1 - m^2}\, e_k) \cdot x = e_k \cdot (\pm e_j - \epsilon e_k - x), \quad j \neq k,$$

whence

$$\nu \cdot e_j = me_j \cdot x, \quad j \neq k, \tag{9}$$

$$\epsilon\nu \cdot e_k + \sqrt{1 - m^2}\, e_k \cdot x = \epsilon + e_k \cdot x. \tag{10}$$

Inserting the value of m from (7) into (10) we find

$$\nu \cdot e_k = 1 + me_k \cdot x. \tag{11}$$

Combining (2), (9), and (11) yields $mx^2 + 2e_k \cdot x = 0$, which is the equation of a sphere of radius $1/m$ centered at $(0, \ldots, 0, -1/m)$. This completes the proof.

References

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2. H. Rademacher and O. Toeplitz, *The enjoyment of mathematics*, section 24, Princeton University Press, Princeton, New Jersey, 1957.

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