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ACTIONS OF COMPACT GROUPS ON AF C^*-ALGEBRAS[1]

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Introduction

Let G be a compact group, and $\bar{A}$ an AF (approximately finite dimensional) C^* algebra. Suppose α: $G \to \mathrm{Aut}(\bar{A})$ is a point norm continuous group homomorphism such that (i) $\alpha(G)$ leaves globally invariant a dense locally finite dimensional *-subalgebra; and (ii) $\alpha(G)$ is locally representable (i.e., there exists a dense locally finite dimensional *-subalgebra $A = \bigcup A_i$, with A_i finite dimensional and invariant, *and* the action of α restricted to each A_i arises from a representation of G in the unitary group of A_i). Then a complete invariant for conjugacy in $\mathrm{Aut}(\bar{A})$ of α may be given in terms of a suitable partially ordered module over the representation ring of G, together with a specified positive element.

This classification applies (as a very special case) to product type actions on UHF algebras, and also to groups of prime order of approximately inner automorphisms acting on arbitrary AF algebras, but satisfying (i) (§IV). Moreover, the invariant can be used to construct weird actions, which are certainly not of product type, on UHF algebras, for any compact group (§III, VI).

We also show that if $\bar{A}$ has unique trace, then the complex vector space generated by the traces of $\bar{A} \times_\alpha G$ (the crossed product) is a cyclic module over the complexified representation ring of G, when G is finite. This is true for any unital C^* algebra $\bar{A}$ with unique trace, and any action of G, and is obtained by directly constructing all the traces on $R \times_\alpha G$, where R is the tracial completion of $\bar{A}$.

Our emphasis here is on the crosssed product $\bar{A} \times_\alpha G$ (as opposed to the fixed point algebra $\bar{A}^G$ studied in [11]). The Grothendieck group (K_0) of this admits a natural ordered module structure over the representation ring of G. The first key result, II.1 and II.2, describe the ordered module as a limit of finitely generated ones (this is precisely analogous to dimension groups arising as limits of simplicial groups; here **Z**, the ordered ring, is replaced by the

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representation ring of G, as an ordered ring, and maps between free modules are described by matrices whose entries are characters). This description can be used to construct strange actions, even for groups of order 2, and may be considered a prescription for constructing all possible actions satisfying (i) and (ii).

The next theorem (III.1) shows that the ordered module invariant together with a single element in the positive cone, is a complete conjugacy invariant for the action of G in $\operatorname{Aut}(\bar{A})$. This is achieved, essentially via an iteration of a Noether Skolem theorem adapted for finite dimensional dynamical systems.

We also consider (§VI) how the complete invariant behaves under tensoring; that is, if α: $G \to \operatorname{Aut}(\bar{A})$, β: $G \to \operatorname{Aut}(\bar{B})$ are actions of the compact group G, then $\alpha \otimes \beta$: $G \to \operatorname{Aut}(\bar{A} \otimes \bar{B})$ is an action of G on $\bar{A} \otimes \bar{B}$. The corresponding invariant ordered module is a tensor product (over the representation ring of G), of the modules arising from α and β, if the latter are locally representable. Infinite tensor products of actions can be used to construct very highly nonproduct type actions on UHF algebras.

For the convenience of the reader, there is an appendix which deals with examples of representation rings of compact groups.

I. Definitions

Let G be a compact group. We denote by $K_0(G)$ its representation ring (that is, $K_0(C^*(G))$); $K_0(G)$ is the ring of differences of (finite dimensional) characters, with multiplication defined via

$$(\chi_1 \cdot \chi_2)(g) = \chi_1(g)\chi_2(g),$$

corresponding to the tensor product of the representations, if χ_1, χ_2 are actual characters. There is a natural partial ordering on $K_0(G)$, making it into a partially ordered ring (take as positive cone, the characters). Then as a partially ordered abelian group, $K_0(G)$ is order-isomorphic to a direct sum of copies of $\mathbf{Z}$ (coordinatewise ordering), one for each irreducible character of G.

We shall assume as familiar the equivalence between characters, finite dimensional representations, and finite dimensional G-modules, and treat them interchangeably as elements of $K_0(G)^+$.

If G is abelian, then $K_0(G)$ is ring isomorphic to the (discrete) integral group ring, $\mathbf{Z}\hat{G}$, over the dual group, and the positive cone consists of elements of the form $\Sigma n_i \chi_i$ with n_i nonnegative integers, and χ_i in $\hat{G}$ (the linear characters), where almost all of the n_i are zero (see the appendix).

Let $\bar{A}$ be any (unital) C^* algebra (or if G is finite, it suffices that $\bar{A}$ be an algebra over $\mathbf{C}$), and let α: $G \to \operatorname{Aut}(\bar{A})$ be a point-norm continuous group homomorphism. We may form the crossed product $\bar{A} \times_\alpha G$: If G is finite, this is easiest to treat as the algebra of formal sums

$$\{\Sigma a_g g | g \in G,\ a_g \in \bar{A}\},$$

with multiplication defined via $ga = a^{\alpha(g)}g$; if G is infinite, $\bar{A} \times_\alpha G$ is the C^* norm closure of the set of continuous functions $f\colon G \to \bar{A}$, with multiplication defined via a twisted convolution:

$$(f_1 * f_2)(g) = \int_G f_1(h)\alpha_h(f_2(h^{-1}g))\,dh,$$

where α_h is $\alpha(h)$, and the measure on G is normalized Haar measure.

Recall the definition of (pre-ordered) K_0 of a C^* algebra, e.g., [7], [2], [4]. As G is compact and A is locally finite dimensional, the crossed product $\bar{A} \times_\alpha G$ is AF, and $C(G, A)$ contains a dense locally finite dimensional subalgebra thereof. Hence the natural inclusion $C(G, A) \to \bar{A} \times_\alpha G$ induces an order-isomorphism on K_0 (e.g., [9, Section 4]), so that when G is infinite, we may restrict to the dense subalgebra of continuous functions on G with values in $\bar{A}$ (and if $\bar{A}$ is AF, we may restrict to functions with values in the dense locally finite dimensional subalgebra A).

Now $K_0(\bar{A} \times_\alpha G)$ admits a natural structure as a module over $K_0(G)$ (e.g., see [21]). If V is a finite dimensional G-module with corresponding character χ, and P is a finitely generated projective left $\bar{A} \times_\alpha G$ module, then the equivalence class in K_0 of the $\bar{A} \times_\alpha G$-module, $P \otimes_{\mathbf{C}} V$ (the G-action on P and V results in an action of $\bar{A} \times_\alpha G$ on the whole tensor product) yields $[P] \cdot \chi$. It is routine to verify that this turns $K_0(\bar{A} \times_\alpha G)$ into a $K_0(G)$-module. Furthermore, if $\bar{A} \times_\alpha G$ is stably finite (as will be the case when A is AF), then $K_0(\bar{A} \times_\alpha G)$ admits a natural partial ordering [7] with positive cone $\{[P] | P$ finitely generated projective $K_0(\bar{A} \times_\alpha G)$-modules$\}$, and the cone is preserved by the action of the positive elements of $K_0(G)$ (the characters). Thus in this case, $K_0(\bar{A} \times_\alpha G)$ is a partially ordered module over the partially ordered ring $K_0(G)$.

Our first main result (II.1 and II.2) will elaborate on this module structure.

If A is a *-algebra $U(A)$ will denote the group of unitary elements of A.

Let $A = \pi A_j$ be a finite dimensional C^* algebra expressed as a product of its simple factors, and let $\alpha\colon G \to \operatorname{Aut}(A)$ be a (continuous) group homomorphism of a special form: There exists a representation $\gamma\colon G \to U(A)$ such that $\operatorname{Ad}\gamma(g) = \alpha(g)$ for all g in G. We call such an action α *representable*. If A is now an algebra written as a unital union of finite dimensional *-subalgebras, $A = \bigcup A^i$, and $\alpha\colon G \to \operatorname{Aut}(A)$ (or $G \to \operatorname{Aut}(\bar{A})$) is a point-norm continuous homomorphism such that for some increasing nest $B^1 \subset B^2 \subset B^3 \subset \ldots$ of finite dimensional C^*-subalgebras of $\bar{A}$, with dense union, and we have that both $\alpha(G)B^i \subset B^i$ and α/B^i is representable for all i, then we say α is *locally representable*. (Since G is compact, $\alpha(G)(A) \subset A$ entails the existence of a nest of finite dimensional $\alpha(G)$-invariant *-subalgebras $B^i \subset B^{i+1}$ with $A = \bigcup B^i$.)

Locally representable actions can be rather fare from being inner, as is well-known. The standard examples analysed in [5, 2] and [11] are *product type* *actions*. Put $A = \otimes M_{n(i)}\mathbf{C}$, and let γ_i: $G \to U(n(i), \mathbf{C})$ be a sequence of representations; then $\alpha(g) = \otimes \operatorname{Ad} \gamma_i(g)$ yields α: $G \to \operatorname{Aut}(\bar{A})$; this is obviously locally representable, but inner only when almost all of the γ_i's are scalar-valued. There are many other *non-product* type actions on this tensor product algebra, and we shall construct and classify the locally representable ones.

A C^* algebra is AF if it contains as a dense *-subalgebra, an algebra $A = \bigcup A^i$, where each A^i is finite dimensional. If additionally any one such dense A can be expressed in the form $\otimes M_{n(i)}\mathbf{C}$ for some sequence of positive integers $\{n(i)\}$, then $\bar{A}$ is a *UHF algebra*. For the AF algebra $\bar{A}$ we shall only be studying those actions α: $G \to \operatorname{Aut}(\bar{A})$ that leave invariant a dense locally finite *-subalgebra. Whether this assumption is redundant for AF algebras is an open question for every compact group G. Since for A^i finite dimensional, $A^i \times_{\alpha/A^i} G$ is itself AF, it easily follows that both inclusions

$$A \times_\alpha G \to \overline{A \times_\alpha G} \quad \text{and} \quad A^G \to \bar{A}^G$$

induce order-isomorphisms on K_0, in the former case as $K_0(G)$ ordered modules. Hence in what follows, we need not discuss $\bar{A}$, but concern ourselves only with A, the dense locally finite dimensional *-subalgebra.

Very often, the subscripted symbol denoting the action of a group on an algebra, e.g., α or $\alpha^{(i)}$ or α/A^i will be suppressed in writing the crossed product (for typographical reasons); thus $A \times_\alpha G$ could be written as $A \times G$ where no confusion would be likely to arise.

II. Finite dimensional dynamical systems

In this section, we discuss the construction of the general locally representable action, and how it translates when K_0 is applied. We deduce that $(K_0(A \times_\alpha G), [A_\alpha])$ is a complete invariant for locally representable actions α (where A_α is A viewed as an $A \times_\alpha G$-module, and $[A_\alpha]$ its class in $K_0(A \times_\alpha G)$). We also show how $(K_0(A^G), [A^G])$ (the complete invariant for the fixed point algebra, qua AF algebra) may be recovered from the crossed product and its K_0.

A *dynamical system* is a triple (A, G, α) where A is a C^* algebra of a *-subalgebra thereof, G is a compact group, and α: $G \to \operatorname{Aut}(A)$ is a group homomorphism with the property that the map $G \to A$, $g \mapsto a^{\alpha(g)}$ is continuous for every a in A. This is precisely the condition that allows the crossed product to be formed. A *morphism* of dynamical systems φ: $(A, G, \alpha) \to (B, G, \beta)$ (G can be allowed to change, but this is never required below) is a *-algebra map φ: $A \to B$ such that

$$\varphi(a^{\alpha(g)}) = \varphi(a)^{\beta(g)} \quad \text{for all } a \text{ in } A,\ g \text{ in } G.$$

Such a morphism φ induces a map $A \times_\alpha G \to B \times_\beta G$ on the crossed product (also denoted φ), given by $ag \mapsto \varphi(a)g$ if G is finite, and by $f \mapsto \varphi \circ f$ if G is infinite. Conversely, any *-algebra map $A \to B$ which extends in this way to a map of the crossed products "is" a morphism $(A, G, \alpha) \to (B, G, \beta)$ of dynamical systems.

If A is finite dimensional, and α: $G \to \mathrm{Aut}(A)$ is representable we use the same symbol α to denote one choice of representation $G \to U(A)$ which implements the action α, provided confusion is unlikely to occur.

Suppose (A, G, α) and (B, G, β) are dynamical systems with A, B *finite dimensional* and α, β *representable*. We wish to describe (up to suitable inner automorphisms) all dynamical systems' maps between them.

Now $K_0(A \times_\alpha G)$ and $K_0(B \times_\alpha G)$ are modules over the commutative ring $K_0(G)$ (what tensor product over $\mathbf{C}$ as multiplication). A morphism

$$\varphi\colon (A, G, \alpha) \to (B, G, \beta)$$

induces a $K_0(G)$-module homomorphism

$$K_0(\varphi)\colon K_0(A \times_\alpha G) \to K_0(B \times_\alpha G)$$

via $[M] \to [(B \times_\beta G) \otimes_{(A \times_\alpha G)} M]$, $B \times_\beta G$ being regarded as a right module over $A \times_\alpha G$ via φ: $A \times_\alpha G \to B \times_\beta G$.

We define A_α to be A, regarded as a left module over $A \times_\alpha G$, via the action

$$(\Sigma a_g g) a = \Sigma a_g a^{\alpha(g)}$$

if G is finite, and

$$fa = \int_G f(g) a^{\alpha(g)}\, dg$$

if G is infinite. (From now on, we shall use the notation for finite G, omitting the obvious modifications for infinite G.)

The map $K_0(\varphi)$ evidently sends the element of $K_0(A \times_\alpha G)$, $[A_\alpha]$, to $[B_\beta]$ in $K_0(B \times_\beta G)$.

In the representable case, we may lift α to a homomorphism $\dot{\alpha}$: $G \to U(A)$; then there is an algebra isomorphism $A \times_\alpha G \to C^*(G, A)$, sending $\Sigma a_g g$ to $\Sigma a_g \dot{\alpha}(g)^{-1} g$. If we write $A = \prod_{i=1}^m A_i$ in terms of simple components, then the $K_0(G)$-module structure of $K_0(AG)$ is especially simple to describe: each simple component contributes a copy of $K_0(G)$, so as a $K_0(G)$-module,

$$K_0(AG) = \sum_{i=1}^{m} K_0(A_i G) = K_0(G)^m.$$

The ordering on $K_0(AG)$ then corresponds to the usual ordering on $K_0(G)^m$. The isomorphism $A \times_\alpha G \to AG$ described above induces an order isomorphism of $K_0(G)$-modules $K_0(A \times_\alpha G) \cong K_0(G)^m$. This isomorphism is not uniquely determined, however, because if $\xi = (\xi_1, \xi_2, \dots \xi_m)$ is an m-tuple of one-dimensional representations of G, then both $\dot\alpha$ and $\xi\dot\alpha$ represent α. Conversely, if $\dot\alpha_1$ and $\dot\alpha_2$: $G \to U(A)$ both represent α: $G \to \mathrm{Aut}(A)$, then they differ only by some such ξ (because $g \to \dot\alpha_1(g)\dot\alpha_2(g)^{-1}$ is a homomorphism of G into the unitary group of the centre of A).

The isomorphism $K_0(A \times_\alpha G) \leftrightarrow K_0(G)^m$ is explicitly given by

$$[M] \mapsto ([\mathrm{Hom}_A(V_i, M)]), \qquad [\Sigma V_i \otimes_{\mathbf{C}} N_i] \leftarrow ([N_i])_i. \tag{1}$$

Here $V_i = \mathbf{C}^{k(i)}$ is the simple A-module corresponding to $A_i \cong M_{k(i)}\mathbf{C}$. The G-module structure on $\mathrm{Hom}_A(V_i, M)$ is given by $(g \cdot f)(v) = g \cdot (f(v))$; the $A \times_\alpha G$-module structure on $V_i \otimes N_i$ is given by $(ag) \cdot (v \otimes n) = (a\dot\alpha(g)v) \otimes gn$. In particular, the element $[A_\alpha]$ of $K_0(A \times_\alpha G)$ corresponds to the element $([V_i{}^*])_i$ of $K_0(G)^m$, where $V^* = \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$ is the dual vector space. (This because

$$\mathrm{Hom}_A(V_i, A) \cong \mathrm{Hom}_A\big(V_i, \mathrm{End}_{\mathbf{C}}(V_i)\big) \cong \mathrm{Hom}_A\big(V_i, V_i \otimes V_i{}^*\big) \cong V_i{}^*$$

as G-modules.)

To simplify notation, it is convenient to introduce the $A \times_\alpha G$-module $V = \Sigma V_i$ and the auxilary algebras

$$E = \mathrm{End}_A(V) = \prod_i E_i = \mathbf{C}^m, \quad E_i = \mathrm{End}_{A_i}(V_i) = \mathbf{C}, \quad EG = \mathbf{C}^m G = (\mathbf{C}G)^m.$$

Then obviously $K_0(EG) = K_0(G)^m$, and the isomorphism (1) above becomes the isomorphism $K_0(A \times_\alpha G) \cong K_0(EG)$ given by

$$\begin{gathered}[M] \to [\mathrm{Hom}_A(V, M)], \\ [V \otimes_E N] \leftarrow [N]\end{gathered} \tag{2}$$

Let $A^G = \{a \in A: a^{\alpha(g)} = a \text{ for all } g \in G]$ be the fixed point subalgebra. Since α is representable, $A^G = \prod A_i^G$. We may regard $K_0(A^G)$ as the subgroup of $K_0(A \times_\alpha G)$ generated by $V \otimes_E M^*$ where M runs over the EG-submodules of V via the embedding

$$K_0(A^G) \hookrightarrow K_0(A \times_\alpha G), \qquad P \to A \otimes_{A^G} P. \tag{3}$$

(The dual space M^* of M is regarded as a G-module via $(g \cdot f)(m) = f(g \cdot m)$; the action of $A \times_\alpha G$ on $A \otimes_{A^G} P$ is given by $(ag) \cdot (b \times p) = (a\dot\alpha(g)b) \otimes p$.) —This is a consequence of the Morita duality between simple A_i^G-modules and simple G-submodules of V_i. —Note that under $K_0(A \times_\alpha G) = K_0(G)^m$, $K_0(A^G)$ becomes the subgroup of $K_0(G)^m$ generated by $([M_i^*])_i$, where M_i

runs over the G-submodules of V_i.

Suppose now we have two finite-dimensional, representable dynamical systems (A, G, α) and (B, G, β) together with a morphism φ: $(A, G, \alpha) \to (B, G, \beta)$. In analogy with A_i, V_i, E etc. as above, we introduce the corresponding objects B_i, W_i, F_i etc. for (B, G, β). The morphism φ: $(A, G, \alpha) \to (B, G, \beta)$ induces maps $A \times_\alpha G \to B \times_\beta G$ and $A^G \to B^C$ and corresponding maps on K_0.

II.1 LEMMA. *The following diagram commutes.*

$$\begin{array}{ccc} K_0(A \times_\alpha G) & \longrightarrow & K_0(B \times_\beta G) \\ \uparrow & & \uparrow \\ K_0(A^G) & \longrightarrow & K_0(B^G). \end{array} \tag{4}$$

Furthermore, under the identification

$$K_0(A \times_\alpha G) = K_0(G)^m \quad \text{and} \quad K_0(B \times_\beta G) = K_0(G)^n$$

the map $K_0(A \times_\alpha G) \to K_0(B \times_\beta G)$ *becomes*

$$K_0(G)^m \to K_0(G)^n, \qquad ([M_i])_i \to \left(\left[\sum_i V_{ji} \otimes_{\mathbf{C}} M_j\right]\right)_j \tag{5}$$

where $V_{ji} = W_j^* \otimes_A V_i = \mathrm{Hom}_A(V_i, W_j)^*$.

Remark. $W_j^* \otimes_A V_i$ is considered as a G-module via

$$g \cdot (f \otimes v) = f \circ \dot{\beta}(g)^{-1} \otimes \dot{\alpha}(g) v;$$

W_j^* is a right A-modula via $f \cdot a = f \circ \varphi(a)$. The maps in the diagram (4) are

$$[M] \mapsto \left[(B \times_\beta G) \otimes_{(A \times_\alpha G)} M\right],$$

$$\begin{array}{cc} [A \otimes_{A^G} P] & [B \otimes_{A^G} Q] \\ \uparrow & \uparrow \\ [P] & [Q], \end{array} \tag{6}$$

$$[P] \mapsto \left[B^G \otimes_{A^G} P\right].$$

Proof. The commutativity of (4) means that

$$(B \times_\beta G) \otimes_{(A \times_\alpha G)} \otimes A \otimes_{A^G} P \quad \text{and} \quad B \otimes_{A^G} \left(B^G \otimes_{A^G} P\right) = B \otimes_{A^G} P$$

are isomorphic as $B \times_\beta G$-modules. The (left) $B \times_\beta G$ actions on these modules are

$$ag \cdot (bh \otimes c \otimes p) = a\dot{\beta}(g) b \dot{\beta}(g^{-1}) gh \otimes c \otimes p,$$
$$ag \cdot (b \otimes p) = a\dot{\beta}(g) b \otimes p.$$

(Here $a, b \in B$, $g, h \in G$, $ag \in B \times_\beta G$, etc.) The required isomorphism is given by

$$bh \otimes c \otimes p \to b\dot{\beta}(h)\varphi(c) \otimes p,$$

with inverse

$$b \otimes 1 \otimes p \leftarrow b \otimes p.$$

To verify (5) it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} K_0(A \times_\alpha G) & \longrightarrow & K_0(B \times_\beta G) \\ \uparrow & & \uparrow \\ K_0(EG) & \longrightarrow & K_0(FG) \end{array} \qquad (7)$$

where the maps are given by

$$[P] \mapsto \left[(B \times_\beta G) \otimes_{(A \times_\alpha G)} P\right],$$

$$\begin{array}{ccc} [V \otimes_E M] & & [W \otimes_F N] \\ \uparrow & & \uparrow \\ [M] & & [N], \end{array}$$

$$[M] \mapsto \left[(\mathrm{Hom}_F(W, F) \otimes_A V) \otimes_E M\right].$$

Indeed, under $K_0(EG) = K_0(G)^m$, $[\Pi_i E_i \otimes_{\mathbf{C}} M_i] \leftrightarrow ([M_i])_i$, the bottom map in (7) becomes exactly the one in (5), because

$$(\mathrm{Hom}_F(W, F) \otimes_A V) \otimes_A M \cong \left(\sum_j \mathrm{Hom}_{F_j}(W_j, F_j) \otimes_A \left(\sum_i V_i\right) \otimes_E \left(\sum_i M_i\right)\right)$$
$$\cong \sum_{ij} \mathrm{Hom}_{\mathbf{C}}(W_j, \mathbf{C}) \otimes_A V_i \otimes_{\mathbf{C}} M_i$$
$$= \sum_{ij} V_{ji} \otimes_{\mathbf{C}} M_i.$$

To prove that (7) commutes we have to check that for

$$N = \left(\operatorname{Hom}_F(W, F) \otimes_A V\right) \otimes_E M$$

the $B \times_\beta G$ modules $(B \times_\beta G) \otimes_{(A \times_\alpha G)} (V \otimes_E M)$ and $W \otimes_F N$ are isomorphic. For this we first note that there are natural bijections

$$B \cong W \otimes_F \operatorname{Hom}(W, F), \quad B \times_\beta G \cong B \otimes_F FG.$$

Under the first bijection, $W \otimes_F \operatorname{Hom}_F(W, F)$ becomes a left and right B-module so that

$$a \cdot (v \otimes f) = av \otimes f, \quad (v \otimes f) \cdot a = v \otimes fa;$$

under the second one, $B \otimes_F FG$ becomes left and right $B \times_\beta G$-module so that

$$ag \cdot (b \otimes h) = a\dot{\beta}(g) b\dot{\beta}(g^{-1}) \otimes gh,$$
$$(b \otimes h) \cdot ag = b\dot{\beta}(h) a\beta(h^{-1}) \otimes hg.$$

We now have to check that with the corresponding $B \times_\beta G$-module structures on

$$\left(W \otimes_F \operatorname{Hom}_F(W, F) \otimes_F FG\right) \otimes_{(A \times_\alpha G)} (V \otimes_E M)$$

and

$$W \otimes_F \left(\left(\operatorname{Hom}_F(W, F) \otimes_A\right) \otimes_E M\right),$$

these $B \times_\beta G$-modules are isomorphic.

The left action of $B \subset B \times_\beta G$ on either one of the modules is on the tensor factor W only. So it suffices to check that

$$\left(\operatorname{Hom}_F(W, F) \otimes_F FG\right) \otimes_{(A \times_\alpha G)} (V \otimes_E M)$$

and

$$\left(\operatorname{Hom}_F(W, F) \otimes_A V\right) \otimes_E M$$

are isomorphic as left G-modules. Here the G-actions are given by

$$g \cdot ((f \otimes h) \otimes (v \otimes m)) = \left(f \circ \dot{\beta}(g^{-1}) \otimes gh\right) \otimes (v \otimes m),$$
$$g \cdot ((f \otimes v) \otimes m) = \left(f \circ \dot{\beta}(g^{-1}) \otimes \dot{\alpha}(g) v\right) \otimes (gm).$$

The required G-isomorphism is given by

$$(f \otimes h) \otimes (v \otimes m) \to (f \circ \dot{\alpha}(h) \otimes v) \otimes hm = (f \otimes \dot{\alpha}(h)v) \otimes hm.$$

Its inverse is given by

$$(f \otimes 1) \otimes (v \otimes m) \leftarrow (f \otimes v) \otimes m.$$

We omit the routine verifications. ■

Thus we have the commuting diagram

$$\begin{array}{ccc} K_0(A \times_\alpha G) & \longrightarrow & K_0(B \times_\beta G) \\ \uparrow & & \uparrow \\ K_0(G)^m & \longrightarrow & K_0(G)^n \\ \uparrow & & \uparrow \\ K_0(A^G) & \longrightarrow & K_0(B^G). \end{array}$$

The upper two vertical maps are (order-)isomorphisms, and the bottom two are order-preserving embeddings. The map $K_0(G)^m \to K_0(G)^n$ described in (5), can be re-interpreted in terms of characters; it is given by an $n \times m$ matrix with entries which are characters (that is, positive elements of $K_0(G)$)—χ_{ij} is the character of the representation afforded by V_{ji}.

Conversely, given α: $G \to U(A)$, a matrix of characters $M = (\chi_{ij})$, and a specified element $\Psi = (\psi_1, \dots, \psi_m)^T$ in $(K_0(G)^m)^+$, and $M\Psi$ in $(K_0(G)^n)^+$, we can construct (explicitly) an algebra B, an action β: $G \to U(B)$, and homomorphism of dynamical systems $(A, G, \alpha) \to (B, G, \beta)$ to yield the matrix M, as follows.

Let $M(1)$ denote the integer matrix $(\chi_{st}(1))$. Define the algebra B as $\oplus_{s=1}^n M_{k(s)}\mathbf{C}$, where $k(s) = \Sigma_t \chi_{st}(1) k'(t)$ (the t-th simple component of A is a matrix ring of size $k'(t) = \psi_t(1)$). In other words, the algebra A is interpreted as the column of integers

$$K' = (k'(1), \dots, k'(m))^T,$$

representing the sizes of the matrix rings, and the corresponding column for B is obtained via the matrix multiplication $K = M(1)K'$. The algebra map φ: $A \to B$ is now obvious: Send the first central idempotent of A into the s'-th component of B with multiplicity $M(1)_{s1}$; iterating this yields a unital algebra map φ: $A \to B$ such that

$$K_0(\varphi): K_0(A) \to K_0(B)$$

is multiplication by the matrix $M(1)$.

Now we must describe the action of β in B. Let $A = \Pi A_i$, $B = \Pi B_j$ be the factorizations in terms of the simple components, and set E^i (F^j) to be the ith (j-th) minimal central idempotent of $A(B)$, and let $e^i = (e_1^i, e_2^i, \dots, e_n^i)$ be $\varphi(E^i)$. Then $F^j\varphi(A_i)F^j$ is a simple unital subalgebra of $e_j^iBe_j^i$, which is itself simple. Hence we may decompose $e_j^iBe_j^i = F^j\varphi(A_i)F^j \otimes X_j^i$, where X_j^i is the centralizer of the former in $e_j^iBe_j^i$, and is of course simple. Then $\dim X_j^i = \chi_{ij}(1)^2$ (by construction), so there exists a representation φ_{ij}: $G \to U(X_j^i)$ such that the character of φ_{ij} is χ_{ij}. Also the map $A_i \to F^j\varphi(A_i)F^j$ is an algebra isomorphism (or the zero map), so if $\gamma = (\gamma_1, \dots, \gamma_m)$ is a representation yielding the action α, we may transfer γ to $F^j\varphi(A_i)F^j$, to obtain a map

$$\gamma_i\colon G \to U\big(F^j\varphi(A_i)F^j\big).$$

Define δ_j: $G \to U(B_j)$ via $\delta_j = \oplus_{i=1}^m \gamma_i \otimes \varphi_{ij}$, and define δ: $G \to U(B)$ as $\delta = (\delta_1, \dots, \delta_n)$.

This construction depends on the choice of representations; there is a method of picking the representations in a manner compatible with limits and homomorphisms; this is implicit in the proof of the results above. The following lemma formalizes the procedure just given, in the language of G-modules. In particular, limits are obtainable by simply writing down a sequence of matrices with characters as entries, of compatible dimensions.

II.2 LEMMA. *Let (A, G, α) and (B, G, β) be finite dimensional, representable dynamical systems. Then any order preserving $K_0(G)$-module homomorphism*

$$\Phi\colon K_0(A \times_\alpha G), [A_\alpha] \to K_0(B \times_\beta G), [B_\beta]$$

is induced by a morphism of dynamical systems

$$\varphi\colon (A, G, \alpha) \to (B, G, \beta).$$

*Furthermore, φ is uniquely determined by Φ up to an inner *automorphism of B commuting with β.*

Proof. The hypothesis $\Phi[A_\alpha] = [B_\beta]$ gives

$$\Phi\Big(\sum_i [V_i] \otimes [V_i^*]\Big) = \sum_i [W_i] \otimes [W_i^*],$$

so

$$\sum_i (\Phi[V_i]) \otimes [V_i^*] = \sum_i [W_i] \otimes [W_i^*], \tag{8}$$

because Φ is a $K_0(G)$-module homomorphism. Write

$$\Phi[V_i] = \sum_j [W_j] \otimes [M_{ij}]$$

with $[M_{ij}] \in K_0(G)^+$. Then (8) gives

$$\sum_{ij} [W_j] \otimes [M_{ij}] \otimes [V_i^*] = \sum_j [W_j] \otimes [W_j^*]$$

so $\Sigma_i [M_{ij}] \otimes [V_i^*] = [W_j^*]$ and

$$W_j \cong \sum_i V_i \otimes M_{ij}^* \tag{9}$$

as G-modules. A choice of such an isomorphism of G-modules gives a G-isomorphism of algebras

$$B_j \cong \sum_i A_i \otimes C_{ij}$$

where $C_{ij} = \operatorname{End}_{\mathbf{C}}(M_{ij}^*)$, and therefore

$$B \cong \sum_{ij} A_i \otimes C_{ij}.$$

So a choice of isomorphism (9) gives a map

$$\varphi\colon A \to B \cong \sum_{ij} A_i \otimes C_{ij}, \qquad a \to \Sigma a_i \otimes e_{ij},$$

e_{ij} the unit of C_{ij}. This is the required morphism $\varphi\colon (A, G, \alpha) \to (B, G, \beta)$ of dynamical systems.

It is clear that *any* such morphism which induces the given

$$\Phi\colon K_0(A \times_\alpha G) \to K_0(B \times_\beta G)$$

is obtained by this procedure with a suitable choice of the G-isomorphism (9) (as one sees from (5)). Since $B_j = \operatorname{End}_{\mathbf{C}}(W_j)$, any two such choices differ by an element b_j of B_j, which must satisfy $b_j \tilde{\beta}(g) = \dot{\beta}(g) b_j$ in order to be a G-map. This b_j can further be chosen to be unitary, as the representatives of elements of $K_0(G)$ can always be chosen to be unitary. It follows that any two morphisms $\varphi\colon (A, G, \alpha) \to (B, G, \beta)$ which induce the same map $\Phi\colon K_0(A \times_\alpha G) \to K_0(B \times_\beta G)$ differ by an inner automorphism of B given by a unitary element $b = \Sigma_j b_j$ which commutes with $\dot{\beta}$. ■

II.3 *Example.* Take $G = \mathbf{Z}_2 = \{1, g\}$. Then (see the appendix),

$$K_0(G) = \mathbf{Z}\hat{G} = 1 \cdot \mathbf{Z} + \hat{g} \cdot \mathbf{Z},$$

with positive cone $K_0(G)^+ = \{m + n\hat{g} \mid m, n \geq 0\}$. Choose a sequence of characters $\{\chi^j = m(j) + n(j)\hat{g} \mid j \in \mathbf{N}\}$. Set $s(j) = m(j) + n(j) = \chi^j(1)$.

Now form the tensor product algebras $A^i = \otimes_{j=1}^i M_{s(j)}\mathbf{C}$. The obvious map, $a \mapsto a \otimes I_{s(i+1)}$ is an embedding $A^i \to A^{i+1}$. An action of G on each A^i, compatible with these algebra maps, can be defined simply via $\otimes_{j=1}^i \varphi^j$: $G \to A^i$, where each φ^j: $G \to M_{s(j)}\mathbf{C}$ is a representation with character χ^j. (Taken to the limit, as we shall in Section III, this example is of a "product type action," and is quite well-known.)

Since each A^i is simple, $K_0(A^j \times G) \simeq K_0(A^j G) = K_0(G)$; i.e., $K_0(A^j \times G)$ is a free rank 1 $K_0(G)$-module. On identification of $K_0(A^j \times G)$ with $K_0(G)$, the diagram translates to

$$\begin{array}{ccc} K_0(A^i \times G) & \longrightarrow & K_0(A^{i+1} \times G) \\ \downarrow & & \downarrow \\ K_0(G) & \xrightarrow{\times \chi^{i+1}} & K_0(G). \end{array}$$

This very simple example will be discussed further in III.4. ∎

II.4 *Example.* Again consider the two-element group $G = \mathbf{Z}_2$. Define a sequence of algebras and maps

$$A^0 = M_1C \times M_1C \to A^1 = M_5C \times M_5C \to \cdots \to A^n = M_{5^n}C \times M_{5^n}C$$

given by

$$(a, b) \mapsto \left(\begin{bmatrix} a & & & & \\ & a & & & \\ & & b & & \\ & & & b & \\ & & & & b \end{bmatrix}, \begin{bmatrix} a & & & & \\ & a & & & \\ & & b & & \\ & & & b & \\ & & & & b \end{bmatrix} \right).$$

Each A^i satisfies $K_0(A^i) \simeq \mathbf{Z}^2$ as ordered groups, via the map on equivalence classes of idempotents, $[(e, f)] \mapsto (\operatorname{rank} e, \operatorname{rank} f)$. Transposing these pairs to obtain columns (so matrices will act on the left), we see that K_0 of the map $A^i \to A^{i+1}$ is given by the matrix

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix},$$

which we call $M(1)$.

We wish to impose a G-action ($G = \mathbf{Z}_2$) on each A^i compatible with the algebra maps. Select 4 characters of G, $\chi_{11}, \chi_{12}, \chi_{21}, \chi_{22}$, so that if $M = [\chi_{ij}]^T$, then M evaluated at 1 should be what we have called $M(1)$. One choice for M is

$$M = \begin{bmatrix} 1 + \hat{g} & 2 + \hat{g} \\ 2 & 1 + 2\hat{g} \end{bmatrix}$$

where $\hat{g}$ is the character sending the non-identity element to -1. The corresponding representations χ_{ij} of G are given by the formulae:

$$\varphi_{11}(g) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad \varphi_{21}(g) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix},$$

$$\varphi_{12}(g) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \varphi_{22}(g) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}.$$

Describe an action of G on A^1 as follows. The image of A^0 in A^1 is

$$\left\{ \left(\begin{bmatrix} a & & & & \\ & a & & & \\ & & b & & \\ & & & b & \\ & & & & b \end{bmatrix}, \begin{bmatrix} a & & & & \\ & a & & & \\ & & b & & \\ & & & b & \\ & & & & b \end{bmatrix} \right) \middle| a, b \in C \right\}.$$

The *first* component (in A^1) of the image of the *first* component of A^0 is the upper 2×2 block $\left\{ \begin{bmatrix} a & \\ & a \end{bmatrix} \right\}$, and corresponds to the $(1,1)$ entry of M. The centralizer in the whole 2×2 matrix ring is of course M_2C again, so we use φ_{11} to obtain a map from G to the upper left 2×2 block, $g \mapsto \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$.

The *first* component in A^1 of the image of the *second* component of A^0 is $\{\text{diag}(b, b, b) | b \in C\}$; its centralizer is M_3C, and corresponds to the $(1,2)$ entry of M^T, or in otherwords, to $\chi_{21} = 2 + \hat{g}$, and its representation $g \mapsto \text{diag}(1, 1, -1)$. We use this last to fill in the image of G in the first component of A^1.

Working in the second component of A^1, the representation are φ_{12} and φ_{22}. Combining all four, we obtain the representation:

$$g \mapsto \left(\begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & -1 \end{bmatrix} \right).$$

This happened to be especially easy, since the action on A^0 was trivial. Now let us extend the action to A^2.

The first component in A^2 of the image of the first component of A^1 is $\left\{\begin{bmatrix} a & \\ & a\end{bmatrix} \middle| a \in M_5C\right\}$; its centralizer in $e_1^1A^2e_1^1$ (where

$$e_1^1 = \left(\begin{bmatrix} I_5 & & \\ & I_5 & \\ & & O_{15}\end{bmatrix}, [O_{25}]\right)$$

is isomorphic to M_2C, and selecting φ_{11}, we obtain a representation $G \to e_1^1A^2e_1^1 \simeq M_{10}C$, by

$$g \mapsto \begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -1\end{bmatrix} \otimes \begin{bmatrix} 1 & \\ & -1\end{bmatrix}$$
$$= \operatorname{diag}(1, -1, 1, 1, -1, -1, 1, -1, -1, 1).$$

The first component in A^2 of the image of the second component of A^1 is isomorphic to

$$\left\{\begin{bmatrix} b & & \\ & b & \\ & & b\end{bmatrix} \middle| b \in M_5C\right\}.$$

Setting

$$e_1^2 = \left(\begin{bmatrix} O_{10} & \\ & I_{15}\end{bmatrix}, [O_{25}]\right),$$

the centralizer in $e_1^2A^2e_1^2$ is isomorphic to M_3C. Selecting φ_{21} define the map $G \to e_1^2A^2e_1^2 = e_1^2(\text{Image } A^1)e_1^2 \otimes M_3C$ via

$$g \mapsto \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & -1\end{bmatrix} \otimes \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1\end{bmatrix}$$
$$= \operatorname{diag}(1, 1, 1, -1, -1, 1, 1, 1, -1, -1, -1, -1, -1, 1, 1).$$

The other two pieces are similarly computed, and the map $G \to A^2$ is given by

$$g \mapsto \left(\begin{bmatrix} \begin{bmatrix} 1 &&&& \\ & -1 &&& \\ && 1 && \\ &&& 1 & \\ &&&& -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} & \\ & \begin{bmatrix} 1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& -1 & \\ &&&& -1 \end{bmatrix} \otimes \begin{bmatrix} 1 && \\ & -1 & \\ && -1 \end{bmatrix} \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \begin{bmatrix} 1 &&&& \\ & -1 &&& \\ && 1 && \\ &&& 1 & \\ &&&& -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} & \\ & \begin{bmatrix} 1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& -1 & \\ &&&& -1 \end{bmatrix} \otimes \begin{bmatrix} 1 && \\ & -1 & \\ && -1 \end{bmatrix} \end{bmatrix} \right)$$

It is straightforward to iterate this (the page is not large enough to hold the diagram describing $G \to A^3$): If $G \to A^i$ is given by $g \mapsto (\pi_1(g), \pi_2(g))$, then $G \to A^{i+1}$ is given by

$$g \mapsto \left(\begin{bmatrix} \pi_1(g) \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} & \\ & \pi_2(g) \otimes \begin{bmatrix} 1 && \\ & 1 & \\ && -1 \end{bmatrix} \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \pi_1(g) \otimes \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} & \\ & \pi_2(g) \otimes \begin{bmatrix} 1 && \\ & 1 & \\ && -1 \end{bmatrix} \end{bmatrix} \right).$$

If $\alpha^{(i)}$ describes $G \to A^i$, then identifying $K_0(A^i \times G)$ (suppressing the $\alpha^{(i)}$ for typographical reasons) with $K_0(G)^2$ in the manner described earlier, identifies

$[A^0_{\alpha^{(0)}}]$ with $(1, 1)^T$;
$[A^1_{\alpha^{(1)}}]$ with $(3 + 2g, 3 + 2g)^T$;
$[A^2_{\alpha^{(2)}}]$ with $(12 + 13g, 12 + 13g)^T$, etc.

These were obtained by taking the characters, on each component, of $\alpha^{(i)}$.

Observe that

$$M\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 + 2g^{+} \\ 3 + 2g^{+} \end{pmatrix}, \quad M\begin{pmatrix} 3 + 2g^{\wedge} \\ 3 + 2g^{\wedge} \end{pmatrix} = \begin{pmatrix} 12 + 13g^{\wedge} \\ 12 + 13g^{\wedge} \end{pmatrix}, \ldots$$

as is to be expected, since M must send $[A^0_{\alpha^{(0)}}]$ to $[A^1_{\alpha^{(1)}}]$, the latter to $[A^2_{\alpha^{(2)}}]$, and so on. ■

This example can be generalized in a number of ways. First, the group G can be altered to any compact group; second, there is no need for the character matrix (here denoted M) to be the same at each level; it can be changed provided the evaluations at 1 yield unital algebra maps; third, the initial action (here, on A^0) need not be trivial (this would alter $[A^0_{\alpha^{(0)}}]$, but not the module structure).

The limit algebra and the corresponding action, together with computations on the K_0-modules will be discussed in Section III.

III. Classification

We now turn from finite dimensional algebras to AF algebras. We shall show that the invariant $K_0(A \times_\alpha G), [A]$ is a complete invariant for the locally representable action on the AF algebra A up to automorphisms. More precisely, we have the following result.

III.1 THEOREM. *Let (A, G, α) and (B, G, β) be direct limits of finite dimensional, representable dynamical systems. Then any order preserving $K_0(G)$-module homomorphism*

$$\Phi\colon K_0(A \times_\alpha G), [A_\alpha] \to K_0(B \times_\beta G), [B_\beta]$$

is induced by a morphism of dynamical systems

$$\varphi\colon (A, G, \alpha) \to (B, G, \beta).$$

*Finally, (A, G, α) and (B, G, β) are isomorphic as dynamical systems (i.e., there is a *isomorphism φ: $A \to B$ so that $\varphi\alpha = \beta\varphi$) if and only if $K_0(A \times_\alpha G), [A_\alpha]$ and $K_0(B \times_\beta G), [B_\beta]$ are isomorphic as ordered $K_0(G)$-modules with specified elements.*

Proof. Consider the first assertion. Let

$$\Phi\colon K_0(A \times_\alpha G), [A_\alpha] \to K_0(B \times_\beta G), [B_\beta]$$

be an order preserving $K_0(G)$-module homomorphism. Write (A, G, α) and (B, G, β) as limits of finite dimensional, representable dynamical systems:

$$\begin{gathered}\cdots \to (A^k, G, \alpha^k) \to (A^{k+1}, G, \alpha^{k+1}) \to \cdots \to (A, G, \alpha),\\ \cdots \to (B^K, G, \beta^k) \to (B^{k+1}, G, \beta^{k+1}) \to \cdots \to (B, G, \beta).\end{gathered}$$

After suitable telescoping and relabelling, the given map Φ can be written as the limit of a commuting diagram of order-preserving $K_0(G)$-module homomorphisms:

$$\begin{array}{ccccccc}\to K_0(A^k \times G),[A^k] & \to & K_0(A^{k+1} \times G),\ [A^{k+1}] & \to & \cdots & \to & K_0(A \times_\alpha G),[A_\alpha]\\ \downarrow & & \downarrow & & & & \downarrow\\ \to K_0(B^k \times G),[B^k] & \to & K_0(B^{k+1} \times G),\ [B^{k+1}] & \to & \cdots & \to & K_0(B \times_\beta G),[B_\beta]\end{array} \tag{1}$$

By Lemmas II.2, we get a diagram of morphisms of dynamical systems:

$$\begin{array}{ccccccc}\cdots \to (A^k, G, \alpha^k) & \to & (A^{k+1}, G, \alpha^{k+1}) & \to & \cdots & \to & (A, G, \alpha),\\ \downarrow & & \downarrow & & & & \\ \cdots \to (B^k, G, \beta^k) & \to & (B^{k+1}, G, \beta^{k+1}) & \to & \cdots & \to & (B, G, \beta)\end{array} \tag{2}$$

This last diagram (2) commutes initially only up to automorphisms of the (B^k, G, β^k). However, modifying the vertical maps successively by such automorphisms, we may assume that the diagram commutes properly, and therefore passes to a vertical map in the limit. This is the required morphism $(A, G, \alpha) \to (B, G, \beta)$ of dynamical systems.

If $K_0(A \times_\alpha G), [A_\alpha]$ is order-isomorphic to $K_0(B \times_\beta G), [B_\beta]$, then the diagram (1) can be further telescoped so that we obtain positive maps in both directions inducing the map and its inverse:

$$\begin{array}{ccccc}\cdots \to K_0(A^k \times G),[A^k] & \to & K_0(A^{k+1} \times G),[A^{k+1}] & \to & \cdots\\ \downarrow & \nearrow & \downarrow & \nearrow & \\ \cdots \to K_0(B^k \times G),[B^k] & \to & K_0(B^{k+1} \times G),[B^{k+1}] & \to & \cdots .\end{array} \tag{1'}$$

Then the corresponding map of dynamical systems $(A, G, \alpha) \to (B, G, \beta)$ *and its inverse* can be simultaneously built up by the usual interweaving argument that arises in direct limits. ∎

Remarks. (1) If $G = \{1\}$, then the theorem and its proof reduce to those of [4], classifying AF algebras via dimension groups, in the unital case.

(2) If α is known to be locally representable, and the $K_0(G)$-module structure of $K_0(A \times_\alpha G)$ is known, then $K_0(A)$ (and thus A, as well as α) can be recovered, as in [21; p. 101] (where a special case is considered). Let

$I = \{x \in K_0(G) | x(1) = 0\} = \{x \in K_0(G) | x \cdot \chi_{\text{reg}} = 0\}$ (χ_{reg} is the character of the regular representation—this applies if G is finite). Then the map, $K_0(A \times_\alpha G) \to K_0(A)$ which simply regards every $A \times_\alpha G$-module as an A-module, has kernel $J = I \cdot K_0(A \times_\alpha G)$. If we impose the trivial $K_0(G)$ action on $K_0(A)$ (so $y \cdot \chi = \chi(1) \cdot y$ for χ in $K_0(G)$), then $K_0(A \times_\alpha G)/J \simeq K_0(A)$ as ordered $K_0(G)$-modules, because the isomorphism holds locally (at the finite dimensional level). ■

Let G be a finite group, and $\alpha_{\text{reg}}\colon G \to U(|G|, \mathbf{C})$ the regular representation, with corresponding character χ_{reg}. If α, β are two actions of G (on a C^* algebra A), we say they are *stably conjugate* if $\alpha \otimes \alpha_{\text{reg}}$, $\beta \otimes \alpha_{\text{reg}}$ are conjugate as actions on $M_n A = A \otimes M_n\mathbf{C}$, where $n = |G|$.

III.2 COROLLARY. *Let $\alpha, \beta\colon G \to \mathrm{Aut}(A)$ be two locally representable actions of the finite group G. Then α and β are stably conjugate if and only if there is an order isomorphism $\rho\colon K_0(A \times_\alpha G) \to K_0(A \times_\beta G)$ such that*

$$\left(\rho([A_\alpha])\right)\chi_{\text{reg}} = \left[A_\beta\right]\chi_{\text{reg}}.$$

Proof. Simply observe that $[M_n A_{(\alpha \otimes \alpha_{\text{reg}})}] = [A_\alpha]\chi_{\text{reg}}$, and apply the previous theorem. ■

Remarks. (3) If, in the situation of the corollary, both α and β decompose as $\alpha' \otimes \alpha_{\text{reg}}$, and $\beta' \otimes \alpha_{\text{reg}}$, then stable conjugacy of α, β implies actual conjugacy, because $\chi_{\text{reg}}^2 = \chi_{\text{reg}}(1) \cdot \chi_{\text{reg}}$.

(4) In [11], analogous results were established for outer, product type actions on UHF algebras, with $K_0(A \times_\alpha G)$ replaced by $K_0(A^G)$. For G finite, the $A \times_\alpha G - A^G$ bimodule A implements a Morita duality between $A \times_\alpha G$ and A^G, so that their corresponding dimension groups are order-isomorphic.

This admits a considerable generalization. Drop the hypotheses of outer, product type, and UHF (so we have a locally representable action on an AF algebra). Then the pair $(K_0(A^G), [A^G])$ can always be recovered from $(K_0(A \times_\alpha G), [A_\alpha])$ (the former pair is the complete invariant, in the category of AF algebras, for A^G). This contrasts sharply with the situation for more general C^* algebras, where it is possible to have $K_0(A \times_\alpha G)$ being zero, while $K_0(A^G)$ is not—so it might be difficult to recover the latter from the former! We are indebted to Claude Schochet for this last remark, due to Alain Connes.

Let $\chi_0\colon G \to \mathbf{C}$ denote the trivial character of G, regarded as an element of $A \times_\alpha G$. Then χ_0 is a projection, and A^G is isomorphic to $\chi_0 * (A \times_\alpha G) * \chi_0$ [20], and χ_0 of course serves as the identity of the latter. As the latter is C^* Morita equivalent to the closure of the ideal generated by χ_0 (in $A \times_\alpha G$), and everything in sight is an AF algebra (and closed ideals in AF algebras correspond in a natural way with order ideals in the dimension group), we have

$$K_0(A^G) \simeq K_0\left(\chi_0 * (A \times_\alpha G) * \chi_0\right) = K_0\left(\mathrm{cl}((A \times_\alpha G)\chi_0(A \times_\alpha G)\right),$$

which is the order ideal in $K_0(A \times_\alpha G)$ generated by $[\chi_0 * (A \times_\alpha G)]$, abbreviated $[\chi_0]$. In fact, the right $A \times_\alpha G$-ideal $\chi_0 * (\overline{A} \times_\alpha G)$ equals $\overline{A^G}$. Thus

$$\left(K_0(A^G), [A^G]\right) = \left(\{s \in K_0(A \times_\alpha G) | \text{ there exists a positive integer } n \text{ with } -n[\chi_0] \leq s \leq n[\chi_0]\}, [\chi_0]\right).$$

For example, if we take a fixed representation of a compact group G, φ: $G \to M_n\mathbf{C}$, with character χ, and take the product type action (so $A = \otimes M_n\mathbf{C}$, and $\alpha = \otimes\varphi$), the diagram for $K_0(A \times_\alpha G)$ consists of a limit of 1×1 matrices, i.e., characters,

$$K_0(G) \overset{\times\chi}{\to} K_0(G) \overset{\times\chi}{\to} K_0(G) \overset{\times\chi}{\to} \cdots.$$

The limit is a ring, $R = (K_0(G)/J)[\chi^{-1}]$ (localize at the annihilator of χ, J, and invert χ; if G is connected, J is $\{0\}$). Then 1 in R corresponds to $[\chi_0]$ if it does so at the first level; this occurs if for example χ is irreducible, so that $(M_n\mathbf{C})^G = \mathbf{C}$. Then $K_0(A^G)$ is just the bounded subring of R, that is the elements of R that are bounded above and below by an integer multiple of 1. This is a ring, and the ordering on it is partially determined by its multiplicative positive homomorphisms to the reals (observed by Wassermann in his thesis, inter alia). Under some circumstances (investigated in a forthcoming paper by one of us), all states of R are extensions of states of the bounded subring.

It is also possible to determine the generating interval, $D(A \times_\alpha G)$ (see [4] for this notation). If ψ is an irreducible character of G, form $\varphi = \psi/\psi(1)$: $G \to \mathbf{C}$ regarded as an element both of $C^*(G)$ and $A \times_\alpha G$. Each φ is a projection, and since $C^*(G)$ has an approximate identity consisting of finite sums of the orthogonal projections equivalent in $K_0(C^*(G))$ to $\psi(1)[\varphi]$, $A \times_\alpha G$ has a similar approximate identity, and so $D(A \times_\alpha G)$ is the interval generated by $\{\psi(1)[\varphi]\}$; that is,

$$D(A \times_\alpha G) = \{r \in K_0(A \times_\alpha G) | 0 \leq r \leq \Sigma\psi(1)[\varphi], \text{ some finite set of } \psi\text{'s}\}$$

III.3 COROLLARY. *Let α: $G \to \mathrm{Aut}(\overline{A})$ be a locally representable outer action of the finite group G (on the* AF *algebra $\overline{A}$) such that $\overline{A} \times_\alpha G$ is* UHF. *Then $\overline{A}$ is* UHF, *and Morita equivalent to $\overline{A} \times_\alpha G$ as well as $\overline{A^G}$. Moreover, α is equivalent to an infinite tensor product of multiples of the regular representation of G (that is, α is equivalent to $\otimes\alpha_i$, where $\alpha_i = \alpha_{\mathrm{reg}} \otimes I_{t(i)}$, where α_{reg}: $G \to$* End *$L^2(G)$ denotes the regular representation, and $I_{t(i)}$ is the trivial representation of multiplicity $t(i)$).*

Proof. We have that $K_0(A \times_\alpha G)$ is rank 1 as a $\mathbf{Z}$-module; necessarily the action of $R = K_0(G)$ is trivial (that is, $\chi \cdot x = \chi(1)x$), as the map

$K_0(A \times_\alpha G) \to K_0(A)$ must be onto, and the abelian group rank is one so the map is an isomorphism. So $K_0(A)$ has rank 1, and thus A is UHF; since $K_0(A \times_\alpha G)$ is isomorphic (simply as an abelian group is sufficient) to it, $A \times_\alpha G$ is Morita equivalent to A. Since the action is outer, $A \times_\alpha G$ is Morita equivalent to A^G (via the bimodule A).

Writing $K_0(A \times_\alpha G)$ as a limit of the sequence of free $K_0(G)$-modules obtained from the local representability, we have

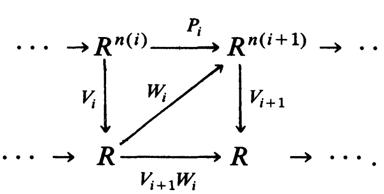
$$K_0(A \times_\alpha G) = \lim P_i\colon R^{n(i)} \to R^{n(i+1)} \quad (R = K_0(G)).$$

We may assume (telescoping if necessary), that every P_i is rank 1 as an *abelian group homomorphism*. Regarding each P_i as a matrix of size $n(i+1) \times n(i)$, each entry must be zero at every non-identity conjugacy class (otherwise, modulo one of the minimal prime ideals other than the kernel of $\chi \mapsto \chi(1)$, the matrix would not be zero, so the abelian group rank of P_i would exceed 1). This means each entry must be an integer multiple of the character of the regular representation.

Any rank 1 matrix with integer entries factors as wv, where w is a column and v is a row; if the matrix has only non-negative entries, then the same may be assumed for the column and row (consider the nonzero rows of the matrix; because $\mathbf{Z}$ is a principal ideal domain, one of them will divide all the others, yielding v). In the case of P_i, we can write $P_i = \chi_{\text{reg}} Q_i'$, where χ_{reg} is the character of the regular representation, and Q_i' is a non-negative matrix with integer entries. An additional telescoping allows us to assume

$$P_i = \chi_{\text{reg}}^2 Q_i = \chi_{\text{reg}} |G| Q_i.$$

Factoring $Q_i = w_i v_i$ as a non-negative column times a non-negative row with integer entries, we have a corresponding factorization $P_i = W_i V_i$, where $W_i = \chi_{\text{Reg}} w_i$, $V_i = \chi_{\text{reg}} v_i$ have entries from $K_0(G)$. This yields an R-module order-isomorphism between the limits of the rows of



The bottom row of course would come from the product type action corresponding to $V_i W_{i+1}$, which is an integer multiple of the regular character, and therefore corresponds to a multiple of the regular representation. The element $[A_\alpha]$ is sent to some positive element in the limit of the bottom row, so we have an order isomorphism to the $K_0(G)$-module with a specified element arising from a product type action. By the theorem, the two actions are conjugate. ■

III.4 *Example* (Continuation of II.3). Taking the limit of the algebra and of the group action(s) constructed in II.3, we obtain an action of $G = \mathbf{Z}_2$ on the UHF algebra $A = \otimes_1^\infty M_{s(j)}\mathbf{C}$. The action is given simply as $\alpha(g) = \otimes \operatorname{Ad} \varphi^j(g)$. The module $K_0(A \times_\alpha G)$ is the limit of the maps

$$\cdots \to K_0(G) \overset{\chi^{i+1}}{\to} K_0(G) \to \cdots .$$

In particular, $K_0(A \times_\alpha G)$ will have rank one over $K_0(G)$. Similar considerations apply to any compact group (replacing $\mathbf{Z}_2$); the resulting action is called product type. For $\mathbf{Z}_2$ and $\mathbf{Z}_p$, these actions are classified in [5], [6]. For other finite groups and outer product type actions, classification is possible via the fixed point algebra [11].

Returning to our example(s), we analyze the limit module $K_0(A \times_{\hat{\alpha}} G)$, which we abbreviate to Q. Denoting the ideals of $\mathbf{Z}G\hat{}$ generated by $\hat{g} - 1$, $\hat{g} + 1$ by P_1, P_2 (see the appendix), Q embeds subdirectly in $Q_1 \oplus Q_2$, where $Q_i = Q/P_iQ$. Sending $\hat{g}$ to 1 yields Q_1 as the limit of $\mathbf{Z} \to \mathbf{Z}$, where the maps are multiplication by $m(i) + n(i)$; similarly Q_2 is the limit of $\mathbf{Z} \to \mathbf{Z}$, with maps multiplication by $m(i) - n(i)$. The embedding $Q \to Q_1 \oplus Q_2$ is onto (that is, Q decomposes as a direct sum of invariant subgroups under the action of $G\hat{}$) if and only if $m(i) + n(i)$ is even for infinitely many i (it is possible to verify that Q_1 or Q_2 is flat as a $\mathbf{Z}G\hat{}$-module if and only if it is 2-divisible; since Q is necessarily a flat module, being a limit of free modules, the result follows).

As a final remark, we observe that $A \times_\alpha G$ has unique trace (and therefore $K_0(A \times_\alpha G)$ has a unique state), if and only if for all j,

$$\prod_{i=j}^{\infty} \frac{|m(i) - n(i)|}{m(i) + n(i)} = 0 \quad [11].$$

Otherwise it has two pure traces (states). In any case the ordering is determined by the states on $K_0(A \times_\alpha G)$ by [3; 1.4]. ■

III.5 *Example* (Conclusion of II.4). Recalling notation from II.4, we have a limit of algebras, $A = \lim A^i$, together with compatible actions of $G = \mathbf{Z}_2$ obtained from $\alpha^{(i)}\colon G \to A^i$. We compute the relevant K_0-groups and modules.

First, we see that the algebra A is isomorphic to $\otimes M_5$ via the maps, $A^i \to M_{5^{i+1}}$, given by

$$(a, b) \mapsto \begin{bmatrix} a & & & & \\ & a & & & \\ & & b & & \\ & & & b & \\ & & & & b \end{bmatrix}.$$

This can also be deduced by computing $K_0(A)$ (keeping track of $[A]$)—$K_0(A)$

is the limit (as ordered abelian groups) of

$$(*) \qquad \mathbf{Z}^2 \xrightarrow{\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}} \mathbf{Z}^2 \xrightarrow{\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}} \mathbf{Z}^2 \longrightarrow \cdots;$$

because

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (2 \quad 3),$$

the limit is isomorphic to the limit of $\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z} \to \cdots$ where the maps between the $\mathbf{Z}$'s are multiplication by

$$(2 \quad 3) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 5.$$

On the crossed product level, the maps $K_0(A^i \times G) \to K_0(A^{i+1} \times G)$ after identification with $K_0(G)^2$, are given by $P = M^T$: $K_0(G)^2 \to K_0(G)^2$. Let Q denote the limit module. We compute the rank of Q as a $\mathbf{Z}$-module by utilizing the subdirect decomposition of $K_0(G)$ as a subdirect product (as rings) of two copies of $\mathbf{Z}$ (see the appendix). Modulo the augmentation ideal P_1, Q/P_1Q is simply the limit of the diagram in $(*)$ (just evaluate the characters at 1), so $Q/P_1Q \simeq \mathbf{Z}[1/5]$ as abelian groups.

On the other hand, modulo I_2 $(\hat{g} \mapsto -1)$, Q/P_2Q, the maps are

$$\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} : \mathbf{Z}^2 \to \mathbf{Z}^2;$$

this has determinant -2, so the limit has $\mathbf{Z}$-rank 2. As the $R = K_0G$-module rank is just the maximum of the two ranks, $\operatorname{rank}_R Q = 2$; the rank of Q as an abelian group is the sum of the two ranks, so it is 3.

A product type action would necessarily yield R-module rank 1 and $\mathbf{Z}$-module rank 1 or 2, so this action cannot be of product type.

In Section VI, we shall see how to build from this example to an action where $K_0(A \times G)$ has infinite rank.

More can be said about the module structure. Since Q/P_1Q is not 2-divisible, and Q is flat, the subdirect sum decomposition

$$Q \to Q/P_1Q \oplus Q/P_2Q$$

is not onto, but of index 2 (it is straightforward to verify that Q/P_1Q as a trivial G-module is not flat).

The ordering can also be described. Identify Q/P_1Q with $\mathbf{Z}[1/5] \subset \mathbf{R}$; then the map $Q \to Q/P_1Q$ is a state. Since the action of $G = \mathbf{Z}_2$ on A is outer, $A \times_\alpha G$ is simple. Thus $Q = K_0(A \times_\alpha G)$ is a simple dimension group. Writing $R = K_0(G) = \mathbf{Z}^2$ as an ordered abelian group, we have Q order isomorphic to the limit

$$\mathbf{Z}^4 \xrightarrow{X} \mathbf{Z}^4 \xrightarrow{X} \mathbf{Z}^4 \xrightarrow{X} \cdots$$

for some positive 4×4 matrix X (easily computed from M). Since the limit is simple, the matrix X must be aperiodic irreducible, and thus Q admits ONLY one state. So $Q \to \mathbf{Z}[1/5]$ arising from the quotient map is the only state, and thus by [3; 1.4],

$$Q^{+} = \{0\} \cup \{m \in Q \mid q + P_1 Q \geq 0 \quad \text{in } \mathbf{Z}[1/5]\}. \qquad \blacksquare$$

Such non-product type actions can be constructed for any compact Lie group G in a very simple fashion, as follows. Find a projectively faithful character χ of $G(|\chi(g)| = \chi(1)$ implies $g = 1)$ (start with a faithful representation, and add the trivial one). Say $\chi(1) = d$. Select any integer $n > d$, and consider the matrix with entries in $R = K_0(G)$,

$$M = \begin{bmatrix} \chi & (n-d)\chi_0 \\ d\chi_0 & (n-d)\chi_0 \end{bmatrix}$$

(where χ_0 is the trivial character). Under the map $K_0(G) \to \mathbf{Z}$ given by $\chi \mapsto \chi(1)$, $\overline{M}$: $\mathbf{Z}^2 \to \mathbf{Z}^2$ given by

$$\begin{bmatrix} d & n-d \\ d & n-d \end{bmatrix}$$

results, The limiting abelian group is $\mathbf{Z}[1/n]$, so we define an action of G on $A = EM_nC$ by regarding A as the limit of the corresponding finite dimensional algebras obtained by iterating $\overline{M}$. The action is obtained as in Example II.4. To see that it is not of product type, we compute the determinant of M as an element of $K_0(G) = R$: It is

$$(n-d)(\chi - d(n-d)\chi_0) \neq 0.$$

Hence regardless of whether G is infinite, the rank of the limit module is 2 (over R), so the corresponding action is not of product type.

Much more complicated actions can be given, with specified ranks, over some UHF algebra—for rank k, set

$$M = \begin{pmatrix} \chi & d & d & \cdots & d \\ d & \chi & d & \cdots & d \\ d & d & \chi & \cdots & d \\ & & & & \vdots \\ d & d & d & \cdots & \chi \end{pmatrix}$$

(the cumbersome $d\chi_0$ has been replaced by d). The determinant of M is

$$(\chi - d\chi_0)^{k-1}(\chi - d(1-k)\chi_0).$$

If k exceeds 2, this vanishes only at 1 in G (as $|\chi(g)| \leq \chi(1)$). The R-module rank of the resulting module is exactly k. The algebra on which G acts is that arising from the $k \times k$ matrix with solid d's, namely $\otimes M_{d^k}C$, a UHF algebra.

IV. Automatic local representability

While the results of the previous sections give a very general method for constructing locally representable actions, it would be pleasant to give some criteria for local representability, so that the invariants obtained there could be applied ($K_0(A \times_\alpha G)$ is always an invariant for the action—but with local representability, it becomes part of a complete invariant). Given a compact group G, an AF algebra $\bar{A}$, and an action α: $G \to \mathrm{Aut}(\bar{A})$, for α to be locally representable, the following (redundant) three properties must hold:

(i) There is a dense locally finite dimensional *-subalgebra that is globally invariant under $\alpha(G)$;

(ii) The action α is locally inner (that is, there exists a nest of invariant finite dimensional *-subalgebras with dense union, $A = \bigcup A_i$, such that for all g in G, $\alpha(g)$, restricted to each A_i, is inner);

(iii) The action α is locally representable (that is, there is a nest of finite dimensional *-subalgebras, each invariant, with dense union, such that α restricted to each can be implemented by a representation of G).

Of course (i) and (ii) may be suppressed, but their presence helps clarify matters. It is a well known open question whether (i) holds for any group, and say, any UHF algebra. One consequence of (ii) is that for all g in G, $\alpha(g)$ is *approximately inner*, meaning that there exists a sequence of inner automorphisms $\mathrm{Ad}\, u_j$ (u_j unitaries in $\bar{A}$) such that $\{(\mathrm{Ad}\, u_j)(a)\}$ converges in norm to $\alpha(g)(a)$ for all a in $\bar{A}$ (or A). It is, however, easy to test for approximate innerness in terms of K_0—$\alpha(g)$ is approximately inner if and only if $K_0(\alpha(g))$: $K_0(\bar{A}) \to K_0(\bar{A})$ is the identity, or what amounts to the same thing, for all projections p in $\bar{A}$, $\alpha(g)(p)$ is equivalent to p, via a unitary (e.g., [1], [4]).

If $\bar{A}$ is UHF, more generally if $K_0(\bar{A})$ is totally ordered and $\bar{A}$ is simple, then all automorphisms are approximately inner.

We shall show that if $G = \langle g \rangle$ is a prime order group of automorphisms of the AF algebra $\bar{A}$, and α: $G \to \mathrm{Aut}(\bar{A})$ satisfies (i) and is approximately inner, then α is locally representable, so that our classification results apply.

It almost follows directly from the definitions, that if G is simply connected, then (i) implies (iii) already, and if G is merely connected, then (i) implies (ii).

IV.1 THEOREM. *Let $\bar{A}$ be an* AF *algebra, with dense locally finite dimensional *-subalgebra A. Suppose t is a (*-) automorphism of A, of prime order p, that is additionally approximately inner. Then t is locally inner, and so $G = \langle t \rangle$ is locally representable.*

Proof. We may find a cofinal nest $A^1 \subset A^2 \subset A^3 \subset \cdots$ of unital finite dimensional *-subalgebras with union A, such that $t(A^i) \subset A^i$. We may additionally assume (by taking a suitable telescoping) that if e, f are projections in A^i that are equivalent in A, then e, f are unitarily equivalent in A^{i+1}. We shall construct an interstitial nest of *-subalgebras B^i with $A^1 \subset B^1 \subset A^2 \subset B^2 \subset \cdots$ where $t(B^i) \subset B^i$, and t/B^i is inner.

Start at some A^i; then the orbits of the minimal central idempotents of A^i under t are either singletons or cycles of order p. If all are singletons, then t/A^i is already inner; assume at least one p-cycle arises. Writing $A^i = \prod F_j^i A^i$ where F_j^i run over all the minimal central invariant idempotents (so F_j^i is either a minimal central idempotent, or a sum of p such), we may in what follows work cycle by cycle, and for notational convenience, we may assume exactly one orbit occurs.

Then $A^i = M_r \times M_r \times \cdots \times M_r$ (p times), and if E_j^i denotes

$$(0,0,\dots,0,1,0,\dots,0) \quad (j\text{-th position}),$$

we may assume $t(E_j^i) = E_{j+1}^i$, indexing mod p. Let e_j denote the image of E_j^i in A^{i+1}. Since $\Sigma e_j = 1$, and the e_j are mutually equivalent in A (as t is approximately inner), we may find a unitary u in A^{i+1} such that $ue_ju^* = e_{j+1}$.

Now consider the map $A^i \to A^{i+1}$. Since the minimal central idempotents of A^i have become mutually equivalent in A^{i+1}, the mapping must be given by a selection of integers (multiplicities), $n(1), n(2), \dots$; and $s(1), s(2), \dots$; where $(a_1, a_2, \dots, a_p)$ in $A^i = M_r \times \cdots \times M_r$ is sent to the element of A^{i+1} (itself a product of matrix rings) with

$$\operatorname{diag}\left(a_1 \otimes I_{n(1)}, a_2 \otimes I_{n(1)}, \dots, a_p \otimes I_{n(1)}\right)$$

in the first $s(1)$ components, the same with $n(1)$ replaced by $n(2)$ in the next $s(2)$ components etc. Here we may assume that each $s(j)$ represents an orbit of a minimal central idempotent in A^{i+1} (so $s(j) = 1$ or p, and the $n(\ )$ are allowed to include repetitions). We wish to find a unitary v in A^{i+1} so that

$$(*) \qquad \operatorname{Ad} v/A^i = t/A^i \quad \text{and} \quad t(v) = v.$$

Then we set B^i to be the *-algebra generated by A^i and v.

In any case, on any orbit, the action of t on A^{i+1} is given by the cyclic permutation followed by an inner automorphism, say $\operatorname{Ad}(w_1, \dots, w_p)$. Thus on a non-trivial orbit,

$$(x_1, x_2, \dots, x_p) \xrightarrow{t} \operatorname{Ad}(w_1, w_2, \dots, w_p)(x_p, x_1, \dots, x_{p-1})$$
$$= \left(w_1 x_p w_1^*, w_2 x_1 w_2^*, \dots, w_p x_{p-1} w_p^*\right).$$

Since t is of order p, the product $w_p w_{p-1} \cdots w_1$ is a scalar; hence so is any cyclic rearrangement of the product of the w's. The action of t is given by the following diagram,

$$
\begin{array}{ccc}
(a_1, a_2, \dots, a_p) \in A^i & \xrightarrow{\quad t \quad} & (a_p, a_1, \dots, a_{p-1}) \\
\downarrow & & \downarrow \\
\left(\dots, \underbrace{\begin{bmatrix} a_1 \otimes I_{n(k)} & & & \\ & a_2 \otimes I_{n(k)} & & \\ & & \dots & \\ & & & a_p \otimes I_{n(k)} \end{bmatrix}}_{s(k) \text{ times}}, \dots \right) \in A^{i+1} & \xrightarrow{t} & \left(\dots, \underbrace{\begin{bmatrix} a_p \otimes I_{n(k)} & & & \\ & a_1 \otimes I_{n(k)} & & \\ & & \dots & \\ & & & a_{p-1} \otimes I_{n(k)} \end{bmatrix}}_{s(k) \text{ times}}, \dots \right).
\end{array}
$$

Fix k (that is, the k-th orbit) and assume $s(k)$ is p and not 1. Now $\mathrm{Ad}(w_1, \dots, w_p)$ implements t (since the image of A^i is invariant under the permutation!), and the following yields information about the w's.

IV.2 LEMMA. *Let W be a unitary matrix of size mnp. Suppose that for all p-tuples of $m \times m$ matrices $(a_1, \dots, a_p)$ we have*

$$(\mathrm{Ad}\, W)\big(\mathrm{diag}(a_1 \otimes I_n, \dots, a_p \otimes I_n)\big) = \mathrm{diag}\big(a_p \otimes I_n, a_1 \otimes I_n, \dots, a_{p-1} \otimes I_n\big).$$

Then W is of the form

$$(**) \qquad \begin{bmatrix} 0_{mn} & I_m \otimes X_1 & & & \\ & 0_{mn} & I_m \otimes X_2 & & \\ & & \vdots & & \\ & & & \cdots & \\ I_m \otimes X_p & & & & 0_{mn} \end{bmatrix}$$

where the X_i are arbitrary unitaries of size n.

Proof. Set V to be the matrix $(**)$ with all the X_i's replaced by I_n. Then $\mathrm{Ad}\, V$ and $\mathrm{Ad}\, W$ agree on the subalgebra $M_m \times \cdots \times M_m$ (p times) embedded (via— $\otimes I_n$) in M_{mnp}. Thus VW^{-1} centralizes all of these matrices. Now we show that the centralizer of

$$\{\mathrm{diag}(a_1, a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_p, \dots, a_p) | a_i \in M_n, \quad n\text{-fold repetitions}\}$$

is simply

$$\left\{ \left. \begin{bmatrix} I_m \otimes x_1 & & & \\ & I_m \otimes x_2 & & \\ & & \cdots & \\ & & & I_m \otimes x_p \end{bmatrix} \right| x_i \quad \text{in } M_n\mathbf{C} \right\}.$$

Write $\mathbf{C}^{mnp} = U$ as a left module over $A = M_m \times \cdots \times M_m = A_1 \times \cdots \times A_p$ embedded unitally in $M_{mnp}\mathbf{C}$, which acts naturally on U. Then $A^U = \oplus_{A_i}(U_i)^n$, where U_i is the standard irreducible of the algebra A_i. The centralizer of A in $\mathrm{End}\, U = M_{mnp}\mathbf{C}$ is

$$\mathrm{End}\big(\oplus_{A_i}(U_i)^n\big) = \oplus \mathrm{End}_{A_i}(U_i)^n \simeq \oplus^p M_n\mathbf{C}.$$

As VW^{-1} is unitary, the result follows. ■

Returning to our proof of IV.1, we have that each w_j has the form

$$(***) \qquad w_j = \begin{bmatrix} 0 & I_r \otimes X_{1j} & & & \\ & 0 & I_r \otimes X_{2j} & & \\ & & 0 & & \\ & & & \ddots & \\ I_r \otimes X_{pj} & & & & 0 \end{bmatrix}_{r \cdot n(k) \cdot p}$$

with unitary X's in $M_{n(k)}\mathbf{C}$.

To construct our $v = (v_1, \dots, v_p)$ to satisfy $(*)$ on this orbit, in particular so that $t(v) = v$, we compute

$$\begin{aligned} t(v) &= \mathrm{Ad}(w_1, \dots, w_p)(v_p, v_1, \dots, v_{p-1}) \\ &= \left(w_1 v_p w_1^*, w_2 v_1 w_2^*, \dots, w_p v_{p-1} w_p^*\right); \end{aligned}$$

so we require $v_1 = w_1 v_p w_1^*, v_2 = w_2 v_1 w_2^*, \dots, v_p = w_p v_{p-1} w_p^*$. Set v_p to be the matrix V in the proof of the lemma with $m = r$, and define $v_1, \dots, v_{p-1}$ via the first $p - 1$ equations. Then

$$v_{p-1} = w_{p-1} \cdots w_1 v_p w_1^* \cdots w_{p-1}^*,$$

so $w_p v_{p-1} w_p^* = v_p$ (as $w_p \cdots w_1$ is a scalar). So we have constructed a candidate v which at least satisfies $t(v) = v$. Now $\mathrm{Ad}\, v$ restricted to the image of A^i agrees with t on this orbit; putting all the orbits in A^{i+1} together (stringing the corresponding v's) into a unitary in A^{i+1}, we obtain V_0 in $U(A^{i+1})$ such that $t(V_0) = V_0$, and $\mathrm{Ad}\, V_0/A^i$ agrees with t/A^i. Define B^i to be the subalgebra of A^{i+1} generated by A^i and V_0. It is clearly an invariant *-subalgebra, and t/B^i is given by $\mathrm{Ad}\, V_0$. Thus t is inner on each B^i, and since the group is generated by t, the action is locally representable. ■

We have the following very easy observations:

IV.3 PROPOSITION. *Let α: $G \to \mathrm{Aut}(A)$ be an action of the compact group G in the locally finite dimensional *-algebra A.*

(a) *If G is connected, then α is locally inner.*

(b) *If G is simply connected, then α is locally representable.*

(*Thus for G connected,* (i) *implies* (ii); *for simply connected G,* (i) *implies* (iii).)

Proof. (a) If A^i is a finite dimensional subalgebra left invariant by $\alpha(G)$, then $\alpha(G)$ must leave invariant each of the simple components (else the kernel of the permutation representation would be a proper closed subgroup of finite index, violating connectedness). Thus each $\alpha(g)/A^i$ is inner.

(b) Any projective finite dimensional representation $G \to PGL(n, \mathbf{C})$ of a simply connected compact group lifts to an ordinary representation, $G \to GL(n, \mathbf{C})$. ■

For other compact groups, to what extent is it true that properties (i) and (ii) imply (iii), say with A simple and infinite dimensional? It certainly holds if G is a d-torus. If (or when) the following conjecture holds, then (i) and (ii) do imply local representability:

CONJECTURE. If $\alpha: G \to \operatorname{Aut}(A)$ is an outer locally inner (i.e., satisfies (ii)) action of a compact group G on a simple infinite dimensional locally finite dimensional *-algebra over $\mathbf{C}$, then the ordered $K_0(G)$ module $K_0(A \times_\alpha G)$ is a direct limit of free $K_0(G)$ modules with positive maps.

The assertion in the conjecture is analogous to the characterization of dimension groups given in [3; 2.2]. This says that if a directed partially ordered abelian group satisfies Riesz decomposition (op. cit.), and $nx \geq 0$ for some positive integer n implies $x \geq 0$, then it is a limit of free abelian groups (with the componentwise ordering) with positive maps. The analogous question for ordered $K_0(G)$ modules would be to decide which they arise as limits arising in our situation, that is, limits of free $K_0(G)$-modules (with the componentwise ordering) and positive module maps between them. There is an additional property that must be added to the properties given above for dimension groups—the module must be *flat* [14; Prop. 3, p. 133] (flat modules can be characterized precisely as direct limits of free modules).

For example, if M is a faithful $\mathbf{Z}[\mathbf{Z}_2]$ submodule of $\mathbf{Q}[\mathbf{Z}_2]$ ($\simeq \mathbf{Q} \times \mathbf{Q}$ as rings), then M is flat if and only if either (a) M is 2-divisible, or (b) M is *not* a direct sum of two non-trivial submodules (which means M does not decompose into its two homomorphic images obtained from the two homomorphisms $\mathbf{Z}[\mathbf{Z}_2] \to \mathbf{Z}$. Set $M = \mathbf{Z}[1/3] \oplus \mathbf{Z}[1/3]$ where $G = \mathbf{Z}_2$ acts on the first component trivially, And on the second by multiplication by -1 at the non-trivial element of G. This module is not flat, but it has an ordered module structure on it which is a dimension group ordering (e.g., first component strictly positive ordering). By IV.1, this cannot arise as $K_0(A \times_\alpha G)$ for any action satisfying (i) of $\mathbf{Z}_2 = G$ on any AF algebra $\bar{A}$ which has all of its automorphisms approximately inner. Hence any action on such an algebra which yields M as $K_0(A \times_\alpha G)$ could not leave a dense locally finite dimensional *-subalgebra invariant!

There still is the question of whether an ordered $K_0(G)$-module that is flat and is a dimension group is a limit (and so arises from a locally representable action).

Here is a partial result in this direction:

IV.4 PROPOSITION. *Let G be a finite abelian group of order n. Equip $R = \mathbf{Z}G$ with the ordering, $R^+ = \{\Sigma a_g g \mid a_g \geq 0\}$. Suppose M is a countable,* *ordered R-module such that*

(a) *M is a dimension group,*

(b) *M is n-divisible as an abelian group.*

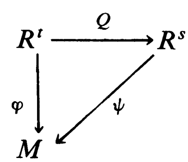
Then M arises as an order direct limit of a system

$$R^{k(1)} \to R^{k(2)} \to \cdots$$

In particular, $M = K_0(A \times_\alpha G)$ for some locally representable action of G on the AF *algebra $\bar{A}$, where $K_0(A) = M/IM$ (I being the augmentation ideal of R).*

Remark. The group G and its dual $G\hat{}$ have been identified for ease of notation.

Proof. We show that if an ordered module homomorphism φ: $R^t \to M$ is given, then there exists a positive integer s, and ordered module homomorphisms Q: $R^t \to R^s$, ψ: $R^s \to M$ such that $\ker\varphi \subset \ker Q$, and the diagram



commutes.

Let $\{e_1, \dots, e_t\}$ denote the standard basis for R^t, and set $m_i = \varphi(e_i)/n$ in M (recall that M is n-divisible and torsion-free as an abelian group, so it is uniquely n-divisible). Define φ_1: $R^t \to M$ via $e_i \mapsto m_i$ (so $n\varphi_1 = \varphi$).

Since M is a dimension group and R^t is simplicial, there exists an integer s and order-preserving group homomorphisms q: $R^t \to \mathbf{Z}^s$, σ: $\mathbf{Z}^s \to M$ such that $\sigma q = \varphi_1$, and $\ker q \supset \ker\varphi_1 = \ker\varphi$. We extend σ to a map ψ: $R^s \to M$ in the obvious way (if $\{f_j\}$ is the standard basis for $\mathbf{Z}^s$, write $R^s = \oplus f_j R$, and define $\psi(f_j g) = \sigma(f_j)g$, etc.). Then ψ is a positive R-module homomorphism, (as $\sigma(f_j) \geq 0$). The inclusion $\mathbf{Z}^s \to R^s$, $f_j \mapsto f_j$ induces a positive *group* homomorphism Φ: $R^t \to R^s$ (not a module homomorphism), with $\psi\Phi = \varphi_1$. Moreover $\ker\Phi \supset \ker q \supset \ker\varphi_1$. Now we apply the standard averaging argument to obtain a positive R-module homomorphism Q: $R^t \to R^s$.

Set $Q(x) = \Sigma_G \Phi(xg^{-1})g$. Since Φ is positive, so is Q. Moreover, for h in G,

$$\begin{aligned} Q(xh) &= \sum_G \Phi(xhg^{-1})g = \sum_{u=g^{-1}h} \Phi(xu^{-1})uh \\ &= h\Big(\sum_G \Phi(xu^{-1})u\Big) = Q(x)h \end{aligned}$$

so Q is an R-module homomorphism. As $\varphi_1 = \psi\Phi$ is also an R-module homomorphism, we have $\psi Q = n\psi\Phi = \varphi$. Finally, $\varphi_1(y) = 0$ implies $\varphi(y) = 0$ which yields that $\varphi(yg) = 0$ for all g in G, and therefore $q(yg) = 0$ for all g in G, so $\Phi(yg) = 0$ for all g in G, and thus $Q(y) = 0$; hence $\ker\varphi \subset \ker Q$.

Since M is countable and generated by M^+, the standard direct limit argument (this time as ordered R-modules) yields that M is a limit in the desired form. ∎

V. Traces on the crossed product

Here we wish to establish some results on the trace space of the crossed product $A \times_\alpha G$, at least for finite groups G which leave invariant an extremal trace of A. By completing at an extremal trace which is invariant to obtain the corresponding finite factor R, the problem is reduced to studying the traces on $R \times_\alpha G$) (because the traces on $R \times_\alpha G$ are in natural bijection with those of $A \times_\alpha G$ which agree on A with the original invariant trace).

Hence we are in the situation of a finite type I or II factor, with some action of the finite group G on R. We wish to write down all traces, and show that there is a single trace that generates the others in a fashion to be described below.

For S a C^* algebra, let $T_{\mathbf{R}}(S), T_{\mathbf{C}}(S)$ denote the real and complex (respectively) vector subspaces of the dual space of S generated by the trace space $T(S)$ of S. We may form $K_{\mathbf{R}}(G) = K_0(G) \otimes \mathbf{R}$ (characters with real coefficients), and $K_{\mathbf{C}}(G) = K_0(G) \otimes \mathbf{C} = L^2_c(G)$. If $S = B \times G$ is a crossed product, then $T_{\mathbf{R}}(S), T_{\mathbf{C}}(S)$ become respectively $K_{\mathbf{R}}(G), K_{\mathbf{C}}(G)$-modules as follows: with τ in $T_\cdot(S)$, and χ in $K_\cdot(G)$ ($\cdot$ denotes $\mathbf{R}$ or $\mathbf{C}$), define $\tau \cdot \chi$ by

$$\tau \cdot \chi\Big(\sum_G b_g g\Big) = \sum_G \tau(b_g g)\chi(g).$$

If $\tau \geq 0$ (i.e., $\tau/\tau(1)$ is a trace), and $\chi \in K_{\mathbf{R}}(G)+$, then $\tau \cdot \chi \geq 0$ as well.

An element of $T_{\mathbf{C}}(B)$ ($T_{\mathbf{R}}(B)$) will be called a (real) trace-like functional of B. If f belongs to the dual space of B, then it lies in $T_{\mathbf{C}}(B)$ if and only if $f(xy) = f(yx)$ for all x, y in B; it is a real trace-like functional provided it additionally satisfies $f(x^*) = \overline{f(x)}$.

Related results to those below are obtained in work of Rieffel [19].

V.1 LEMMA. *Let ρ be a trace-like functional on $R \times_\alpha G$, and χ an element of $K_\cdot(G)$. Then the functional $\rho\chi$ defined by*

$$\rho\chi\Big(\sum_G r_g g\Big) = \sum_G \rho(r_g g)\chi(g)$$

is trace-like. If ρ and χ are real (positive), so is $\rho\chi$.

Proof. Set $x = \Sigma r_g g$, $y = \Sigma s_g g$; then

$$\begin{aligned}\rho\chi(xy) &= \rho\chi\Big(\sum_{g,h} r_g s_h^{\alpha(g)} gh\Big)\\ &= \sum \rho\big(r_g s_h^{\alpha(g)} gh\big)\chi(gh)\\ &= \sum \rho\big(r_g g s_h h\big)\chi(hg)\\ &= \sum \rho\big(s_h h r_g g\big)\chi(hg)\\ &= \sum \rho\big(s_h r_g^{\alpha(h)} hg\big)\chi(hg)\\ &= \sum \rho\chi\big(s_h r_g^{\alpha(h)} hg\big)\\ &= \rho\chi(yx).\end{aligned}$$

Reality of the product (if both are real) is clear, and positivity follows from the usual Hadamard product trick, as in [18; proof of 7.1.10]. ∎

We will show that if G is abelian, then $T_{\mathbf{R}}(R \times_\alpha G)$ is a *cyclic* $K_{\mathbf{R}}(G)$ module, as is $T_{\mathbf{C}}(R \times_\alpha G)$ without the abelian hypothesis. This means that there is a trace-like functional τ_G on $R \times_\alpha G$, such that for any other trace-like functional τ, there is χ in $K_{\mathbf{C}}(G)$ with $\tau = \tau_G \chi$ (so any trace can be obtained from τ_G by multiplying by a suitable complexified character of G). The same result holds of course, if R is replaced by any C^* algebra with unique trace. The generating trace is very far from the trivial trace, which sends $\Sigma b_g g$ to $\tau_0(b_1)$, where τ_0 is the trace on R—in fact the vector space dimension of the $K_{.}(G)$-module generated by this trace is 1.

Define a subset N of G as follows:

$$N = \big\{ g \in G \,|\, \text{there exists } u \text{ in } U(R) \text{ with } \alpha(g) = \operatorname{Ad} u, \\ \text{and } ghg^{-1}h^{-1} = 1 \text{ implies } u^{\alpha(h)} = u \big\}.$$

(Observe that if g belongs to N, and is represented by u, then any other unitary which implements g will have the invariance property specified in the definition of N, as well.) Then N is closed under conjugation (but is not a subgroup in general), and if G is abelian,

$$N = \big\{ g \in G \,|\, \text{there exists } u \text{ in } U(R^G) \text{ with } \alpha(g) = \operatorname{Ad} u \big\}$$

(so in this case N is a subgroup). Since R is simple, it is easy to verify by direct calculation that each G-conjugacy class in N contributes exactly one dimension to the centre of $R \times_\alpha G$, and the central elements so obtained span the centre. If G is abelian $\{u_g^{-1} g \,|\, g \in N,\ \alpha(g) = \operatorname{Ad} u_g\}$ is a basis for the centre, $Z(R \times_\alpha G)$. These calculations are done in [9] and [10; 1.6].

The subset N is related to but distinct from Jones' $N(\psi)$ [12], which is $\{g \in G | \alpha(g) \text{ is inner}\}$ (so $N \subset N(\psi)$). The construction of the traces given below is not explicitly in [12], but can be deduced from I.3.1 and II.2.2 of [12].

Because G is finite, $R \times_\alpha G$ is a finite product of factors, so

$$Z(R \times_\alpha G) = \dim_{\mathbf{R}}(T_{\mathbf{R}}(G)) = \dim_{\mathbf{C}}(T_{\mathbf{C}}(G));$$

there is a bijection between the pure traces of $R \times_\alpha G$, and the minimal central idempotents of $R \times_\alpha G$.

Assume from now until further notice that G is abelian (what is necessary is that N be a subgroup of the centre of G). We construct a group homomorphism θ: $N \to U(R)$ (the codomain could be taken to be $U(R^G)$), so that the following diagram commutes:

$$\begin{array}{ccc} N & \xrightarrow{\ \theta\ } & U(R) \\ \downarrow & & \downarrow \pi \\ G & \xrightarrow{\ \alpha\ } & \mathrm{Aut}(R). \end{array}$$

Here $\pi(u) = \mathrm{Ad}\, u$; thus $\mathrm{Ad}\, \theta(g) = \alpha(g)$ for all g in N.

Write $G = \oplus \langle g_i \rangle$, a direct sum of cyclic groups, each with generator g_i. We may select u_i in $U(R)$ such that $\alpha(g_i) = \mathrm{Ad}\, u_i$; as g_i belongs to N and R is simple, u_i lies in $U(R^G)$; we may additionally assume $u_i^{\mathrm{order}(g_i)} = 1$, by multiplying by a suitable scalar. The assignment $g_i \mapsto u_i$ will extend to a group homomorphism $N \to U(R)$ provided $u_i u_j = u_j u_i$ for all i, j. But this follows from $u_i = u_i^{\alpha(g_j)} = u_j u_i u_j^*$.

Having thus obtained θ, we have an isomorphism

$$R \times_\alpha N \to RN \simeq R \otimes \mathbf{C}N; \qquad r \mapsto r; \qquad g \mapsto \theta(g) g.$$

Identifying RN with $R \otimes CN$, we may explicitly write down all traces on RN, and transfer them back to $R \times_\alpha N$, via the isomorphism. Then a sleight of hand will show that these traces extend uniquely and exhaustively to traces on $R \times_\alpha G$, via the completely positive map (e.g., see [22]) $R \times_\alpha G \to R \times_\alpha N$.

Let τ_0 be the unique trace on R. Then all traces on $R \otimes \mathbf{C}N$ are of the form

$$\sum_N s_g \otimes g \overset{\tau_\chi}{\mapsto} \sum_N \tau_0(s_g)\chi(g)$$

for χ in $K_{\mathbf{R}}(N)+$ satisfying $\chi(1) = 1$; similar comments apply for (real) trace-like functionals. In particular $\mathrm{Dim}\,.T.(RN) = |N|$, and the state

$$\rho_N\colon \sum_N s_g \otimes g \mapsto \sum_N \tau_0(s_g)$$

is a generator for $T_{\cdot}(RN)$, in the sense that every other element of $T_{\cdot}(RN)$ is of the form $\rho_N\chi$ for a unique χ in $K_{\cdot}(N)$. Via the isomorphism $R \times_\alpha N \simeq RN$ described above, all traces on $R \times_\alpha N$ are of the form

$$\rho_\chi\Big(\sum_N r_g g\Big) = \tau_\chi\Big(\sum_N r_g\theta(g)g\Big) = \sum_N \tau_0\big(r_g\theta(g)\big)\chi(g)$$

for a unique χ in $K_{\mathbf{R}}(N)^+$ with $\chi(1) = 1$, and all ρ's so defined are traces; parallel remarks apply to trace-like functionals.

Now we consider trace-like functionals on $R \times_\alpha G$. Let

$$Q\colon R \times_\alpha G \to R \times_\alpha N$$

be the completely positive map which restricts elements of $R \times_\alpha G$ to N; that is, $Q(\Sigma_G r_g g) = \Sigma_N r_g g$. Given ρ_χ in $T(R \times_\alpha N)$ (or $T_{\cdot}(R \times_\alpha N)$), we obtain P_χ in the dual space of $R \times_\alpha G$ via $P_\chi = \rho_\chi \circ Q$; in other words,

$$P_\chi\Big(\sum_G r_g g\Big) = \sum_N \rho_\chi(r_g g) = \sum_N \tau_0\big(r_g\theta(g)\big)\chi(g).$$

We shall prove that P_χ is trace-like, although we were not able to find a direct computational proof of this. We shall also show that all trace-like functionals arise in this fashion.

V.2 Proposition. *Let α: $G \to \operatorname{Aut}(R)$ be an action of a finite abelian group on a II_f or I_n factor. Define $N = \alpha^{-1}(\pi U(R^G))$, where π: $U(R) \to \operatorname{Aut}(R)$ is the map $u \mapsto \operatorname{Ad} u$. With θ: $N \to U(R^G)$ as defined above, we have:*

(i) *Every (real) trace-like functional on $R \times_\alpha G$ is of the form $P_\chi = \rho_\chi \circ Q$, where Q: $R \times_\alpha G \to R \times_\alpha N$ is the restriction map, χ belongs to $K_{\cdot}(G)$, and*

$$\rho_\chi\Big(\sum_N r_g g\Big) = \sum_N \tau_0\big(r_g\theta(g)\big)\chi(g).$$

(ii) *Every ρ_χ so defined is a (real) trace-like functional on $R \times_\alpha N$, and all elements of $T_{\cdot}(R \times_\alpha N)$ admit such a form for a unique χ in $K_{\cdot}(N)$; ρ_χ is a trace if and only if χ belongs to $T_{\mathbf{R}}(N)^+$ and $\chi(1) = 1$.*

(iii) *All functionals of the form P_χ on $R \times_\alpha G$ are trace-like; they are traces if and only if $\chi(1) = 1$ and $\chi \in T_{\mathbf{R}}(N)^+$.*

Proof. The isomorphism $R \times_\alpha N \to RN$ yields (ii) as described earlier. We prove (i) and (iii) by a trick.

The centres of $R \times_\alpha N$ and $R \times_\alpha G$ coincide—they are spanned by the basis $\{\theta(g)^{-1}g \mid g \in N\}$ as was indicated above. Since both $R \times_\alpha N$ and $R \times_\alpha G$

must be finite products of simple algebras, and each simple component is a factor so has unique trace, it follows that for the restriction maps,

$$T(R \times_\alpha G) \to T(R \times_\alpha N), \quad T_{\cdot}(R \times_\alpha G) \to T_{\cdot}(R \times_\alpha N)$$

the first is an affine homeomorphism and the second and third (. indicates either $\mathbf{R}$ or $\mathbf{C}$) are vector space isomorphisms (the pure traces are determined by $\tau_j(e_i) = \delta_{ij}$ where $\{e_i\}$ is a complete list of minimal central idempotents, which are the same for the two algebras).

Let P be an element of $T_{\cdot}(R \times_\alpha G)$. Define ψ in $K_{\mathbf{C}}(G) = L_c^2(G)$ to be the characteristic function of N. By Lemma V.1, $P\psi$ belongs to $T_{\mathbf{C}}(R \times_\alpha G)$. However, $P/R \times_\alpha N = (P\psi)/R \times_\alpha N$; as the restriction map is one to one, $P = P\psi$. By (ii), $P/R \times_\alpha N = \rho_\chi$ for a unique χ in $K_{\cdot}(N)$. Then

$$P\Big(\sum_G r_g g\Big) = P\psi\Big(\sum_G r_g g\Big) = \sum_G P(r_g g)\psi(g) = \sum_N P(r_g g) = \rho_\chi\Big(\sum_N r_g g\Big)$$
$$= \rho_\chi Q\Big(\sum_G r_g g\Big) = P_\chi\Big(\sum_G r_g g\Big).$$

Thus $P = P_\chi$, yielding (i).

Now we show that for any χ in $K_{\mathbf{C}}(N)$, P_χ is trace-like. Form ρ_χ in $T_{\mathbf{C}}(R \times_\alpha N)$. From the isomorphism $T_{\mathbf{C}}(R \times_\alpha G) \to T_{\mathbf{C}}(R \times_\alpha N)$ (obtained by restriction), there exists P in $T_{\mathbf{C}}(R \times_\alpha G)$ with $P/R \times_\alpha N = \rho_\chi$. By (i) and (ii), there exists a unique ψ in $K_{\mathbf{C}}(N)$ such that $P = \rho_\psi \circ Q$, so that $P/R \times_\alpha N = \rho_\psi$. As the restriction map is one to one, $\rho_\psi = \rho_\chi$, so that $\psi = \chi$, and thus $P = P_\chi$. Hence P_χ belongs to $T_{\mathbf{C}}(R \times_\alpha G)$!

To complete the proof of (iii), simply observe that Q is a completely positive map, so ρ_χ being a trace implies P_χ is positive, and the converse is trivial. Now (ii) applies. ■

We now drop our assumption that G be abelian.

V.3 Corollary. *Let α: $G \to \operatorname{Aut} R$ be an action of a finite group G on a II_f factor R. With $N = \{g \in G \mid$ there exists u in $U(R)$ such that $\alpha(g) = \operatorname{Ad} u$, and for all h in G commuting with g, $u^{\alpha(h)} = u\}$, and τ a trace on $R \times_\alpha G$, we have $\tau(rg) = 0$ for all r in R and g in $G \setminus N$.*

Proof. If g does not belong to N, either (i) $\alpha(g)$ is outer, or (ii) there exists u in R with $\alpha(g) = \operatorname{Ad} u$ and h in G commuting with g such that $u^{\alpha(h)} \neq u$.

If (i) holds, set $H = \langle g \rangle$. Then g does not belong to $N(H)$, and H is abelian, so the previous result applies to $R \times_\alpha H$. The restriction of τ to $R \times_\alpha H$ must be of the form $\rho_\chi \circ Q_{N(H)}$, and $Q_{N(H)}(rg) = 0$ so $\tau(rg) = 0$.

If (ii) holds, set $H = \langle g, h \rangle$. Then H is abelian, g does not belong to $N(H)$, and the same idea as in the previous paragraph finishes the proof. ■

We now wish to obtain the results analogous to V.2 for nonabelian G. Unfortunately, N is not a subgroup in general, but simply a union of G-conjugacy classes. Define a set function, θ: $N \to U(R)$ as follows. For each G-conjugacy class in N, $\mathscr{C}$, select a representative $g \in \mathscr{C}$. There exists u_g in $U(R)$ such that $\alpha(g) = \operatorname{Ad} u_g$. If h belongs to G, then we may attempt to define θ: $C \to U(R)$ via $\theta(hgh^{-1}) = u_g^{\alpha(h)}$; this is well defined as $hgh^{-1} = jgj^{-1}$ implies $j^{-1}h$ commutes with g, and thus $u_g^{\alpha(j^{-1}h)} = u_g$; applying $\alpha(h)^{-1}$ yields $u_g^{\alpha(h)} = u_g^{\alpha(j)}$. This gives us a function θ: $N \to U(R)$ with the property that for all g in N, h in G, $\operatorname{Ad}\theta(g) = \alpha(g)$ and $\theta(g^h) = \theta(g)^{\alpha(h)}$. We may also select θ to satisfy additionally, $\theta(g)^{\mathrm{order}(g)} = 1$.

V.4 LEMMA. *Let α: $G \to \operatorname{Aut}(R)$ be an action of G on R. If P is a trace-like functional on $R \times_\alpha G$, there exists χ in $K_{\mathbf{C}}(G)$ such that*

$$P\Big(\sum_G r_g g\Big) = \sum_N \tau_0\big(r_g\theta(g)\big)\chi(g),$$

where θ is defined just above.

Proof. For g not in N, $P(rg) = 0$ by the corollary. Define χ: $G \to \mathbf{C}$ via

$$\chi(g) = \begin{cases} P\big(\theta(g)^{-1}g\big) & \text{if } g \in N \\ 0 & \text{if } g \notin N. \end{cases}$$

As $(\theta(g)^{-1})^{\alpha(h)} = \theta(g^h)^{-1}$, χ is a class function, so χ belongs to $K_{\mathbf{C}}(G)$. It suffices to show $P(rg) = \tau_0(r_g\theta(g))\chi(g)$ for every g in N.

Fix g in N, set $H = \langle g\rangle$, and let E: $H \to \mathbf{C}$ be the characteristic function of g. Let p in $T_{\mathbf{C}}(R \times_\alpha H)$ be $P/R \times_\alpha H$, and $\psi = \chi/H$. Then pE belongs to $T_{\mathbf{C}}(R \times_\alpha H)$ by IV.1, and $N(H) = H$. We may define Θ: $H \to U(R)$ via $\Theta(g^k) = (\theta(g))^k$. This will play the role of theta in Proposition V.2. Hence there exists φ in $K_{\mathbf{C}}(H)$ with

$$p\psi\Big(\sum_H r_h h\Big) = \sum \tau_0\big(r_h\Theta(h)\big)\varphi(h).$$

But $p\psi(\Sigma r_h h) = p(r_g g) = P(r_g g)$; thus $\varphi(h) = 0$ if $h \neq g$, and

$$\tau_0\big(r_g\theta(g)\big)\varphi(g) = P(r_g g).$$

Setting $r_g = \theta(g)^{-1}$, we obtain $\varphi(g) = P(\theta(g)^{-1}g) = \chi(g)$.

Hence for all g in N, $P(rg) = \tau_0(r_g\theta(g))\chi(g)$. ∎

Define f_0 in the dual space of $R \times G$ by $f_0(\Sigma_G r_g g) = \Sigma_N \tau_0(r_g\theta(g))$. The lemma just established shows that every trace-like functional on $R \times_\alpha G$ is of the form $f_0\chi$ for suitable χ in $K_{\mathbf{C}}(G)$.

V.5 LEMMA. *Let $G, \alpha, R, \theta, f_0$ be as above. Let χ be an element of $K_{\mathbf{C}}(G)$. Then every functional of the form $f_0\chi$: $R \times_\alpha G \to \mathbf{C}$,*

$$f_0\chi\Big(\sum_G r_g g\Big) = \sum_N \tau_0\big(r_g\theta(g)\big)\chi(g),$$

is an element of $T_{\mathbf{C}}(R \times_\alpha G)$.

Proof. We have $f_0\chi = f_0\chi'$ if and only if $\chi/N = \chi'/N$; that is, $\chi - \chi'$ belongs to $N^\perp$. Hence the space $F = \{f_0\chi | \chi \in K_{\mathbf{C}}(G)\}$ (a subspace of the dual space of $R \times_\alpha G$), has dimension exactly the number of G-conjugacy classes in N. But this is also the dimension of the centre of $R \times_\alpha G$, hence $\dim_{\mathbf{C}} F = \dim T_{\mathbf{C}}(R \times_\alpha G)$. Thus the map $T_{\mathbf{C}}(R \times_\alpha G) \to F$ given by the previous lemma is a vector space homomorphism between equidimensional spaces. The map (an identification of $T_{\mathbf{C}}(R \times G)$ with a subspace of the dual space) is clearly one to one, so must be an isomorphism. Hence every $f_0\chi$ (including f_0 itself!) belongs to $T_{\mathbf{C}}(R \times_\alpha G)$. ■

The combination of the previous two lemmas yields

$$T_{\mathbf{C}}(R \times_\alpha G) = f_0 K_{\mathbf{C}}(G); \tag{1}$$

that is, $T_{\mathbf{C}}(R \times_\alpha G)$ is a cyclic $K_{\mathbf{C}}(G)$-module. The involution on $T_{\mathbf{C}}(R \times_\alpha G)$, $f \mapsto f^*$, where $f^*(g) = f(g^{-1})^-$, leaves fixed exactly $T_{\mathbf{R}}(R \times_\alpha G)$. It seems plausible that $\frac{1}{2}(f_0 + f_0^*) + \frac{1}{2}i(f_0 - f_0^*)$ should generate a $T_{\mathbf{R}}(R \times_\alpha G)$-module as a $K_{\mathbf{R}}(G)$-module, but we have not been able to find a proof. On the other hand, if G is abelian, our original choice for θ yields an f_0 with $f_0 = f_0^*$, so $T_{\mathbf{R}}(R \times_\alpha G)$ is a cyclic $K_{\mathbf{R}}(G)$-module with generator f_0. (More is true when G is abelian—f_0 is a trace; after passing to $R \times_\alpha N$, then to $R \otimes \mathbf{C}N$, f_0 corresponds to the trivial character on N, so arises out of a trace on $R \times_\alpha N$, and is thus itself a trace.)

When G is abelian, the number of pure traces on $R \times_\alpha G$ is the order of a subgroup, N, of G, so must divide the order of G. This was obtained by Kishimoto [13; Lemma 3.9] for UHF algebras, and can also be obtained by duality (since $\hat{G}$ must act transitively on the pure traces).

Recalling our reduction to a factor R, we may summarize some of the results of this section as follows:

V.6 THEOREM. *Let B be a unital C^* algebra, and α: $G \to \mathrm{Aut}(B)$ an action of a finite group on B. Let τ be a pure trace of B that is left invariant by $\alpha(G)$. Let $T_{\mathbf{C}}(B \times_\alpha G, \tau)$ denote the complex vector subspace of the dual space of $B \times_\alpha G$ generated by the traces of $B \times_\alpha G$ that restrict to τ on B. Then $T_{\mathbf{C}}(B \times_\alpha G, \tau)$ is a cyclic $K_{\mathbf{C}}(G)$-module. If G is abelian, the dimension of $T_{\mathbf{C}}(B \times_\alpha G, \tau)$ divides the order of G.*

VI. Tensor products of actions

Let G be a compact group, and suppose that α: $G \to \mathrm{Aut}(A)$, β: $G \to \mathrm{Aut}(B)$ are locally representable actions (so A and B are locally semisimple algebras). We may construct a new action, this time on $A \otimes B$, $\alpha \otimes \beta$: $G \to \mathrm{Aut}(A \otimes B)$ via

$$(\alpha \otimes \beta)(g)(a \otimes b) = (\alpha(g)(a)) \otimes (\beta(g)(b)).$$

If $A = \mathrm{Lim}\, A^i$, $B = \mathrm{Lim}\, B^j$ ($\{A^i\}_i, \{B^j\}_j$ families of finite dimensional algebras, indexed by the positive integers), then the (algebraic) tensor product $A \otimes B$ is given by $\mathrm{Lim}\, A^i \otimes B^i$, and it follows that $\alpha \otimes \beta$ is locally representable. Moreover, the crossed products on the finite dimensional levels, $(A^i \otimes B^i) \times G$ (the subscript indicating the action has been deleted for legibility) are themselves AF algebras, so that the (algebraic) limit will have the same K_0-theory as the crossed product on $A \otimes B$. In particular,

$$K_0((A \otimes B) \times G) \simeq \lim K_0((A^i \otimes B^i) \times G).$$

Now at the finite dimensional levels, we may assume that A^i, B^i are G-invariant, and moreover that the corresponding restrictions $\alpha(i), \beta(i)$ are representable. Thus $\alpha(i) \otimes \beta(i)$ is representable, so that

$$(A^i \otimes B^i) \times G \simeq C^*(G, A^i \otimes B^i)$$

(but *not* $C^*(G, A^i) \otimes C^*(G, B^i)$). If A^i has t simple components, and B^i has u, then $A^i \otimes B^i$ has tu, so that K_0 of $C^*(G, A^i \otimes B^i)$, as a $K_0(G)$-module is $K_0(G)^{tu}$. However, much more can be said, especially about the ordered module structure; to this end, we now consider tensor products of ordered modules.

Let R be a commutative partially ordered ring , and M, N ordered R-modules. We may form a new ordered module, $M \otimes_R N$ (e.g., [14; Section 5.2]), with positive cone

$$(M \otimes_R N)^+ = \{\Sigma m_i \otimes n_i | m_i \in M^+, n_i \in N^+\}.$$

It is routinely verified that $M \otimes_R N$ becomes an ordered module, and if $M = R^n$ (with the coordinatewise ordering), then $M \otimes_R N$ is order-isomorphic to N^n (with the coordinatewise ordering). In particular, $R^t \otimes_R R^u$ is order-isomorphic to R^{tu}, and the obvious map is the order-isomorphism. Furthermore, if

$$M = \lim P_i\colon R^{t(i)} \to R^{t(i+1)} \quad \text{and} \quad N = \lim Q_i\colon R^{u(i)} \to R^{u(i+1)},$$

then

$$M \otimes_R N = \lim P_i \otimes Q_i \colon R^{t(i)} \otimes_R R^{u(i)} \to R^{t(i+1)} \otimes_R R^{u(i+1)}.$$

Now for any two AF algebras, or dense locally finite dimensional *-subalgebras thereof, $C, D, K_0(C \otimes D)$ is order-isomorphic to $K_0(C) \otimes_{\mathbf{Z}} K_0(D)$ (viewed as ordered **Z**-modules, that is, ordered abelian groups). In the cases under consideration, A^i, B^i finite dimensional, it is readily verified that $K_0(C^*(G, A^i \otimes B^i))$ is order-isomorphic to

$$K_0\big(C^*(G, A^i)\big) \otimes_R K_0\big(C^*(G, B^i)\big)$$

as *ordered* $R = K_0(G)$*-modules*: Each class of minimal idempotents in $A^i \otimes B^i$ contains a pure tensor of such from A^i, respectively B^i, and these form an R-basis for $K_0 C^*(G, A^i \otimes B^i)$. The identification of $K_0(C^*(G, A^i \otimes B^i))$ with the R-module tensor product also results in the commuting square

$$\begin{array}{ccc} K_0\big((A^i \otimes B^i) \times G\big) & \longrightarrow & K_0\big((A^{i+1} \otimes B^{i+1}) \times G\big) \\ \downarrow & & \downarrow \\ K_0\big(A^i \times_{\alpha(i)} G\big) \otimes_R K_0\big(B^i \times_{\beta(i)} G\big) & \longrightarrow & K_0(A^{i+1} \times G) \otimes_R K_0(B^{i+1} \times G). \end{array}$$

(The lower horizontal map is the R-module tensor product of the module homomorphisms

$$K_0(A^i \times G) \to K_0(A^{i+1} \times G) \quad \text{and} \quad K_0(B^i \times G) \to K_0(B^{i+1} \times G).$$

As limits commute with tensor products (of free modules), we deduce that

$$K_0\big((A \otimes B) \times_{\alpha \otimes \beta} G\big) \to K_0(A \times_\alpha G) \otimes_R K_0(B \times_\beta G)$$

as ordered R-modules. The element $[(A \otimes B)_{\alpha \otimes \beta}]$ is carried over to $[A_\alpha] \otimes [B_\beta]$.

A special case occurs if $B = \otimes M_{n(i)}$ and the action $\beta \colon G \to \operatorname{Aut}(B)$ is trivial. Then $\beta(i)$ corresponds to the character $(\Pi_{j=1}^{i} n(j))\chi_0$ (χ_0 is the trivial character), and the map

$$K_0\big(B^i \times_{\beta(i)} G\big) \to K_0\big(B^{i+1} \times_{\beta(i+1)} G\big)$$

is simply multiplication by $n(i)\chi_0$ as a map $R \to R$. In this case, $K_0((A \otimes B) \times_{\alpha \otimes \beta} G)$ is simply $K_0(A \times_\alpha G) \otimes_R (R \otimes_{\mathbf{Z}} E)$, where E is the subgroup of the rational numbers associated to the sequence $\{n(i)\}$ (the "supernatural number").

If α, β are both product type actions, then obviously so is $\alpha \otimes \beta$. If the respective sequences of characters are given by $\{\chi_i\}$, $\{\chi'_i\}$, with $\alpha = \otimes \alpha_i, \beta = \otimes \beta_i$, then the maps in K_0 of the crossed product of the tensor product action are simply multiplication by $\chi_i \cdot \chi'_i$ ($R \otimes_R R \to R$ via $r_1 \otimes r_2 \mapsto r_1 r_2$).

More interesting phenomena occur if neither $K_0(A \times_\alpha G)$ nor $K_0(B \times_\beta G)$ is rank 1 as an R-module (product type actions automatically lead to rank 1 modules). Consider the non-product type action of $G = \mathbf{Z}_2$ discussed in III.5, and tensor it with itself. If M is the R-module arising from the original action, then the module corresponding to the tensor product action is $M \otimes_R M$. Iterations yield multifold tensor products $M \otimes_R \cdots \otimes_R M$, and it even makes sense to consider the limit of the multifold tensor product actions, and obtain the "infinite tensor product" of the actions, and the corresponding infinite tensor product of the modules. We wish to compute some of the module invariants attached to these tensor products, specifically, the rank as an abelian group ($\text{rank}_{\mathbf{Z}}$) and the rank as an R-module (rank_R). This we do in considerably more generality than is necessary to calculate this example.

Let M, N be modules over a ring R; assume R is a subdirect product of domains $D_1, \ldots, D_k$ corresponding to ideals $P_1, \ldots, P_k$, that is, $D_i = R/P_i$, $\bigcap P_i = \{0\}$, but no intersection of a proper subset of $\{P_1, \ldots, P_k\}$ is zero. Suppose that $M_i = M/P_iM$, $N_i = N/P_iN$ are torsion-free modules over D_i. As $M(N)$ is a subdirect sum of the $M_i(N_i)$, with rank_i denoting the rank over D_i,

$$\begin{aligned}\text{rank}_R M \otimes_R N &= \max_i\{\text{rank}_i M_i \otimes_i N_i\}\\ &= \max_i\{(\text{rank}_i M_i)(\text{rank}_i N_i)\}.\end{aligned}$$

On the other hand, if $\text{rank}_R M_i = m_i$, and $\text{rank}_R N_i = n_i$, then

$$\text{rank}_{\mathbf{Z}} M \otimes_R N = \sum (\text{rank}_{\mathbf{Z}} D_i) m_i n_i.$$

If $R = K_0(\hat{G})$ with $G = \mathbf{Z}_2$, then let P_1 denote the augmentation ideal (generated by $\hat{g} - 1$). The other prime ideal is generated by $\hat{g} + 1$; call it P_2. The corresponding D_1, D_2 are isomorphic to $\mathbf{Z}$ (see the appendix).

In the example of III.5 with non-product type action, $M = N$, $M_1 = \mathbf{Z}[1/5]$, M_2 has rank (over $\mathbf{Z}$) 2, and therefore

$$\text{rank}_R M \otimes_R M = \max\{1 \cdot 1, 2 \cdot 2\} = 4$$

and

$$\text{rank}_R \otimes_R^n M = \max\{1^n, 2^n\} = 2^n.$$

The abelian group ranks are $2^n + 1$. It also makes sense to discuss infinite tensor products of actions. In this case, the underlying algebra remains the same (because $\otimes_{\mathbf{Z}} \mathbf{Z}[1/5] \simeq \mathbf{Z}[1/5]$), and the module is the infinite tensor product of copies of M. This is a limit of the finite tensor products, and has infinite R-rank (and therefore infinite $\mathbf{Z}$-rank). These actions of $\mathbf{Z}_2$ are thus very far removed from product type actions.

The simple computation of ranks over $K_0(G)$ as above works for finite groups G. If instead, G is compact and connected, the representation ring is a commutative domain, so the R-module rank of R-torsion free modules is given by multiplication of the ranks. Since flat modules over a commutative domain are automatically torsion-free [14; Prop. 3, p. 133], this applies to modules arising from locally representable actions of G.

Appendix. Representation rings (examples)

Recall that for G a compact group, $K_0(G)$ is a partially ordered ring with positive cone the set of characters, and multiplication determined as functions $G \to \mathbf{C}$.

First consider the case that G be finite abelian. Then $\hat{G}$, the dual group is isomorphic to G, and $K_0(G) \simeq \mathbf{Z}\hat{G}$ (the integral group ring), with

$$K_0(G)^+ = \left\{ \sum a_g \hat{g} \,|\, a_g \geq 0 \right\},$$

as characters are sums of the irreducible characters, and multiplication of the irreducibles is simply the usual group multiplication in $\hat{G}$.

If $G = \mathbf{Z}_2$, then $K_0(G) \simeq \mathbf{Z}[\mathbf{Z}_2]$, which is embeddable in the ring direct product $\mathbf{Z} \oplus \mathbf{Z}$ via $g \mapsto (1, -1)$. As $a + bg \mapsto (a + b, a - b)$, $K_0(G)$ has image in $\mathbf{Z} \oplus \mathbf{Z}$ given by $A = \{(r, s) \in \mathbf{Z} \oplus \mathbf{Z} \,|\, r + s \text{ is even}\}$, with positive cone $\{(r, s) \in A \,|\, r \geq |s|\}$. Notice that $(\mathbf{Z} \oplus \mathbf{Z})/A = \mathbf{Z}_2$.

If $G = \mathbf{Z}_p$, p an odd prime, set $Z^p = \mathbf{Z}[\xi\}$, where ξ is a primitive p-th root of unity. As rings,

$$K_0(G) = \mathbf{Z}\hat{G} \simeq A = \left\{ (a, b) \in \mathbf{Z} \oplus Z^p \,|\, \text{if } b = \sum_0^{p-1} b_j \xi^j, \sum b_j = a \right\},$$

$$g \mapsto (1, \xi),$$

where g is a fixed generator of $\hat{G}$. (Since $\{1, , \dots, \xi^{p-2}\}$ is a $\mathbf{Z}$-basis of Z^p over $\mathbf{Z}$, the expression $b = \sum_0^{p-1} b_j \xi^j$ is not unique. However, $1 + \xi + \cdots + \xi^{p-1} = 0$ is the only relation among the roots of unity, so $\sum b_j$ is independent of the choice of b_j's.)

If G is cyclic of order p^n (p a prime), the ring structure of $\mathbf{Z}G$ is somewhat more complicated. Select a primitive p^n-th root of unity w. Then define the rings of integers,

$$\mathbf{Z}[w] \supset \mathbf{Z}[w^p] \supset \mathbf{Z}\left[w^{p^2}\right] \supset \cdots \supset \mathbf{Z}\left[w^{p^{n-1}}\right] \supset \mathbf{Z}[w^{p^n}] = \mathbf{Z}.$$

There is an obvious ring homomorphism $\mathbf{Z}G \to \oplus_0^n \mathbf{Z}[w^{p^i}]$ given by

$$g \mapsto (w, w^p, \ldots, w^{p^{n-1}}, 1)$$

(g a fixed generator), yielding a subdirect product representation.

Finally, if G is an arbitrary finite abelian group, it is a direct sum of groups of the form, cyclic of order p^n, so $\mathbf{Z}G$ is the corresponding tensor product, with the tensor product ordering.

Now consider the situation that $G = T^d$, so that $\hat{G} = \mathbf{Z}^d$. Then

$$K_0(G) = \mathbf{Z}[\mathbf{Z}^d] = \mathbf{Z}[x_1, x_1^{-1}, \ldots, x_d, x_d^{-1}],$$

the polynomial ring in d variables and their inverses. The irreducible characters of G are exactly the monomials, so an element (that is, a polynomial) is positive here precisely if all of its coefficients are non-negative.

A more exotic compact connected group is a solenoid, a group G with $\hat{G}$ being embeddable in the rationals. If for example $\hat{G} = \mathbf{Z}[\frac{1}{2}]$ then $K_0(G)$ is somewhat complicated: $K_0(G) = \mathbf{Z}[x^{2^{-n}}, x^{-2^{-n}}]$, with the obvious relations. Every element of $K_0(G)$ can be regarded as a polynomial in $x^{2^{-m}}$ for some fixed m, and the characters again correspond to those elements which can be realized as a positive combination of the generators and 1.

If G is a compact (connected) Lie group, then $K_0(G)$ as a ring is a polynomial ring (possibly with some inverses), but the ordering is often difficult to describe. For $G = \mathrm{SO}(3)$, $K_0(G) = \mathbf{Z}[x + x^{-1}] \subset \mathbf{Z}[x, x^{-1}]$, with irreducibles $1, x + 1 + x^{-1}, x^2 + x + 1 + x^{-1} + x^{-2}, \ldots$.

Finite non-abelian groups may have complicated representation rings (although these rings are still subdirect products of abelian rings of integers). If $G = S_n$, $K_0(G)$ is a ring subdirect product of copies of $\mathbf{Z}$, because all irreducible complex representations are realizable over the rational numbers. In particular, for $G = S_3$, there are three irreducible characters, two of dimension one (corresponding to the trivial character (χ_0) and the non-trivial one arising from the homomorphism $S_3 \to \mathbf{Z}_2$, (χ_1), and a two-dimensional one (χ_2) emanating from the irreducible representation

$$(1\ \ 2) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (1\ \ 2\ \ 3) \mapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Since $\chi_1\chi_2$ corresponds to an irreducible character of dimension two (as χ_1 is linear), $\chi_1\chi_2 = \chi_2$. Now $\chi_2^2((1\ \ 2)) = 0$, $\chi_2^2((1\ \ 2\ \ 3)) = 1$; as χ_2^2 is a combination of χ_2, χ_1 and χ_0, we deduce $\chi_2^2 + \chi_2 + \chi_1 + \chi_0$. We have the multiplication table

	χ_0	χ_1	χ_2
χ_0	χ_0	χ_1	χ_2
χ_1	χ_1	χ_0	χ_2
χ_2	χ_2	χ_2	$\chi_2 + \chi_1 + \chi_0$

;

as $K_0(G) = \chi_0\mathbf{Z} + \chi_1\mathbf{Z} + \chi_2\mathbf{Z}$ (as abelian groups), the ring structure is completely determined. The ring map $K_0(G) \to \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ is given by

$$\chi_0 \mapsto (1,1,1), \quad \chi_1 \mapsto (1,-1,1), \quad \chi_2 \mapsto (2,0,-1).$$

In all the examples (and in all cases), there is a map $K_0(G) \to \mathbf{Z}$ given by $\chi \mapsto \chi(1)$. For G abelian, this corresponds to factoring out the augmentation ideal. Call the kernel of the map I. If M is a $K_0(G)$-module, then M/MI is a $\mathbf{Z}$-module (and in the abelian case, a trivial $\hat{G}$-module); if M is of the form $K_0(A \times_\alpha G)$ arising from a locally representable action on an AF algebra, then $K_0(A) = M/MI$ with the obvious ordering on the latter (at any level, $K_0(G)^{n(i)} \to K_0(G)^{n(i+1)}$ is transformed to $\mathbf{Z}^{n(i)} \to \mathbf{Z}^{n(i+1)}$, etc.).

All of the above is of course well known to algebraists, but may not be so well known to people interested in group actions on C^* algebras.

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