ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 1, Spring 1984

POSSIBLE BRAUER TREES

BY

WALTER FEIT[1]

1. Introduction

Let G be a finite group and let p be a prime. Let K be a finite extension field of $\mathbf{Q}_p$, the p-adic numbers, and let R be the ring of integers in K. Let B be a p-block of $R[G]$ with a cyclic defect group $D \neq \langle 1 \rangle$. Let $\tau = \tau(B)$ be the Brauer tree associated to B. We will say that τ *belongs* to G.

The object of this paper is to show that most trees do not occur as Brauer trees. This answers a question raised in [3]. The proof relies on the classification of finite simple groups. As far as I know not a single tree can be eliminated without using this classification. We first prove a result which reduces the question to the study of simple groups and then use the classification of simple groups to get information about trees belonging to simple groups.

Let τ be a tree and let P_0 be a vertex of τ. Let n be a natural number. Then $(\tau, P_0)^n$ is defined to be the union of n copies of τ with the vertices P_0 identified. (A more precise definition can be found in Section 2.) Observe that $\tau \simeq (\tau, P_0)^1$ for any vertex P_0 of τ.

Two trees α and β are *similar* if there exists a tree γ such that

$$\alpha \simeq (\gamma, P_0)^m \quad \text{and} \quad \beta \simeq (\gamma, P_0')^n$$

for integers m,n. It is easily seen that similarity is an equivalence relation. See Lemma 2.2.

THEOREM 1.1. *Let G be a finite group and let τ be a tree which belongs to G. Then there exists a simple group H which is involved in G and a group $\tilde{H}$ where*

$$\tilde{H} = H \quad \text{if } |H| = p,$$
$$\tilde{H}' = \tilde{H} \text{ and } \tilde{H}/\mathbf{Z}(\tilde{H}) \simeq H \quad \text{if } |H| \neq p,$$

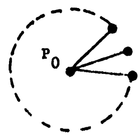
such that τ is similar to a tree that belongs to $\tilde{H}$. If furthermore $\tau = (\gamma, P_0)^n$ for some $n > 1$ then P_0 is the exceptional vertex of τ if τ has an exceptional vertex.

In case G is p-solvable, $|H| = p$. Thus Theorem 1.1 asserts the well-known

Received September 23, 1981.

[1]The work in this paper was partially supported by a National Science Foundation grant.

fact that τ is similar to . Hence τ is



where P_0 is the exceptional vertex.

Somewhat more precise results are proved in Section 4.

THEOREM 1.2. *Let G be a finite group and let τ be a tree which belongs to G. Then $\tau \simeq (\gamma, P_0)^n$ for some natural number n, where P_0 is the exceptional vertex (if there is an exceptional vertex). Furthermore, one of the following holds.*

(i) *γ has at most* 248 *edges.*

(ii) *γ is an open polygon* (*i.e., a straight line segment*).

It is of course in the proof of Theorem 1.2 that the classification of finite simple groups is needed. We actually use only rather superficial properties of simple groups. In case $G = GL_n(q)$ more precise results have been obtained by Fong and Srinivasan [6].

By looking more closely at various classes of finite simple groups it should be possible to improve Theorem 1.2 and eventually to give a complete description of all possible Brauer trees.

In Theorem 1.2 (ii) neither n nor the number of edges of γ needs to be bounded as is shown in the following examples.

Let r be a prime and let m, n be integers with $mn > 2$. Let p be a prime so that r has order mn modulo p. (Such a p always exists if $r^{mn} \neq 2^6$; for example, see E. Artin, Comm. Pure Appl. Math., vol. 8 (1955), p. 358, Corollary 2.) Let $G_0 = GL_m(r^n)$ and let $G = G_0 \langle \sigma \rangle$, where $\sigma^n = 1$ and σ induces the Frobenius automorphism of order n on G_0. Let τ be the Brauer tree of the principal p-block of G and let γ be the Brauer tree of the principal p-block of G_0. By [6], γ is an open polygon with m edges and the exceptional vertex P_0 at one end. Lemma 4.3 below implies that $\tau = (\gamma, P_0)^n$.

I wish to thank E. C. Dade and R. Solomon for pointing out some gaps in the first version of this paper.

2. Some properties of trees

By a *tree* we will always mean a finite connected graph with no cycles. An *object* of a tree is either an edge or a vertex. An isomorphism from a tree τ to a tree τ' is a bijection from the set of all objects of τ onto the set of all objects of τ' which preserves edges, vertices and incidence.

If a tree τ has exactly e edges then it has exactly $e + 1$ vertices.

Let P_0 be a vertex on the tree α. Let n be a positive integer. Then we will write $\alpha = (\tau, P_0)^n$ if τ is a tree with P_0 as a vertex and there exist n graph isomorphisms $f_1, \ldots, f_n$ of τ onto subgraphs of α such that the following hold:

(1) $f_s(P_0) = P_0$ for $1 \leq s \leq n$.

(2) $f_s(\tau) \cap f_t(\tau) = \{P_0\}$ if $1 \leq s \neq t \leq n$.

(3) $\bigcup_{s=1}^{n} f_s(\tau) = \alpha$.

Observe that if τ is a tree with P_0 as a vertex and n is a natural number then there always exists a tree $\alpha = (\tau, P_0)^n$, and any two such trees are isomorphic by an isomorphism which fixes P_0.

LEMMA 2.1. *Let $m,n > 1$. If $(\tau,P_0)^n = (\tau', P_0')^m$ then $P_0 = P_0'$.*

Proof. Let γ be the graph obtained from $(\tau,P_0)^n$ by removing a vertex P and all the edges incident to P. If $P \neq P_0$ then the connected component of γ which contains P_0 has more than half the edges of γ. If however $P = P_0$ then no connected component of γ has more than half the edges of γ. Thus P_0 is uniquely determined. ∎

LEMMA 2.2. *Let α, β be trees and let s, t be integers such that*

$$(\alpha, P_0')^s = (\beta, P_0'')^t.$$

Then there exists a tree γ and integers m,n with

$$\alpha \simeq (\gamma, P_0)^m, \quad \beta \simeq (\gamma, P_0)^n$$

for some vertex P_0 of γ and $ms = nt$. In particular similarity is an equivalence relation.

Proof. If $s = 1$ or $t = 1$ the result is clear. Suppose that $s > 1$ and $t > 1$. Then $P_0' = P_0''$ by Lemma 2.1. Let $P_0 = P_0'$ and let $\tau = (\alpha, P_0)^s$.

Let $\{E_i\}$ be the set of all edges of τ which are incident to P_0. For each i let $\eta(E_i)$ be the subgraph of τ consisting of P_0 and all edges and vertices which are connected to P_0 via E_i. Then $\eta(E_i)$ is a tree for each i, $\tau = \bigcup_i \eta(E_i)$ and $\eta(E_i) \cap \eta(E_j) = \{P_0\}$ for all $i \neq j$. Let $\eta_1, \ldots, \eta_k$ be a complete set of representatives of the isomorphism classes of the trees $\eta(E_i)$. Let a_j be the number of trees $\eta(E_i)$ which are isomorphic to η_j. Let a be the greatest common divisor of all the a_j. Define γ to be a subtree of τ containing P_0 which is the union of various $\eta(E_i)$ such that for each j, exactly a_j/a of these are isomorphic to η_j. Then $\tau = (\gamma, P_0)^a$; s and t divide a; $(\alpha, P_0) \simeq (\gamma, P_0)^m$, $(\beta, P_0) \simeq (\gamma, P_0)^n$ where $ms = nt = a$. ∎

Let σ be an automorphism of the tree τ. Then σ is *admissible* if it satisfies the following conditions.

(I) If $\sigma(E) = E$ for an edge E of τ then $\sigma(P) = P$ for every vertex P of τ incident to E.

(II) If $\sigma(P) = P$ for a vertex P of τ and $E_1, \ldots, E_k$ are all the edges of τ incident to P then σ defines a permutation of $\{E_1, \ldots, E_k\}$ which is a product of disjoint cycles, all of which have the same length.

Condition (II) is clearly equivalent to:

(II)′ If $\sigma(P) = P$ for a vertex P of τ and σ^s fixes an edge incident to P for some s then σ^s fixes every edge incident to P.

Suppose that $\tau \simeq (\gamma, P_0)^n$. Let A be a group of order n. For $x \in A$ let γ_x be a subtree of τ which contains P_0 so that $\tau = \cup_{x \in A} \gamma_x$ and there exists an isomorphism f_x from γ to γ_x which fixes P_0. For $y \in A$ define the automorphism α_y of τ by

$$\alpha_y = f_{zy} f_z^{-1} : \gamma_z \to \gamma_{zy}.$$

Then it is easily seen that $\{\alpha_y \mid y \in A\} \simeq A$ is a group of admissible automorphisms of τ. In the rest of this section we will be primarily concerned with proving the converse of this result.

LEMMA 2.3. *Let σ be an admissible automorphism of the tree τ.*

(i) *If $\sigma(E) = E$ for some edge E then $\sigma = 1$.*

(ii) *If σ fixes two distinct vertices of τ then $\sigma = 1$.*

Proof. (i) If $\sigma \neq 1$ then there exist edges E, E' with a common vertex P such that $\sigma(E) \neq E$ and $\sigma(E') = E'$. Hence $\sigma(P) = P$ by condition (I). This contradicts condition (II)′.

(ii) Suppose that σ fixes the vertices P and Q of τ. There exist objects $O_0 = P, \ldots, O_k = Q$ such that O_{i-1} is incident with O_i for $i = 1, \ldots, k$. Thus also $\sigma(O_{i-1})$ is incident with $\sigma(O_i)$ for $i = 1, \ldots, k$. As τ contains no cycles this implies that $\sigma(O_i) = O_i$ for $i = 0, \ldots, k$ and so if $P \neq Q$ then σ fixes an edge of τ. Hence $\sigma = 1$ by (i). ■

THEOREM 2.4. *Let A be a group of admissible automorphisms of the tree τ. Then the following hold*

(i) *Each orbit of A on the set of edges of τ has length $|A|$.*

(ii) *There exists a vertex P_0 of τ fixed by all the elements of A. If $\{P_u\}$ is the set of all vertices of τ distinct from P_0 then each orbit of A on $\{P_u\}$ has length $|A|$.*

Proof. (i) By Lemma 2.3(i) no element of $A - \{1\}$ fixes any edge. This implies the result.

(ii) Let e be the number of edges in τ. By (i), $|A| \mid e$. Let $e = |A| t$. Let V be the set of all vertices of τ. Thus $|V| = e + 1$. Let θ be the character afforded by the permutation representation of A on V. By Lemma 2.3(ii),

$0 \le \theta(x) \le 1$ for $x \in A - \{1\}$ and $\theta(1) = t|A|+1$. As $\Sigma_{x \in A}\theta(x) \equiv 0 \pmod{|A|}$ it follows that $\theta(x) = 1$ for all $x \in A - \{1\}$. Hence A has exactly $t+1$ orbits on V. Let their lengths be $d_1, \ldots, d_{t+1}$. Thus $d_i \mid |A|$ for each i, and so for each i either $d_i = |A|$ or $d_i \le \frac{1}{2}|A|$. As $\Sigma_{i=1}^{t+1} d_i = |A|t + 1$, this implies that after a possible change of notation $d_i = |A|$ for $i = 1, \ldots, t$ and $d_{t+1} = 1$. ■

LEMMA 2.5. *Let A be a group of admissible automorphisms of the tree τ and let P_0 be a vertex of τ fixed by all elements of A. Let $\sigma \in A$ and let O be an object of τ with $\sigma(O) \neq O$.* Let $O = O_0, \ldots, O_k = \sigma(O)$ *be objects of τ such that O_{i-1} is incident to O_i for $1 \le i \le k$. Then $O_j = P_0$ for some j with $1 \le j \le k$.*

Proof. Suppose that the result is false. Choose σ and O so that k is as small as possible. Then $\sigma^s(O_i) \neq O_j$ for $1 \le i < j \le k$ and any s. Let σ have order n. Consider

$$O = O_0, \ldots, O_k = \sigma(O_0), \ldots, \sigma(O_k) = \sigma^2(O), \ldots$$
$$\ldots \sigma^{n-1}(O_{k-1}), \sigma^{n-1}(O_k) = \sigma^n(O_0) = O.$$

Any two consecutive objects in this sequence are incident and no object in this sequence is equal to P_0. By Theorem 2.4 each orbit of $\langle\sigma\rangle$ on the set of objects in the sequence has length n. Thus no two objects in the sequence are equal except the first and the last. Hence the sequence is a cycle in τ contrary to the fact that τ is a tree. ■

Let τ be a tree and let A be a group of admissible automorphisms of τ. For an object O of τ let O^A denote the orbit of A containing O. Let P_0 be a vertex of τ fixed by all elements of A. Define the graph τ^A as follows:

$\{E^A\}$ is the set of edges of τ^A as E ranges over all edges of τ.

$\{P^A\}$ is the set of vertices of τ^A as P ranges over all vertices of τ.

E^A is incident with P^A if and only if some edge in E^A is incident with some vertex in P^A in τ.

It is easily seen that each E^A is incident with exactly two P^A so that τ^A is a graph. (This for instance follows from the proof of Theorem 2.7.)

LEMMA 2.6. *Let τ be a tree and let A be a group of admissible automorphisms of τ. Then τ^A is a tree.*

Proof. Clearly τ^A is connected. Suppose that τ has e edges. By Theorem 2.4, $e = |A|t$ for some integer t. Furthermore τ^A has t edges and $t+1$ vertices. Thus τ^A has no cycles. ■

THEOREM 2.7. *Let τ be a tree and let A be a group of admissible automorphisms of τ. Let P_0 be a vertex of τ fixed by all the elements of A. Then τ is isomorphic to $(\tau^A, P_0)^{|A|}$.*

Proof. If $|A| = 1$ there is nothing to prove. Suppose that $|A|>1$. By Theorem 2.4, P_0 is uniquely determined. Let $E_1,\ldots,E_n$ be all the edges of τ incident with P_0. By Theorem 2.4, $n = |A|t$ for some integer t and A has t orbits on $\{E_i\}$. Choose the notation so that $E_1,\ldots,E_t$ is a complete set of representatives of the orbits of A on $\{E_i\}$.

For $1 \leq i \leq t$ define the subgraph γ_i of τ to consist of all objects O of τ such that there exists a sequence of objects

$$O = O_0,\ldots,O_{s-1} = E_i, O_s = P_0$$

with $O_j \neq P_0$ for $1 \leq j \leq s-1$ such that O_{j-1} is incident with O_j for $1 \leq j \leq s$. Let $\gamma = \cup_{i=1}^{t}\gamma_i$.

Clearly γ is connected and so γ is a tree. Furthermore $\tau = \cup_{\sigma\in A}\sigma(\gamma)$. By Lemma 2.5 $\sigma(\gamma) \cap \sigma'(\gamma) = \{P_0\}$ for $\sigma \neq \sigma'$ in A. Therefore $\tau = (\gamma, P_0)^{|A|}$.

Since $\sigma(\gamma) \cap \gamma = \{P_0\}$ for $\sigma \neq 1$ it follows that the map sending an object O of γ onto the object O^A in τ^A defines an isomorphism from γ onto τ^A. ∎

COROLLARY 2.8. *Let τ be a tree and let A, A' be groups of admissible automorphisms of τ with $|A|, |A'| > 1$. Let P_0, P_0' be the vertex of τ fixed by all the elements of A, A' respectively. Then $P_0 = P_0'$.*

Proof. Clear by Lemma 2.1 and Theorem 2.7. ∎

3. Brauer trees

Throughout this section the following notation will be used.

G is a finite group.

p is a prime.

K is a finite extension of $\mathbf{Q}_p$, the p-adic numbers, which is a splitting field for G. R is the ring of integers in K and π is a prime in R.

$\overline{R} = R/\pi R$ and $\overline{V} = V/\pi V$ for an R module V.

B is a block with a cyclic defect group $D \neq \langle 1 \rangle$ and $\tau = \tau(B)$ is the Brauer tree associated to B.

We will freely use the notation and results of [4], Chapter VII. Thus $\{\varphi_1,\ldots,\varphi_e\}$ is the set of all irreducible Brauer characters in B. If B contains no exceptional character then $\chi_0,\ldots,\chi_e$ is the set of all irreducible characters in B. If B contains exceptional characters $\chi_\lambda, \lambda \in \Lambda$ then $\chi_0 = \Sigma_{\lambda\in\Lambda}\chi_\lambda$ and $\{\chi_1,\ldots,\chi_e\}$ is the set of all non-exceptional irreducible characters in B. Let $\delta_u = \pm 1$ be the sign corresponding to χ_u defined in [4], Chapter VII, above Theorem 2.15.

Theorems 2.20 and 2.25 of [4], Chapter VII, imply the following statements.

Suppose that $d_{ui} \neq 0$. Then there exists an R-free $R[G]$ module X_{ui} such that

$(X_{ui})_K$ affords χ_u and $\overline{X}_{ui}$ is a serial module whose socle affords φ_i. If furthermore X is an R-free $R[G]$ module such that X_K affords χ_u and $\overline{X}$ is indecomposable then $X \simeq X_{ui}$ for some i with $d_{ui} \neq 0$. Hence in particular X_{ui} is determined up to isomorphism by u and i.

LEMMA 3.1. *Let σ be an automorphism of the field K or of the group G with $B^\sigma = B$. Then σ defines an admissible automorphism of $\tau = \tau(B)$.*

Proof. Clearly σ defines an automorphism of τ. This will also be denoted by σ.

Let E be an edge of τ with $\sigma(E) = E$. Let φ_i correspond to E and let χ_u, χ_v correspond to the vertices incident with E. By [4], Theorem VII. 2.15, $\delta_u \neq \delta_v$. If $|D| > 2$ then the definition of δ_u, δ_v implies that $\chi_u^\sigma \neq \chi_v, \chi_v^\sigma \neq \chi_u$. Hence $\chi_u^\sigma = \chi_u, \chi_v^\sigma = \chi_v$ as $\varphi_i^\sigma = \varphi_i$.

Suppose that $|D| = 2$. Hence τ is

$$\chi_1 \ \overset{\varphi_1}{\circ\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ} \ \chi_0$$

If σ is an automorphism of K then $\chi_1^\sigma = \chi_1$ by Theorem VII. 2.19 of [4]. Suppose that σ is an automorphism of G. Replacing σ by a power it may be assumed that $\sigma^{2^n} = 1$ for some $n > 1$. Let H be the semi-direct product $G\langle\sigma\rangle$. Then there exists a unique irreducible Brauer character $\tilde{\varphi}$ of H such that φ_1 is a constituent of $\tilde{\varphi}_G$. Furthermore $\tilde{\varphi}_G = \varphi_1$. For example, see [4], Corollary IV. 3.15. Suppose that $\chi_0^\sigma = \chi_1$. By Lemma V. 2.4 of [4] any irreducible character $\tilde{\chi}$ in the 2-block $\tilde{B}$ of H which contains $\tilde{\varphi}$ has the property that $\tilde{\chi}_G = c(\chi_0 + \chi_1)$ for some integer c. This implies that every decomposition number for $\tilde{B}$ is even contrary to the fact that the elementary divisors of the decomposition matrix are all 1. Thus $\chi_u^\sigma = \chi_u$ for $u = 0, 1$.

Let P be a vertex of τ and let χ_u correspond to P. Suppose that some power σ^k of σ fixes some φ_j with $d_{uj} \neq 0$. Then $X_{uj}^{\sigma k} \simeq X_{uj}$. As $\overline{X}_{uj}$ is serial this implies that $\varphi_i^{\sigma^k} = \varphi_i$ for all i with $d_{ui} \neq 0$. As this holds for every k it follows that σ defines a permutation on the set $\{\varphi_i | d_{ui} \neq 0\}$ which is a product of pairwise disjoint cycles all of which have the same length. ■

In case σ is an automorphism of K, Lemma 3.1 is implicit in [4], Chapter VII.

LEMMA 3.2. *Suppose that B contains exceptional characters and the vertex P_0 of τ corresponds to the exceptional characters in B. Let σ be an automorphism of K or of G which fixes B. Then $\sigma(P_0) = P_0$.*

Proof. The result follows from considering the higher decomposition numbers. See [4], Theorem VII. 2.17. ■

Suppose that $|D| \neq 2$ and σ is an automorphism of K or of G which fixes B such that σ induces the automorphism $\sigma \neq 1$ on τ. By Lemma 3.1 and

Theorem 2.4 there is a unique vertex P_0 of τ with $\sigma(P_0) = P_0$. Define P_0 to be the *exceptional vertex of* τ.

If B contains exceptional characters then by Lemma 3.2 the exceptional vertex corresponds to the exceptional characters. By Corollary 2.8 the exceptional vertex is independent of the choice of σ.

In case σ is an automorphism of K the exceptional vertex was defined essentially in this way in [4], Chapter VII, Section 2.

4. Proof of Theorem 1.1

The notation of Section 3 will be used throughout this section. Furthermore e will denote the index of inertia of B. In view of Theorem 2.7 and Lemma 3.1 it may be assumed for the proof of Theorem 1.1 that K is a splitting field for G and all its subgroups.

We first prove some preliminary results.

LEMMA 4.1. *Let $H \lhd G$ and let B_0 be a block of H which is covered by B. Let $T(B_0)$ denote the inertia group of B_0. Then there exists a block B_1 of $T(B_0)$ which covers B_0 and has defect group D such that $\tau(B) \simeq \tau(B_1)$.*

Proof. This follows from known results of Fong and Reynolds. For example, see [4], Theorem V. 2.5. ■

LEMMA 4.2. *Let $H \lhd G$. There exists a block B_0 of H which is covered by B such that $D \cap H$ is a defect group of B_0. Furthermore the defect group of every block covered by B is contained in a conjugate of D.*

Proof. See [1], (0.1c). ■

LEMMA 4.3. *Let $D \subseteq H \lhd G$. Let B_0 be a block of H which is covered by B and has D as a defect group. Then $\tau(B)$ is similar to $\tau(B_0)$.*

Proof. Induction on $|G : H|$. If $|G : H| = 1$ the result is clear. If $p = 2$ then $\tau(B) = \tau(B_0)$ is $\circ\!\!-\!\!\!-\!\!\circ$.

Thus it may be assumed that $p \neq 2$. By induction it may be assumed that G/H is simple. By Lemma 4.1 it may be assumed $G = T(B_0)$. Therefore $G = H\mathbf{N}_G(D)$.

Let ψ be an irreducible Brauer character in B_0. By Lemma 3.1, G acts by conjugation as a group of admissible automorphisms of $\tau(B_0)$. Thus $T(\psi) = T(\psi_i)$ for every irreducible Brauer character ψ_i in B, where $T(\psi)$ denotes the inertia group of ψ. Hence $T(\psi) \lhd G$. Thus $T(\psi) = H$ or $T(\psi) = G$. If $T(\psi) = H$ then Theorem 2.7 implies that $\tau(B)$ is similar to $\tau(B_0)$. Thus it may be assumed that $T(\psi_i) = G$ for all irreducible Brauer characters ψ_i in B_0.

As $H \subseteq H\mathbf{C}_G(D) \lhd G$ it follows that either $H = H\mathbf{C}_G(D)$ or $G = H\mathbf{C}_G(D)$.

Suppose that $H = H\mathbf{C}_G(D)$. As $p \neq 2$, G/H is cyclic. Thus any nonexceptional irreducible character θ_u in B_0 extends to an irreducible character ζ_u of G. Furthermore if α is a faithful linear character of G/H then $\zeta_u\alpha^i, 1 \leq i \leq |G:H|$ is the set of all irreducible characters of G whose restriction to H have θ_u as a constituent, and they are all distinct. For example, see [2], (9.12). Thus $p \nmid |G:H|$ as every character in B vanishes on all elements whose p-parts are not in H. Hence $\mathbf{C}_G(y) \subseteq H$ for every element $y \in D - \{1\}$. Therefore if λ_B is the central character of $\overline{R}[G]$ corresponding to B then λ_B vanishes on all class sums not in H. Thus $\zeta_u\alpha^i$ is in B for all u, i. Multiplication by any α^i defines an admissible automorphism of $\tau(B)$ with $\tau(B)^{<\alpha>} \simeq \tau(B_1)$. Hence Theorem 2.7 implies that

$$\tau(B) \simeq (\tau(B_1), P_0)^{|G:H|}$$

where P_0 is the exceptional vertex of $\tau(B_0)$ (which necessarily exists as $G \neq H$). Hence $\tau(B)$ is similar to $\tau(B_0)$. Thus it may be assumed that $G = H\mathbf{C}_G(D)$ and $G = T(\psi)$ for every irreducible Brauer character ψ in B_0.

Let e_0 be the index of inertia of B_0. As $G = H\mathbf{C}_G(D)$, $e \leq e_0$. By [4], Lemma V.2.3, and Clifford's theorem, $e = e_0$. Thus the map sending irreducible Brauer characters φ_i and non-exceptional irreducible characters χ_u in B to the unique irreducible constituents of $(\varphi_i)_H, (\chi_u)_H$ respectively, defines an isomorphism from $\tau(B)$ onto $\tau(B_0)$. ∎

LEMMA 4.4. *Let $H \lhd G$ with G/H simple and $<1> \neq D \cap H \neq D$. Let B_0 be a block of H which is covered by B and has $D \cap H$ as a defect group. Assume that $G = T(B_0)$. Then $\tau(B)$ is similar to* $\circ$—$\circ$

Proof. As $G = T(B_0)$ it follows that $G = H\mathbf{N}_G(D \cap H)$ and so

$$DH \subseteq H\mathbf{C}_G(D \cap H) \lhd G.$$

As G/H is simple this implies that $G = H\mathbf{C}_G(D \cap H)$.

By [4], (V.3.5), there exists a unique block B_1 of HD which covers B_0. As $T(B_0) = G$ it follows that B_0 is the unique block of H which is covered by B or by B_1.

Let V be an indecomposable $\overline{R}[G]$ module in B with vertex D. Then there exists an indecomposable component V_1 of V_{HD} with vertex D. As V_H is a direct sum of modules in B_0, this is also the case for $(V_1)_H$. Hence the uniqueness of B_1 implies that V_1 is in B_1. Hence there exists a defect group D_1 of B_1 with $D \subseteq D_1$. This implies that $HD = HD_1$.

As B_0 is the unique block of H which is covered by B_1, Lemma 4.2 implies that $D \cap H = D_1 \cap H$. Hence

$$|D| = |D \cap H|\,|HD:H| = |D_1 \cap H|\,|HD_1:H| = |D_1|.$$

Thus $D = D_1$ as $D \subseteq D_1$.

Let x be a p'-element in $\mathbf{N}_{HD}(D)$. Then $x \in H$ and so $[x, D] \subseteq H \cap D \subsetneq D$. Thus $x \in \mathbf{C}_{HD}(D)$. Therefore $|\mathbf{N}_{HD}(D) : \mathbf{C}_{HD}(D)|$ is a power of

p. Thus the inertial index of B_1 is 1. Let φ be the unique irreducible Brauer character in B_1.

Let e_0 be the inertial index of B_0 and let $\{\psi_i \mid 1 \leq i \leq e_0\}$ be all the irreducible Brauer characters in B_0. As B_1 is the unique block of HD which covers B_0, it follows that each ψ_i is a constituent of φ_H. As B_0 is the unique block of H which is covered by B_1, Clifford's theorem implies that $\varphi_H = s\Sigma\psi_i$ for some natural number s. Furthermore $|HD : T(\psi_i) \cap HD| = e_0 < p$. As HD/H is a p-group this implies that $HD \subseteq T(\psi_i)$ for all i. Hence $\psi = \psi_1$ is the unique irreducible Brauer character in B_0. Thus $e_0 = 1$.

If $p = 2$ the result is clear. Suppose that $p \neq 2$.

Let $\{\zeta_\mu\}$ be the exceptional characters in B_0. These exist as $p \neq 2$. Let $\zeta_0 = \Sigma\zeta_\mu$ and let $\zeta = \zeta_1$ be the unique nonexceptional irreducible character in B_0.

Let $\{\chi_\lambda\}$ be the exceptional characters in B. These exist as $|D| > p$. Let $\chi_0 = \Sigma\chi_\lambda$ and let $\{\chi_u \mid 1 \leq u \leq e\}$ be the nonexceptional irreducible characters in B.

As $G = T(B_o)$ it follows that $G = T(\zeta)$ and if $z \in G$ and μ is given then $\zeta_\mu^z = \zeta_\nu$ for some ν. Thus if $0 \leq u \leq e$ then either $(\chi_u)_H = m_u\zeta$ or $m_u\zeta_0$ for some natural number m_u.

If $\chi_u + \chi_v$ is projective then, after changing notation,

$$(\chi_u)_H = m_u\zeta,\ (\chi_v)_H = m_v\zeta_0 \quad \text{with } m_u = m_v.$$

As $\tau(B)$ is connected it follows that $m_u = m$ is independent of u. Define

$$S = \{u \mid (\chi_u)_H = m\zeta\}, \quad T = \{u \mid (\chi_u)_H = m\zeta_0\}.$$

Then $S \cup T = \{0,\ldots,e\}$. Let $S' = S - \{0\}$, $T' = T - \{0\}$. Thus

$$S' \cup T' = \{1,\ldots,e\}.$$

If $u \in S$ then $\chi_u(1) = m\zeta(1)$. If $u \in T$ then $\chi_u(1) = (p^k - 1)m\zeta(1)$ where $|D \cap H| = p^k$. Furthermore $\{|D| - 1\}\chi_\lambda(1) = e\chi_0(1)$ and

$$\Sigma_{u \in S'}\chi_u(1) = \Sigma_{u \in T'}\chi_u(1) + \delta\chi_\lambda(1),$$

where $\delta = -1$ if $0 \in S$ and $\delta = 1$ if $0 \in T$. See [4], Theorem VII 2.15.

Suppose that $0 \in S$. Then $|T'| > 0$ and

$$|S'|m\zeta(1) = |T'|(p^k - 1)m\zeta(1) - \frac{e}{\{|D| - 1\}} m\zeta(1) > (p^k - 1)m\zeta(1) - m\zeta(1).$$

As $|S'| < e \leq p - 1$ this implies that

$$p - 1 \geq |S'| + 1 > p^k - 1$$

which is not the case. Thus $0 \in T$. Hence

$$|S'|m\zeta(1) = |T'|(p^k - 1)m\zeta(1) + \frac{e}{|D| - 1}(p^k - 1)m\zeta(1)$$

Therefore

$$\frac{e}{|D|-1}(p^k-1)+|T'|(p^k-1) = |S'| = e-|T|'$$

and so

$$e = |T'|p^k+\frac{e}{|D|-1}(p^k-1) \geq |T'|(e+1).$$

Hence $|T'| = 0$.

This implies that $\tau(B) = (\tau, P_0)^e$, where τ is . ■

LEMMA 4.5. *Let $H \triangleleft G$ with $D \cap H = \langle 1 \rangle$. Let B_0 be a block of H which is covered by B. Assume that $G = T(B_0)$. Then there exists a group M and a subgroup $Z \subseteq \mathbf{Z}(M)$ so that $M/Z \simeq G/H$ and $\tau(B) \simeq \tau(B_1)$ for a block B_1 of M.*

Proof. Let ζ be the unique irreducible character in B_0 and let M be the representation group of ζ. In this case the result has been proved by E. C. Dade [1a] and I am indebted to him for sending me his proof. In the special case that $p \nmid |H|$ the result had been proved earlier by Fong. For example, see [4], Theorem X.1.2. ■

Proof of Theorem 1.1. Induction on $|G : \mathbf{Z}(G)|$.

If $G = \mathbf{Z}(G)$ then $\tau(B)$ is

Suppose that $G \neq \mathbf{Z}(G)$. Let H be a maximal normal subgroup of G with $\mathbf{Z}(G) \subseteq H$.

Suppose that $D \cap H \neq \langle 1 \rangle$. By Lemma 4.2 there exists a block B_0 of H which is covered by B and has $D \cap H$ as a defect group. By Lemma 4.1 and induction it may be assumed that $G = T(B_0)$. If $D \subseteq H$ then Lemma 4.3 implies that $\tau(B)$ is similar to $\tau(B_0)$ and the result follows by induction. If $D \subsetneqq H$ then Lemma 4.4 implies that $\tau(B)$ is similar to

and the result is proved in this case.

Suppose that $D \cap H = \langle 1 \rangle$. If M is defined as in Lemma 4.5 then $M/\mathbf{Z}(M) \simeq G/H$ is simple. The result follows from Lemma 4.5. ■

5. Proof of Theorem 1.2

Throughout this section G denotes a noncyclic simple group and $\tilde{G}$ denotes its universal central extension. As usual $\mathbf{Z}(H)$ is the center of any group H.

We begin with a simple observation.

LEMMA 5.1. *Let B be a p-block of $\tilde{G}$ with a cyclic defect group D. Suppose* *that $\tilde{G}$ has a representation of degree d over a field of characteristic not p whose kernel is in $\mathbf{Z}(\tilde{G})$. Then $\tau(B)$ has at most d edges.*

Proof. If $D \subseteq \mathbf{Z}(\tilde{G})$ then $e = 1 \leq d$. If $D \not\subseteq \mathbf{Z}(\tilde{G})$ this follows, as a generator of D is conjugate to at most d elements in D. ■

If $p \leq 249$, case (i) of Theorem 1.2 holds. Hence from now on it will be assumed that $p > 249$. In view of Theorem 1.1 it is enough to prove Theorem 1.2 for $\tilde{G}$ as G ranges over all noncyclic simple groups. As $p > 249$, G is not a sporadic group.

Suppose that $G = G(q)$ is of Lie type. If $p \mid q$ and $\tilde{G}$ has a p-block B with a cyclic defect group $D \neq \langle 1 \rangle$ then it follows from the results of [8] that $G \simeq PSL_2(p)$. In this case it is well known $\tau(B)$ is an open polygon. Suppose that $p \nmid q$. By Lemma 5.1 it may be assumed that $\tilde{G}$ does not have a representation in a field of characteristic not p of degree ≤ 248. This implies that the only groups that need to be considered for the proof of Theorem 1.2 are those in Table I. Here A_n denotes the alternating group on n letters and $|\mathbf{Z}(\tilde{A}_n)| = 2$ as $n > 7$.

TABLE I

G	A_n	$PSL_n(q)$	$PSU_n(q)$	$PSp_n(q)$	$PSO_n(q)'$
$\tilde{G}$:	$\tilde{A}_n$	$SL_n(q)$	$SU_n(q)$	$Sp_n(q)$	$Spin_n(q)'$

The following result will be used in considering the groups in Table I.

LEMMA 5.2. *Let $M \subseteq H \subseteq H_0$, where $M \lhd H_0$, H/M is abelian and $|H_0 : H| \leq 2$. Suppose that every element in H is conjugate in H_0 to its inverse. Let B be a p-block of M with a cyclic defect group $D \neq \langle 1 \rangle$. Then $\tau(B)$ is similar to an open polygon.*

Proof. Let $M \subseteq H_1 \subseteq H$ where $p \nmid |H_1 : M|$ and H/H_1 is a p-group. Let B_1 be a block of H_1 which covers B. Then B and B_1 have the same defect. Thus D is a defect group of B_1 by Lemma 4.2. By Lemma 4.3, $\tau(B)$ is similar to $\tau(B_1)$. Thus it suffices to prove the result in case $M = H_1$. Hence it will be assumed that H/M is a p-group.

If $p = 2$ the result is clear. Suppose that $p \neq 2$.

If every irreducible Brauer character in B is real valued then $\tau(B)$ is an open polygon. For example, see [4], Theorem VII.9.2. Suppose that $\varphi \neq \bar{\varphi}$ for some irreducible Brauer character φ in B. By assumption there exists $x \in H_0$ with $\varphi^x = \bar{\varphi}$. Thus $B^x = \bar{B}$ is the contragredient block of B.

Let T_0 be the inertia group of B in H_0. Let T be the inertia group of B in H. Then T/M is a p-group and one of the following occurs.

$$|T_0 : T| = 2, \quad T_0 = \langle T, x \rangle, \; \bar{B} = B, \tag{5.3}$$

$$T_0 = T, \quad B^x = \bar{B} \neq B. \tag{5.4}$$

Furthermore if $y \in H_0$ then $B^y = \bar{B}$ if and only if $y \in T_0 x$.

Let $\{\varphi_i\}$ be the set of all irreducible Brauer characters in B. Then $|\{\varphi_i\}| \leq p - 1$. As the p-group T/M acts as a permutation group on $\{\varphi_i\}$ it follows that T fixes every φ_i. Suppose that $\varphi_j \neq \overline{\varphi_j}$. Then $\varphi_j^y = \overline{\varphi_j}$ for some $y \in T_0x$. Hence $y = y_1x$ for some $y_1 \in T$. Thus $\varphi_j^x = \sigma_j^{y_1x} = \overline{\varphi_j}$. Consequently $\varphi_j + \varphi_j^x$ is real valued for all j and there is a unique block of $<M,x>$ which covers B.

Let $M_0 = <M,x>$. Then $|M_0 : M| = 2$. Let B_0 be the block of M_0 which covers B. Then D is a defect group of B_0 as $p \neq 2$. If (5.4) holds then $\tau(B) \approx \tau(B_0)$ is an open polygon as every irreducible Brauer character in B_0 is of the form $\varphi_i^{M_0}$ and so is real valued. If (5.4) holds then $\varphi_i^{M_0}$ is irreducible for all i by Lemma 2.3 and Lemma 3.1 as there exists $\varphi_j \neq \overline{\varphi_j} = \varphi_j^x$. Hence $\tau(B_0)$ is an open polygon as every irreducible character in B_0 is real valued. By Lemma 4.3, $\tau(B)$ is similar to $\tau(B_0)$. ■

The next result will only be needed in case $|H_0 : H| = 4$.

Lemma 5.5. *Let $H \triangleleft H_0$ such that H_0/H is an elementary abelian 2-group. Suppose that every element in H is conjugate in H_0 to its inverse. Let B be a p-block of H with a cyclic defect group $D \neq <1>$. Then $\tau(B)$ is similar to an open polygon.*

Proof. Let $\{\varphi_i\}$ be the set of all irreducible Brauer characters in B. Let T_i be the inertia group of φ_i in H_0. Then $B^x = B$ for $x \in T_i$. Thus Lemmas 2.3 and 3.1 imply that $T = T_i$ is independent of i.

As H_0/H has exponent 2 there exists a group M with $H \subseteq M \subseteq H_0$ such that $TM = H_0$ and $T \cap M = H$. Then φ_i^M is irreducible for each i and φ_i^M is real valued as it is constant on H_0-classes in H. Hence if B_1 is the unique block of M which covers B then every irreducible Brauer character in B_1 is real valued and so $\tau(B_1)$ is an open polygon. The result now follows from Theorem 2.7 and Lemma 3.1. ■

We will now show that case (ii) of Theorem 1.2 holds for each of the groups $\tilde{G}$ in Table I. The various cases will be treated separately.

$G = A_n$. Let $H = \tilde{A}_n$ and let $H_0 = \tilde{\Sigma}_n$, where $\tilde{\Sigma}_n$ is a central extension of the symmetric group Σ_n.

Let $x \in \tilde{A}_n$ and let $\bar{x}$ denote the image of x in A_n. There exists $y \in \tilde{\Sigma}_n$ such that $\bar{y}^{-1}\bar{x}\bar{y} = \bar{x}^{-1}$. Thus $y^{-1}xy = x^{-1}$ or $x^{-1}z$ where $<z> = \mathbf{Z}(\tilde{A}_n)$. If x has odd order then clearly $y^{-1}xy \neq x^{-1}z$ and so $y^{-1}xy = x^{-1}$. If x has even order then a result of Schur implies that x^{-1} is conjugate to $x^{-1}z$ in $\tilde{\Sigma}_n$. See [9], vol. I, p. 363, Theorem IV. Thus in any case x is conjugate to x^{-1} in $\tilde{\Sigma}_n$. The result now follows from Lemma 5.2 with $H_0 = \tilde{\Sigma}_n$ and $M = H = \tilde{A}_n$.

$G = PSL_n(q)$. Let $M = \overline{G} = SL_n(q)$. Let $H = GL_n(q)$ and let $H_0 = <H,\sigma>$ where $x^\sigma = x'^{-1}$ and the prime denotes transpose. (The group H_0 was considered by Gow [7] in a different context). As every element x in H is conjugate to x' in H it follows that x is conjugate to x^{-1} in H_0. The result follows from Lemma 5.2.

$G = PSU_n(q)$. Let $M = \tilde{G} = SU_n(q)$. Let $H = H_0 = U_n(q)$. Every element in x in H is conjugate to $x' = x^{-1}$. See [10], p. 34. The result follows from Lemma 5.2.

$G = PSp_n(q)$. The result follows from [5], Theorem E and Lemma 5.2.

$G = PSO_n(q)$. The result follows from [5] Theorem B and Lemma 5.2. □

References

1. E. C. Dade, *Block extensions,* Illinois J. Math., vol. 17 (1973), pp. 198-272.
1a. ______, Private communication.
2. W. Feit, *Characters of finite groups,* Benjamin, New York, 1967.
3. ______, *Some consequences of the classification of finite simple groups,* Proc. Sympos. Pure Math. vol. 37, Santa Cruz 1979, Amer. Math. Soc., Providence, R.I., 1980, pp. 175–181.
4. ______, *The representation theory of finite groups,* North Holland-Amsterdam, 1982.
5. W. Feit and G. Zuckerman, *Reality properties of conjugacy classes in Spin groups and symplectic groups.* Contemporary Math., Amer. Math. Soc. vol. 13 (1983), pp. 239–254.
6. P. Fong and B. Srinivasan, *The blocks of finite general linear and unitary groups,* Invent. Math., vol. 69 (1982), pp. 109–153.
7. R. Gow, *Schur indices of some groups of Lie type,* J. Algebra, vol. 42 (1976), pp. 102–120.
8. J. E. Humphreys, *Defect groups for finite groups of Lie type,* Math Zeitschr., vol. 119 (1971), pp. 149–152.
9. I. Schur, *Collected works,* Springer-Verlag, New York 1973.
10. G. E. Wall, *On the conjugacy classes in the unitary,* Symplectic and orthogonal groups, J. Australian Math. Soc., vol. 3 (1961), pp. 1–62.

Yale University
New Haven, Connecticut