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AUTOMORPHIC INTEGRALS AND RATIONAL PERIOD FUNCTIONS FOR THE HECKE GROUPS[1]

BY

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0. Introduction

In [6] Knopp defined the concept of automorphic integrals with rational period functions for Fuchsian groups. Examples of automorphic integrals are provided by Eichler integrals [2], which have polynomial period functions. In [6] Knopp constructed automorphic integrals with rational period functions for the modular group which differ from Eichler integrals. In this article examples of automorphic integrals with rational period functions for the Hecke groups are given, and from certain of these, new examples of automorphic integrals with rational period functions for the modular group are constructed. Also, the effect of the Hecke operators defined for Hecke groups by Bogo and Kuyk [1] on automorphic integrals with rational period functions is studied. Finally some open questions concerning rational period functions are mentioned.

1. Some definitions

Let $\mathscr{H}$ be the complex upper half plane and let Γ be a Fuchsian group acting on $\mathscr{H}$.

DEFINITION 1.1. Let $F(z)$ be a meromorphic function in $\mathscr{H}$ satisfying the transformation formula

$$(cz + d)^{-2k}F(Mz) = F(z) + q_M(z) \tag{1.2}$$

where k is an integer and for each element $M = \binom{*\ *}{c\ d}$ of Γ, $q_M(z)$ is a rational function of z. Further assume that F is meromorphic in the local uniformizing parameter at each cusp of a fundamental region for Γ. Then F is called an *automorphic integral of weight* $-2k$ *for* Γ, *with rational period functions* $q_M(z)$. Note that if $q_M(z) \equiv 0$ for all M in Γ, then F is an automorphic form. Automorphic integrals for the modular group are called *modular integrals.*

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In the sequel automorphic integrals will be found for the Hecke groups. The Hecke groups were first studied by Hecke [4] in his study of Dirichlet series that satisfy functional equations. Let $\lambda_n = 2\cos(\pi/n)$ where n is an integer, $n \geq 3$. The *Hecke group* $G(\lambda_n)$ is the group of all linear fractional transformations generated by the two transformations

$$S_n = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which satisfy the relations

$$T^2 = (S_n T)^n = I. \tag{1.3}$$

Let $U = S_n T$. Then, as is well known (see [3] for instance), the Hecke group $G(\lambda_n)$ is the free product of $\langle T \rangle$ and $\langle U \rangle$. Note that the modular group $\Gamma(1)$ is identical to the Hecke group $G(\lambda_3)$.

Since the Hecke group $G(\lambda_n)$ is generated by S_n and T, (1.2) is equivalent to

$$F(z + \lambda_n) = F(z) + q_{S_n}(z), \quad z^{-2k}F(-1/z) = F(z) + q_T(z). \tag{1.4}$$

The *stroke operator* is defined by

$$F(z) \mid M = (cz + d)^{-2k} F(Mz) \quad \text{where } M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right).$$

With this notation (1.2) becomes $F \mid M = F + q_M$ for $M \in \Gamma$. It is easy to see that for the composition of two transformations $M_1, M_2 \in \Gamma$ one has

$$q_{M_1 M_2} = q_{M_1} \mid M_2 + q_{M_2}. \tag{1.5}$$

Note that (1.4) may be rewritten as

$$F \mid S_n + F + q_{S_n}, \quad F \mid T = F + q_T. \tag{1.6}$$

Since any rational function F trivially satisfies (1.6), the further condition that $q_{S_n} \equiv 0$ is added. This forces F to be periodic with period λ_n. With the added assumption that $q_{S_n} \equiv 0$, we use $q \equiv q_T$ to generate the other period functions by means of the consistency condition (1.5). We define q_M for any $M \in G(\lambda_n)$ as follows. Use the fact that every $M \in G(\lambda_n)$ may be written uniquely as $M = P_1 \cdots P_k$ where the P_i are one of T or U with no two consecutive T's or more than $n - 1$ consecutive U's (this follows from the free product structure $G(\lambda_n)$), and proceed inductively. First define q_U by

$$q_U = q_{S_n} \mid T + q = q.$$

Next, if $M = P_1 P_2$ set

$$q_M = q_{P_1} \mid P_2 + q_{P_2}.$$

In general, if $M = P_1 \cdots P_M$, set

$$q_M = q_{P_1} \mid P_2 \cdots P_M + q_{P_2 \cdots P_M}.$$

Then q_M is well defined. For the rest of this paper it will always be assumed that $q_{S_n} \equiv 0$ and that $q \equiv q_T$ is the generator of the other period functions.

2. Examples of rational period functions for the Hecke groups

Following the construction of Knopp [8, Theorem 3] (which is valid for the Hecke groups in general) one has that the rational function q is the generating period function for an analytic automorphic integral F if and only if q is analytic in $\mathscr{H}$,

$$q \mid T + q = 0, \tag{2.1}$$

and

$$q \mid (S_n T)^{n-1} + q \mid (S_n T)^{n-2} + \cdots + q \mid S_n T + q = 0. \tag{2.2}$$

To find rational functions such that (2.1) and (2.2) hold one needs the following auxiliary result which is an immediate consequence of James' classification [5, Theorem 6, p. 393] of functions automorphic on large domains.

LEMMA 2.3. *Let $M \in SL(2, \mathbf{R})$ be hyperbolic. Then a nonconstant meromorphic function f satisfies $f \mid M = f$ if and only if*

$$f(z) = A(z - \alpha_1)^{-k}(z - \alpha_2)^{-k}, \quad A \in \mathbf{C},$$

where α_1, α_2 are the fixed points of M.

One can now give examples of rational period functions for the Hecke groups. For the sake of notational convenience we write λ for λ_n and S for S_n.

THEOREM 2.4. *Let $G(\lambda)$, $\lambda = 2\cos(\pi/n)$ with $n \in \mathbf{Z}$, $n \geq 3$, be a Hecke group. Let*

$$\alpha = \frac{\lambda + \sqrt{\lambda^2 + 4}}{2}$$

so that α and $-1/\alpha$ are the fixed points of

$$[S, T] = STS^{-1}T^{-1} = \begin{pmatrix} \lambda^2 + 1 & \lambda \\ \lambda & 1 \end{pmatrix}.$$

Then if k is an odd integer, the rational function

$$q_{\lambda,2k,T} = q_T = q = \frac{1}{(z - \alpha)^k \left(z + \dfrac{1}{\alpha}\right)^k} + \frac{1}{(z + \alpha)^k \left(z - \dfrac{1}{\alpha}\right)^k}$$

is the generating period function of some automorphic integral of weight $-2k$ for $G(\lambda)$.

Proof. Let $r(z) = -(z-\alpha)^{-k}(z+1/\alpha)^{-k}$. Then, by Lemma 2.3, $r(z)$ satisfies

$$r \mid [S, T] = r, \tag{2.5}$$

and the only other rational functions which satisfy (2.5) are constant multiples of r. One also verfies that $r \mid T + r = 0$ for k odd. Set

$$q = r \mid ST - r.$$

An easy computation shows that, for k odd,

$$q = \frac{1}{(z-\alpha)^k\left(z+\dfrac{1}{\alpha}\right)^k} + \frac{1}{(z+\alpha)^k\left(z-\dfrac{1}{\alpha}\right)^k},$$

whereas q is identically zero if k is even. To demonstrate that q generates the period functions for an automorphic integral it suffices to show that q satisfies (2.1) and (2.2).

Since $(ST)^n = I$, it follows that

$$\begin{aligned} q \mid (ST)^{n-1} &+ q \mid (ST)^{n-2} + \cdots + q \mid ST + q \\ &= (r \mid (ST)^n - r \mid (ST)^{n-1}) + (r \mid (ST)^{n-1} - r \mid (ST)^{n-2}) + \cdots \\ &\quad + (r \mid (ST)^2 - r \mid ST) + (r \mid ST - r) \\ &= 0. \end{aligned}$$

Hence (2.2) is satisfied. To show that (2.1) also holds note that from (2.5), $r \mid TS = r \mid ST$. Hence

$$\begin{aligned} q \mid T + q &= (r \mid ST - r) \mid T + r \mid ST - r \\ &= r \mid S - r \mid T + r \mid ST - r \\ &= (r \mid T + r) \mid S - (r \mid T + r) \\ &= 0 \end{aligned}$$

since $r \mid T + r = 0$. ∎

It should be remarked that when $n = 3$ the rational period functions of Theorem 2.4 are precisely those found by Knopp. Also, Theorem 2.4 produces rational period functions for all the Hecke groups, not just for the commensurable ones. It should also be noted that when k is even, this construction fails since the resulting function q is identically zero.

3. The construction of period functions for $G(\sqrt{2})$ and $G(\sqrt{3})$ from period functions for the modular group

Leutbecher [9] has shown that of the Hecke groups only for $n = 3$, 4, and 6 are the groups $G(\lambda_n)$ pairwise commensurable. The commensurability of these groups permits construction of automorphic integrals for $G(\sqrt{2})$ and $G(\sqrt{3})$ from modular integrals. This construction is described in the following theorem.

THEOREM 3.1. *Let F_1 be a modular integral of weight $-2k$, $k \in \mathbf{Z}$, with generating period function $q_T = q$. Then*

$$F_\lambda(z) = \lambda^{2k} F_1(\lambda z) + F_1(z/\lambda)$$

is an automorphic integral for $G(\lambda)$, when λ is $\sqrt{2}$ or $\sqrt{3}$. The corresponding generating period function $q_{\lambda,T} = q_\lambda$ is

$$q_\lambda = \lambda^{2k} q(\lambda z) + q(z/\lambda).$$

Remark. It should be pointed out that the definition of $F_\lambda(z)$ is precisely that used by Hecke in his original construction of automorphic forms on $G(\sqrt{2})$ and $G(\sqrt{3})$ from modular forms.

Proof. Note that F_λ as defined above is clearly meromorphic in $\mathscr{H}$. The cusps of $G(\lambda)$ are images of $i\infty$ and are all of the form $a\lambda/c$ with $(a\lambda^2, c) = 1$ or $a/c\lambda$ with $(a, c\lambda^2) = 1$; and hence the behavior of F_λ at a cusp of $G(\lambda)$ corresponds to the behavior of F_1 at two rational points which are cusps of the modular group. Consequently since F_1 is meromorphic at the cusps of the modular group, F_λ is meromorphic at the cusps of $G(\lambda)$.

To see how F_λ behaves under the transformation S note that

$$\begin{aligned} F_\lambda(z) \mid S &= \lambda^{2k} F_1(\lambda z + \lambda^2) + F_1\left(\frac{z+\lambda}{\lambda}\right) \\ &= \lambda^{2k} F_1(\lambda z) + F_1(z/\lambda) \\ &= F_\lambda(z). \end{aligned}$$

When T is applied to F_λ, one calculates

$$\begin{aligned} F_\lambda \mid T &= z^{-2k}[\lambda^{2k} F_1(-\lambda/z) + F_1(-1/z\lambda)] \\ &= z^{-2k}[\lambda^{2k}(\lambda/z)^{-2k} F_1(z/\lambda) + \lambda^{2k}(\lambda/z)^{-2k} q(z/\lambda) \\ &\quad + (1/z\lambda)^{-2k} F_1(z\lambda) + (1/z\lambda)^{-2k} q(z\lambda)] \\ &= F_1(z/\lambda) + \lambda^{2k} F_1(z\lambda) + q(z/\lambda) + \lambda^{2k} q(\lambda z) \\ &= F_\lambda(z) + q_\lambda(z). \end{aligned}$$

This proves Theorem 3.1. ∎

4. The construction of period functions for the modular group from period functions for the Hecke groups $G(\sqrt{2})$ and $G(\sqrt{3})$

To obtain a converse to Theorem 3.1, we note that the pairwise commensurability of the Hecks groups $G(\lambda_n)$, $n = 3$, 4, and 6, permits the construction of modular integrals with rational period functions from automorphic integrals with rational period functions for $G(\sqrt{2})$ and $G(\sqrt{3})$. This is demonstrated by the following theorem which uses a construction introduced by Bogo and Kuyk [1].

THEOREM 4.1. *Let F_λ be an automorphic integral for $G(\lambda)$ where λ is $\sqrt{2}$ or $\sqrt{3}$ of weight $-2k$ with generating period function $q_{\lambda,T} = q_\lambda$. Then*

$$F_1(z) = F_\lambda(\lambda z) + \lambda^{-2k} \sum_{t=0}^{\lambda^2-1} F_\lambda\left(\frac{z+t}{\lambda}\right) \tag{4.2}$$

is a modular integral with generating period function $q_{1,T} = q_1$ where

$$q_1 = q_\lambda(\lambda z) + \lambda^{-2k} q_\lambda(z/\lambda) + \lambda^{-2k} q_\lambda\left(\frac{z-1}{\lambda}\right) + (1-z)^{-2k} q_\lambda\left(\frac{\lambda z}{1-z}\right) \quad \textit{when } \lambda = \sqrt{2}$$

and

$$q_1 = q_\lambda(\lambda z) + \lambda^{-2k} q_\lambda(z/\lambda) + \lambda^{-2k} q_\lambda\left(\frac{z-1}{\lambda}\right) + \lambda^{-2k} q_\lambda\left(\frac{z+1}{\lambda}\right) + (z+1)^{-2k} q_\lambda\left(\frac{\lambda z}{z+1}\right) + (1-z)^{-2k} q_\lambda\left(\frac{\lambda z}{1-z}\right) \quad \textit{when } \lambda = \sqrt{3}.$$

Proof. First note that the function F_1 defined by (4.2) is meromorphic in $\mathscr{H}$. At the cusps of a fundamental region for the modular group the behavior of F_1 is determined by the behavior of F_λ at cusps of a fundamental region for the corresponding Hecke group. Hence F_1 is meromorphic at each cusp of a fundamental region fot the modular group.

It is easy to see that F_1 has period 1. One has

$$\begin{aligned} F_1(Sz) &= F_\lambda(\lambda z + \lambda) + \lambda^{-2k} \sum_{t=0}^{\lambda^2-1} F_\lambda\left(\frac{z+1+t}{\lambda}\right) \\ &= F_\lambda(\lambda z) + \lambda^{-2k} \sum_{t=1}^{\lambda^2-1} F_\lambda\left(\frac{z+t}{\lambda}\right) + \lambda^{-2k} F_\lambda\left(\frac{z+\lambda^2}{\lambda}\right) \\ &= F_\lambda(\lambda z) + \lambda^{-2k} \sum_{t=0}^{\lambda^2-1} F_\lambda\left(\frac{z+t}{\lambda}\right) \\ &= F_1(z). \end{aligned}$$

When one applies the inversion T, one finds

$$\begin{aligned}F_1(z)\mid T &= z^{-2k}F_1(-1/z)\\ &= z^{-2k}\left[F_\lambda(-\lambda/z) + \lambda^{-2k}\sum_{t=0}^{\lambda^2-1} F_\lambda\left(\frac{-(1/z)+t}{\lambda}\right)\right]\\ &= z^{-2k}[(z/\lambda)^{2k}F_\lambda(z/\lambda) + (z/\lambda)^{2k}q_\lambda(z/\lambda)\\ &\quad + \lambda^{-2k}\sum_{t=0}^{\lambda^2-1}\left(\frac{\lambda z}{1-tz}\right)^{2k} F_\lambda\left(\frac{\lambda z}{1-tz}\right)\\ &\quad + \lambda^{-2k}\sum_{t=0}^{\lambda^2-1}\left(\frac{\lambda z}{1-tz}\right)^{2k} q_\lambda\left(\frac{\lambda z}{1-tz}\right)\Bigg].\end{aligned}$$

The cases $\lambda = \sqrt{2}$ and $\lambda = \sqrt{3}$ will now be treated separately. When $\lambda = \sqrt{2}$,

$$\sum_{t=0}^{\lambda^2-1}\left(\frac{\lambda z}{1-tz}\right)^{2k} F_\lambda\left(\frac{\lambda z}{1-tz}\right) = (\lambda z)^{2k}F_\lambda(\lambda z) + \left(\frac{\lambda z}{1-z}\right)^{2k} F_\lambda\left(\frac{\lambda z}{1-z}\right). \tag{4.4}$$

Since

$$\frac{\lambda z}{1-z} = S^{-1}TS^{-1}\left(\frac{z+1}{\lambda}\right),$$

it follows that

$$F_\lambda\left(\frac{\lambda z}{1-z}\right) = F_\lambda\left(\begin{pmatrix}-\lambda & 1\\ 1 & -\lambda\end{pmatrix}\left(\frac{z+1}{\lambda}\right)\right) = F_\lambda\left(S^{-1}TS^{-1}\left(\frac{z+1}{\lambda}\right)\right).$$

Further, one has

$$\begin{aligned}F_\lambda\left(\frac{z+1}{\lambda}\right)\mid S^{-1}TS^{-1} &= \left(\frac{z-1}{\lambda}\right)^{-2k} F_\lambda\left(S^{-1}TS^{-1}\left(\frac{z+1}{\lambda}\right)\right)\\ &= F_\lambda\left(\frac{z+1}{\lambda}\right) + q_{S^{-1}TS^{-1}}\left(\frac{z+1}{\lambda}\right),\end{aligned} \tag{4.5}$$

where

$$q_{S^{-1}TS^{-1}} = q_{S^{-1}}\mid TS^{-1} + q_{TS^{-1}} = q_T\mid S^{-1} + q_{S^{-1}} = q_T\mid S^{-1}$$

and hence

$$q_{S^{-1}TS^{-1}}\left(\frac{z+1}{\lambda}\right) = q_{\lambda,T}\left(\frac{z-1}{\lambda}\right). \tag{4.6}$$

From (4.3), (4.4), (4.5), and (4.6) one concludes that

$$\begin{aligned} F_1(z)\,|\,T &= F_\lambda(\lambda z) + \lambda^{-2k}F_\lambda(z/\lambda) \\ &\quad + \lambda^{-2k}F_\lambda\left(\frac{z+1}{\lambda}\right) + \lambda^{-2k}q_\lambda(z/\lambda) + q_\lambda(\lambda z) \\ &\quad + (1-z)^{-2k}q_\lambda\left(\frac{\lambda z}{1-z}\right) + \lambda^{-2k}q_\lambda\left(\frac{z-1}{\lambda}\right) \\ &= F_1(z) + q_1(z) \end{aligned}$$

where $q_1(z)$ is given in the statement of the theorem.

When $\lambda = \sqrt{3}$, one has

$$\sum_{t=0}^{\lambda^2-1}\left(\frac{\lambda z}{1-tz}\right)^{2k}F_\lambda\left(\frac{\lambda z}{1-tz}\right) \tag{4.7}$$

$$= (\lambda z)^{2k}F_\lambda(\lambda z) + (\lambda z)^{2k}(1-z)^{-2k}F_\lambda\left(\frac{\lambda z}{1-z}\right) + (\lambda z)^{2k}(1-2z)^{-2k}F_\lambda\left(\frac{\lambda z}{1-2z}\right).$$

Since

$$\frac{\lambda z}{1-z} = \begin{pmatrix} -\lambda & 2 \\ 1 & -\lambda \end{pmatrix}\left(\frac{z+2}{\lambda}\right) = S^{-1}TS^{-1}\left(\frac{z+2}{\lambda}\right)$$

and

$$\frac{\lambda z}{1-2z} = \begin{pmatrix} -\lambda & 1 \\ 2 & -\lambda \end{pmatrix}\left(\frac{z+1}{\lambda}\right) = TSTST\left(\frac{z+1}{\lambda}\right),$$

$$\begin{aligned} F_\lambda\left(\frac{z+2}{\lambda}\right)\Big|\,S^{-1}TS^{-1} &= \left(\frac{z-1}{\lambda}\right)^{-2k}F_\lambda\left(S^{-1}TS^{-1}\left(\frac{z+2}{\lambda}\right)\right) \\ &= F_\lambda\left(\frac{z+2}{\lambda}\right) + q_{S^{-1}TS^{-1}}\left(\frac{z+2}{\lambda}\right) \end{aligned}$$

and

$$\begin{aligned} F_\lambda\left(\frac{z+1}{\lambda}\right)\Big|\,TSTST &= \left(\frac{z-2}{\lambda}\right)^{-2k}F_\lambda\left(TSTST\left(\frac{z+1}{\lambda}\right)\right) \\ &= F_\lambda\left(\frac{z+1}{\lambda}\right) + q_{TSTST}\left(\frac{z+1}{\lambda}\right). \end{aligned}$$

As before one has $q_{S^{-1}TS^{-1}} = q_T\,|\,S^{-1}$, so

$$q_{S^{-1}TS^{-1}}\left(\frac{z+2}{\lambda}\right) = q_\lambda\left(\frac{z-1}{\lambda}\right), \tag{4.10}$$

while

$$q_{TSTST} = q_T \,|\, (ST)^2 + q_{(ST)^2} = q_T \,|\, (ST)^2 + q_{ST} \,|\, ST + q_{ST}.$$

Since $q_{ST} = q_S \,|\, T + q_T = q_T$, one has

$$(4.11) \qquad q_{TSTST}\left(\frac{z+1}{\lambda}\right) = z^{-2k} q_\lambda\left(\frac{2z-1}{\lambda z}\right) + \lambda^{2k}(z+1)^{-2k} q_\lambda\left(\frac{\lambda z}{z+1}\right) + q_\lambda\left(\frac{z+1}{\lambda}\right).$$

Also note that since $q_\lambda \,|\, T + q_\lambda = 0$, it follows that

$$(4.12) \qquad q_\lambda\left(\frac{2z-1}{\lambda z}\right) = -\left(\frac{\lambda z}{1-2z}\right)^{2k} q_\lambda\left(\frac{\lambda z}{1-2z}\right).$$

From (4.3), (4.7), (4.8), (4.9), (4.10), (4.11), and (4.12) one sees that

$$\begin{aligned} F_1(z) \,|\, T = {} & F_\lambda(\lambda z) + \lambda^{-2k} F_\lambda(z/\lambda) + \lambda^{-2k} F_\lambda\left(\frac{z+1}{\lambda}\right) \\ & + \lambda^{-2k} F_\lambda\left(\frac{z+2}{\lambda}\right) + q_\lambda(\lambda z) + \lambda^{-2k} q_\lambda(z/\lambda) \\ & + \lambda^{-2k} q_\lambda\left(\frac{z-1}{\lambda}\right) + \lambda^{-2k} q_\lambda\left(\frac{z+1}{\lambda}\right) + (z+1)^{-2k} q_\lambda\left(\frac{\lambda z}{z+1}\right) \\ & + (1-z)^{2-k} q_\lambda\left(\frac{\lambda z}{1-z}\right) \\ = {} & F_1(z) + q_1(z) \end{aligned}$$

where $q_1(z)$ is given in the statement of the theorem. ∎

Remarks. 1. Theorem 3.1 shows how to derive automorphic integrals with rational period functions for $G(\sqrt{2})$ and $G(\sqrt{3})$ from those for the modular group. Theorem 4.1 gives a construction of automorphic integrals with rational period functions for the modular group from those for $G(\sqrt{2})$ and $G(\sqrt{3})$. Combining the two theorems, one can start with a modular integral F_1 and end up with a new modular integral $\tilde{F}_1$. A straightforward calculation reveals that

$$\tilde{F}_1(z) = (\lambda^2 + 1)F_1(z) + (\lambda^2)^{-k+1} T_{2k}(2) F_1(z)$$

where $T_{2k}(2)$ is a Hecke operator of weight $2k$. Knopp [6] showed that Hecke operators induce a mapping of the space of modular integrals with rational period functions into itself. Hence $\tilde{F}_1(z)$ is a linear combination of modular integrals considered previously by Knopp.

However, it is worth remarking that the rational period functions for the modular group which are obtained by Theorem 4.1 from the rational period functions of Theorem 2.4 for $G(\sqrt{2})$ and $G(\sqrt{3})$ are indeed different. The generating period functions discovered by Knopp have poles of order k at points in $Q(\sqrt{5})$ whereas the modular generating period functions obtained from those on $G(\sqrt{2})$ and $G(\sqrt{3})$ have poles in $Q(\sqrt{3})$ and $Q(\sqrt{21})$ respectively. More specifically, one has poles of order k at

$$\pm\left(\frac{1 \pm \sqrt{3}}{2}\right), \quad \pm(1 \pm \sqrt{3}), \quad \pm\sqrt{3}, \quad \pm\frac{1}{\sqrt{3}}$$

and the other has poles of order k at

$$\pm\left(\frac{3 \pm \sqrt{21}}{6}\right), \quad \pm\left(\frac{3 \pm \sqrt{21}}{2}\right), \quad \pm\left(\frac{5 \pm \sqrt{21}}{2}\right),$$
$$\pm\left(\frac{1 \pm \sqrt{21}}{10}\right), \quad \pm\left(\frac{1 \pm \sqrt{21}}{2}\right).$$

2. Marvin Knopp [7] has recently shown that for the modular group (2.2) and (2.3) imply that all poles of a generating rational period function q must be real. It now follows from Theorem 4.1 that this is also true of the generating rational periods for $G(\sqrt{2})$ and $G(\sqrt{3})$.

5. The effect of Hecke operators for the Hecke group on rational period functions

In [1] Bogo and Kuyk introduced Hecke operators for the Hecke groups $G(\sqrt{2})$ and $G(\sqrt{3})$. The operators they studied are defined for a function F as follows. Let $\lambda^2 = l$. When p is prime and $p \neq l$, one has

$$T_{2k}(l, p)F(z) = T_{2k}(p)F(z) = p^{2k-1} \sum_{\substack{ad=p \\ d>0 \\ b(\mathrm{mod}\ d)}} d^{-2k} F\left(\frac{az + b\lambda}{d}\right) \tag{5.1}$$

while

$$\begin{aligned} T_{2k}(l, l)F(z) &= T_{2k}(l)F(z) \\ &= l^{2k-1}\left[\sum_{\substack{ad=l \\ d>0 \\ b(\mathrm{mod}\ d)}} d^{-2k} F\left(\frac{az + b\lambda}{d}\right) + \sum_{t=1}^{l-1} \lambda^{-2k} F\left(\frac{\lambda z + t}{\lambda}\right)\right]. \end{aligned} \tag{5.2}$$

The Hecke operators applied to automorphic integrals with rational period functions induce operators on the space of all rational period functions. These operators are defined in the following theorem.

THEOREM 5.3. *Let F_λ be an automorphic integral for $G(\lambda)$ where λ is $\sqrt{2}$ or $\sqrt{3}$ with a rational generating period function $q_T = q$ and let p be prime. Then $T_{2k}(p)F_\lambda$ is an automorphic integral for $G(\lambda)$ with rational generating period function*

$$\hat{T}_{2k}(p)q = T_{2k}(p)F_\lambda \mid T - T_{2k}(p)F_\lambda.$$

Proof. It is clear from the definition of $T_{2k}(p)$ and the fact that F_λ is an automorphic integral that $T_{2k}(p)F_\lambda(z)$ is meromorphic in $\mathscr{H}$ and at all cusps of $G(\lambda)$ and that $T_{2k}(p)F_\lambda \mid S = T_{2k}(p)F_\lambda$. Thus it suffices to study the behavior of $T_{2k}(p)F_\lambda$ under T. When $p \neq l$, one has

$$T_{2k}(p)F_\lambda(z) = p^{2k-1} \sum_{\substack{ad=p \\ d>0 \\ b(\text{mod } d)}} F_\lambda(z) \mid M_{b,d}$$

where

$$M_{b,d} = \begin{pmatrix} a & b\lambda \\ 0 & d \end{pmatrix}.$$

Hence

$$T_{2k}(p)F_\lambda \mid T = p^{2k-1}\Bigg(\sum_{\substack{ad=p \\ d>0 \\ b(\text{mod } d)}} F_\lambda \mid M_{b,d}\Bigg) \mid T \tag{5.3}$$

$$= p^{2k-1} \sum_{\substack{ad=p \\ d>0 \\ b(\text{mod } d)}} F_\lambda \mid M_{b,d}T.$$

When $p = l$,

$$T_{2k}(l)F_\lambda(z) = l^{2k-1} \sum_{\substack{ad=l \\ d>0 \\ b(\text{mod } d)}} F_\lambda(z) \mid M_{b,d} + l^{2k-1} \sum_{t=1}^{l-1} F_\lambda(z) \mid L_t$$

where

$$M_{b,d} = \begin{pmatrix} a & b\lambda \\ 0 & d \end{pmatrix} \quad \text{and} \quad L_t = \begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix}.$$

Hence

$$T_{2k}(l)F_\lambda \mid T = l^{2k-1} \sum_{\substack{ad=l \\ d>0 \\ b(\text{mod } d)}} (F_\lambda \mid M_{b,d}) \mid T + l^{2k-1} \sum_{t=1}^{l-1} (F_\lambda \mid L_t) \mid T \tag{5.4}$$

$$= l^{2k-1} \sum_{\substack{ad=l \\ d>0 \\ b(\text{mod } d)}} F_\lambda \mid M_{b,d}T + l^{2k-1} \sum_{t=1}^{l-1} F_\lambda \mid L_t T.$$

Now when $p \neq l$, let

$$S_{\lambda,p} = \{M_{b,d} \mid ad = p,\ d > 0,\ b \text{ runs through a full set of residues (mod } d)\}$$

and let

$$S_{\lambda,l} = \{M_{b,d} \mid ad = l,\ d > 0,\ b \text{ runs through a full set of residues (mod } d)\} \cup \{L_t \mid t = 1, 2, \ldots, l-1\}.$$

From (5.3) and (5.4) one has for all primes p,

$$T_{2k}(p)F_\lambda \mid T = p^{2k-1} \sum_{X \in S_{\lambda,p}} F_\lambda \mid XT. \tag{5.5}$$

For every prime p, for each element $X \in S_{\lambda,p}$ there is a unique element $X' \in S_{\lambda,p}$ such that $XT = V_X X'$ where $V_X \in G(\lambda)$. Distinct X give rise to distinct X' (see Bogo and Kuyk [1]). Hence from (5.5)

$$\begin{aligned} T_{2k}(p)F_\lambda \mid T &= p^{2k-1} \sum_{X \in S_{\lambda,p}} F_\lambda \mid V_X X' \\ &= p^{2k-1} \sum_{X \in S_{\lambda,p}} (F_\lambda \mid V_X) \mid X' \\ &= p^{2k-1} \sum_{X \in S_{\lambda,p}} (F_\lambda + q_{V_X}) \mid X' \\ &= p^{2k-1} \sum_{X \in S_{\lambda,p}} F_\lambda \mid X' + p^{2k-1} \sum_{X \in S_{\lambda,p}} q_{V_X} \mid X' \\ &= T_{2k}(p)F_\lambda + q_{\lambda,p,2k,T} \end{aligned}$$

where

$$q_{\lambda,q,2k,T} = T_{2k}(p)F_\lambda \mid T - T_{2k}(p)F_\lambda = p^{2k-1} \sum_{X \in S_{\lambda,p}} q_{V_X} \mid X'$$

is again a rational function. Therefore, $T_{2k}(p)F_\lambda$ is indeed an automorphic integral with rational periods. ▮

The following corollary of Theorem 5.3 is established by a straightforward, yet tedious, argument similar to one found in [6, pp. 56–57]. Actually, when $p = l$, the result is easily verified directly.

COROLLARY 5.6. *Let k be an odd positive integer, let p be prime, and let λ be $\sqrt{2}$ or $\lambda = \sqrt{3}$. The rational period function $\hat{T}_{2k}(p)q_{\lambda,2k,T}$ has a pole at $p\alpha$, where $q_{\lambda,2k,T}$ is the rational period function associated with the automorphic integral for $G(\lambda)$ of weight $-2k$ found in Theorem 2.7 and $\alpha = \frac{1}{2}(\lambda + \sqrt{\lambda^2 + 4})$.*

From Corollary 5.6 and the fact that $p\alpha$ is the largest pole of $\hat{T}_{2k}(p)q_{\lambda,2k,T}$ which is a multiple of α, one can obtain the following result.

COROLLARY 5.7. *When k is an odd positive integer and $\lambda = \sqrt{2}$ or $\lambda = \sqrt{3}$, the set $\{T_{2k}(l, p)q_{\lambda,2k,T} \mid p \text{ prime}\}$ is an infinite set of linearly independent rational period functions of weight $-2k$ for the Hecke group $G(\lambda)$.*

6. Conclusion

By applying the construction of Section 4 to the rational period functions obtained for the Hecke groups $G(\sqrt{2})$ and $G(\sqrt{2})$ in Section 2, one obtains rational period functions for the modular group with poles in the real quadratic fields $Q(\sqrt{3})$ and $Q(\sqrt{21})$. The rational period functions found by Knopp [6] have poles in $Q(\sqrt{5})$. Knopp [7] and Grinstein (personal communication) have shown that rational period functions for the modular group may have poles only at elements of real quadratic fields. Further, Knopp [7] has shown that the only possible rational poles occur at $z = 0$ or ∞. These facts lead to the following questions.

Question 6.1. In which quadratic fields may a rational period function for the modular group have poles?

Question 6.2. What are the possible orders of poles of rational period functions? (All known poles, except for the rational poles, have order k where $-2k$ is the weight of the corresponding modular integral.)

Since the constructions of Knopp and of the authors only give rational period functions with poles in $Q(\sqrt{n})$, $n \geq 2$, when the weight of the corresponding modular integral is $-2k$ when k is odd, the following question arises.

Question 6.3. Are there rational period functions for the modular group with poles in $Q(\sqrt{n})$, $n \geq 2$, when the weight of the corresponding modular integral is $-2k$ with k even?

Finally one can make the following open-ended query.

Question 6.4. What can be said about the zeros of a rational period function?

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