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MOD 2 UNSTABLE EQUIVARIANT SPHERICAL CHARACTERISTIC CLASSES AND POINCARE DUALITY BORDISM GENERATORS

BY

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In this paper we determine the mod 2 characteristic classes for unstable $\mathbf{Z}/2$ and S^1 equivariant spherical fibrations as well as analyze the natural inclusions of the relevant classifying spaces $BSG(\mathbf{Z}/2, n)$ and $BSG(S^1, 2n)$ into their stable analogs $BSG(\mathbf{Z}/2)$ and $BSG(S^1)$. As a consequence of our computations we obtain homological obstructions to endowing a spherical fibration with a fibre preserving involution thus generalizing a result of Hodgson [6], and strengthen our result in [8] on representing polynomial generators of the unoriented Poincaré duality bordism ring by spherical projective bundles.

Before giving a brief description of the organization of this paper we note that our analysis depends in a crucial way on an idea of Reinhard Schultz [16] and we would like to thank Mark Mahowald for first bringing Schultz's idea to our attention. We would also like to thank Jim Becker for communicating to us his proof of Theorem 3.1 and pointing out that similar arguments appear in the work of M. Crabb [5] and L. Woodward [20]. Finally we would like to thank the referee for his comments and suggestions which we trust have greatly improved the readability of this paper.

In Section 1 we recall the Schultz filtrations of the function spaces $G(\mathbf{Z}/2, n)$ and $G(S^1, 2n)$ of unstable equivariant self maps of spheres. This filtration allows us to relate these function spaces to both the unstable non-equivariant function space $G(n)$ and the stable equivariant function spaces $G(\mathbf{Z}/2)$ and $G(S^1)$. We then precede in Section 2 to use the results of Section 1 and [9], [12] to compute

$$H_*(SG(\mathbf{Z}/2, n)),\ H_*(BSG(\mathbf{Z}/2, n)),$$

$$H_*(SG(S^1, 2n)) \text{ and } H_*(BSG(S^1, 2n)).$$

Here as through out the rest of the paper all homology is understood to be with $\mathbf{Z}/2$ coefficients.

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For geometric applications it is necessary to have more detailed information about the following commutative diagrams:

$$(0.1)\qquad \begin{array}{ccc} H_*(SG(\mathbf{Z}/2,n)) & \xrightarrow{i(\mathbf{Z}/2,n)_*} & H_*(SG(\mathbf{Z}/2)) \\ \downarrow{\scriptstyle f(n)_*} & & \downarrow{\scriptstyle f_*} \\ H_*(SG(n)) & \xrightarrow{i(n)_*} & H_*(SG) \end{array}$$

and

$$(0.2)\qquad \begin{array}{ccc} H_*(BSG(\mathbf{Z}/2,n)) & \xrightarrow{Bi(\mathbf{Z}/2,n)_*} & H_*(BSG(\mathbf{Z}/2)) \\ \downarrow{\scriptstyle Bf(n)_*} & & \downarrow{\scriptstyle Bf_*} \\ H_*(BSG(n)) & \xrightarrow{Bi(n)_*} & H_*(BSG) \end{array}$$

where $i(n)$ and $i(\mathbf{Z}/2, n)$ are the natural inclusion maps and $f(n)$ and f are the natural forgetful maps. The remainder of this paper is devoted to analyzing these diagrams and examining the resulting consequences.

In Section 3 we recall that while the stable forgetful map Bf has a stable section (see Theorem 3.1) there is no such splitting for the unstable forgetful map $Bf(n)$ and, in fact, a theorem of Hodgson [6] implies $Bf(n)$ is not onto even in $\mathbf{Z}/2$ homology. None the less it is still possible to compute the effect of $f(n)$ in homology as follows: Results of Milgram [14] and Sections 1 and 2 show both $i(n)_*$ and $i(\mathbf{Z}/2, n)_*$ are monomorphisms. Furthermore results of [9] completely determine f_*. Thus $f(n)_*$ can be computed once $i(\mathbf{Z}/2, n)_*$ is better understood (the arguments of Section 1 and 2 determine $i(\mathbf{Z}/2, n)_*$ only up to a filtration).

Sections 4 and 5 analyze $i(\mathbf{Z}/2, n)_*$ in detail. We first construct explicit families of elements in image($i(\mathbf{Z}/2, n)_*$). Next, given $x \in H_*(SG(\mathbf{Z}/2))$ we give a bound on n to guarantee that x is in image($i(\mathbf{Z}/2, n)_*$). Section 6 then uses these results to explicitly determine image($f(n)_*$) and thus give homological obstructions to giving a spherical fibration a fibre preserving involution. Finally in Section 7 we combine the computations of the first six sections with results of [8] to obtain projective bundle representatives for unoriented bordism ring generators.

Section 1

We begin by defining our fundamental object of interest, $SG(H, n)$, and give a filtration of $SG(H, n)$ which we will use in Section 2 to compute $H_*(SG(H, n))$. Let H be a cyclic group of prime power order or the circle group and let V be a finite dimensional real representation of H so that H acts freely on $V - \{0\}$. Further suppose V has a H-invariant metric and let $k = \dim V$.

DEFINITION 1.1. $G(H, kn) = \mathrm{Map}_H(S(nV), S(nV))$ is the space of H equivariant self-maps of the unit sphere $S(nV)$ of nV.

As V has an H-invariant metric the topological structure of $G(H, kn)$ is given by the metric $\langle f, g\rangle = \max_{x \in S(nV)}\langle f(x), g(x)\rangle$. $G(H, kn)$ also has a natural multiplication, the composition product,

$$\circ : G(H, kn) \times G(H, kn) \to G(H, kn) \tag{1.2}$$

given by the composition of maps. Thus $G(H, kn)$ is an associative H-space with unit.

If $H_1 \to H_2$ is an inclusion of subgroups then there is a natural restriction map

$$R(H_2, H_1): \mathrm{Map}_{H_2}(S(nV), S(nV)) \to \mathrm{Map}_{H_1}(S(nV), S(nV)) \tag{1.3}$$

as any representation V of H_2 which is free on $V - \{0\}$ is trivially a representation of H_1 which is free on $V - \{0\}$. Thus we obtain the canonical "forgetful map"

$$t(H_2, H_1): G(H_2, kn) \to G(H_1, kn). \tag{1.4}$$

There is another natural map, the suspension map,

$$\sigma: G(H, kn) \to G(H, k(n+1)) \tag{1.5}$$

defined by $\sigma(f) = f * \mathrm{id}_V$ where we have used the H-equivariant homeomorphism

$$S(nV) * S(V) \simeq S((n+1)V).$$

σ allows us to pass to the limit over n and obtain the space of stable H-equivariant self maps $G(H)$. This is the space studied by Becker and Schultz [2]. They showed that for any compact Lie group H, $G(H)$ is homotopy equivalent to $Q(BH^{\zeta})$, where $Q(X)$ is the limit over n of $\Omega^n\Sigma^n X$, and BH^{ζ} is the Thom space of the vector bundle ζ over BH associated to the adjoint representation of H on its Lie algebra. If H is finite then $\zeta = 0$, and $BH^{\zeta} = BH^+$, the disjoint union of BH with a distinguished base point $+$. Thus $G(H)$ is independent of the choice of V. Becker and Schultz also showed that $SG(H)$, the degree 1 component of $G(H)$, is, in fact, an infinite loop space under the composition product.

DEFINITION 1.6. $SG(H, n)$ is the subspace of $G(H, kn)$ consisting of all degree 1 maps.

As the composition of degree 1 maps has degree 1, $SG(H, kn)$ is also an associative H-space with unit. Just as in the stable case well known results of Barrett, Gugenheim and Moore [1] and May [13] show that $BSG(H, kn)$ classifies fibrations with a fibre-preserving action of H on the total space such that the fibre is H-equivariantly homotopy equivalent to $S(nV)$. The composition product thus induces a Pontragin product on $H_*(SG(H, kn))$. Classically, when H is the trivial group, $H_*(SG(n))$, $H_*(BSG(n))$, $H_*(SG)$ and $H_*(BSG)$ were determined in the works of Milgram [14], Madsen [7], May [4], Tsuchiya [19], and Cohen [4]. For non-trivial H the calculations for the stable spaces are contained in our joint work with Haynes Miller [9], [10], [12].

We now restrict our attention to the prime 2 and the two cases when H is $\mathbf{Z}/2$ or S^1. If $H \cong \mathbf{Z}/2$ we may let $V = \mathbf{R}^1$ with the standard antipodal action. Thus $S(nV_{\mathbf{Z}/2})$ may be taken to be the standard $n-1$ sphere with the standard antipodal action. If $H \cong S^1$ we may let $V = \mathbf{R}^2 \simeq \mathbf{C}^1$ with the standard S^1 action induced by complex multiplication $S^1 \times \mathbf{C}^1 \to \mathbf{C}^1$. Thus $S(nV_{S^1})$ may be taken to be the standard $2n-1$ sphere with the standard S^1 action.

In the non-equivariant case the J-homomorphism $J\colon SO(n) \to SG(n)$ played a central role at the prime 2. Since a linear orthogonal map commutes with the antipodal action (and a unitary map commutes with the standard S^1 action) on the unit sphere we obtain $\mathbf{Z}/2$ and S^1 equivariant J-homomorphisms

$$J_{\mathbf{Z}/2}\colon SO(n) \to SG(\mathbf{Z}/2, n) \tag{1.7}$$

and

$$J_{S^1}\colon U(n) \to SG(S^1, 2n) \tag{1.8}$$

which cover the classical J-homomorphism. That is, the following four diagrams all commute:

$$\begin{array}{ccc} U(n) & \xrightarrow{J_{S^1}} & SG(S^1, 2n) \\ \downarrow & & \downarrow{\scriptstyle f(S^1, \mathbf{Z}/2)} \\ SO(2n) & \xrightarrow{J_{\mathbf{Z}/2}} & SG(\mathbf{Z}/2, 2n) \end{array} \tag{1.9}$$

$$\begin{array}{ccc} BU(n) & \xrightarrow{BJ_{S^1}} & BSG(S^1, 2n) \\ \downarrow{\scriptstyle Bi} & & \downarrow{\scriptstyle Bf(S^1, \mathbf{Z}/2)} \\ BSO(2n) & \xrightarrow{BJ_{\mathbf{Z}/2}} & BSG(\mathbf{Z}/2, n) \end{array} \tag{1.10}$$

(1.11)
$$\begin{array}{ccc} SO(n) & \xrightarrow{J_{\mathbf{Z}/2}} & SG(\mathbf{Z}/2, n) \\ & {\scriptstyle J}\searrow & \downarrow{\scriptstyle F(\mathbf{Z}/2,\mathrm{id})} \\ & & SG(n) \end{array}$$

(1.12)
$$\begin{array}{ccc} BSO(n) & \xrightarrow{BJ_{\mathbf{Z}/2}} & BSG(\mathbf{Z}/2, n) \\ & {\scriptstyle BJ}\searrow & \downarrow{\scriptstyle Bf(\mathbf{Z}/2,\mathrm{id})} \\ & & BSG(n) \end{array}$$

See [9] and [12] for an analysis of the stable analogs of these commutative diagrams.

We now describe a beautiful idea of Schultz [16] which gives filtrations for $G(\mathbf{Z}/2, n)$ and $G(S^1, 2n)$. We begin with $\mathbf{Z}/2$. Consider the $\mathbf{Z}/2$ equivariant cofibration filtration

(1.13)
$$S^{n-1} \supset S^{n-2} \supset \cdots \supset S^1 \supset S^0$$

Applying $\mathrm{Map}_{\mathbf{Z}/2}(\ \ , S^{n-1})$ to 1.13 we obtain the $\mathbf{Z}/2$ Schultz filtration:

(1.14)
$$\begin{array}{ccc} \mathrm{Map}_{\mathbf{Z}/2}(S^{n-1}, S^{n-1}) & \longrightarrow & \mathrm{Map}_{\mathbf{Z}/2}(S^{n-2}, S^{n-1}) \\ \uparrow & & \uparrow \\ F_0 & & F_1 \end{array}$$
$$\to \cdots \to \mathrm{Map}_{\mathbf{Z}/2}(S^0, S^{n-1})$$

where F_i is the homotopy theoretic fibre

$$F_i \to \mathrm{Map}_{\mathbf{Z}/2}(S^{n-1-i}, S^{n-1}) \to \mathrm{Map}_{\mathbf{Z}/2}(S^{n-1-i-1}, S^{n-1}).$$

LEMMA 1.15. *F_i is homotopy equivalent to $\Omega^{n-1-i}S^{n-1}$.*

Proof. A $\mathbf{Z}/2$ equivariant map $f\colon S^{n-1-i} \to S^{n-1}$ which extends the identity on $S^{n-1-i-1} \to S^{n-1}$ is completely determined by its behavior on the upper hemisphere $D^{n-1-i} \subset S^{n-1-i}$. But $\Omega^{n-1-i}S^{n-1}$ is equivalent to the space of maps $g\colon D^{n-1-i} \to S^{n-1}$ with $g|_{\partial D^{n-1-i} = S^{n-1-i-1}}$ equal to the standard inclusion.

If $H = S^1$ we consider the S^1 equivariant cofibration filtration

(1.16)
$$S^{2n-1} \supset S^{2n-3} \supset \cdots \supset S^3 \supset S^1.$$

Applying $\mathrm{Map}_{S^1}(\ \ , S^{2n-1})$ to 1.16 we obtain the S^1 Schultz filtration:

$$(1.17)\qquad \begin{array}{ccc} \mathrm{Map}_{S^1}(S^{2n-1}, S^{2n-1}) & \longrightarrow & \mathrm{Map}_{S^1}(S^{2n-3}, S^{2n-1}) \\ \uparrow & & \uparrow \\ \overline{F}_0 & & \overline{F}_1 \end{array}$$

$$\rightarrow \cdots \rightarrow \quad \mathrm{Map}_{S^1}(S^1, S^{2n-1})$$

where $\overline{F}_i$ is the homotopy theoretic fibre

$$\overline{F}_i \rightarrow \mathrm{Map}_{S^1}(S^{2n-1-2i}, S^{2n-1}) \rightarrow \mathrm{Map}_{S^1}(S^{2n-1-2(i+1)}, S^{2n-1}).$$

LEMMA 1.18. *$\overline{F}_i$ is homotopy equivalent to $\Omega^{2(n-1-i)}S^{2n-1}$.*

Proof. We regard S^{2j+1} as $S^{2j-1} * S^1$. Choosing a point p in S^1 we note that an S^1 equivariant map

$$f\colon S^{2n-1-2i} \rightarrow S^{2n-1}$$

restricted to $S^{2n-1-2i} - S^{2n-1-2(i+1)}$ is determined by its behavior on

$$\left[\left(S^{2n-1-2i-2} * p\right) - S^{2n-1-2i-2}\right].$$

Now if f restricted to $S^{2(n-1-i)-1}$ is the standard inclusion then f is determined solely by its values on $D^{2(n-1-i)} = S^{2(n-1-i)-1} * p$. The lemma now follows as in 1.15.

If we let n tend to infinity we obtain stable Schultz filtrations:

$$(1.19)\qquad \begin{array}{ccc} \lim\limits_{n\rightarrow\infty} \mathrm{Map}_{\mathbf{Z}/2}(S^{n-1}, S^{n-1}) & \longrightarrow & \lim\limits_{n\rightarrow\infty} \mathrm{Map}_{\mathbf{Z}/2}(S^{n-2}, S^{n-1}) \\ \uparrow & & \uparrow \\ Q(S^0) & & Q(S^1) \end{array}$$

$$\begin{array}{c} \rightarrow \cdots \rightarrow \quad \lim\limits_{n\rightarrow\infty} \mathrm{Map}_{\mathbf{Z}/2}(S^{n-i}, S^{n-1}) \rightarrow \cdots \\ \uparrow \\ Q(S^{i-1}) \end{array}$$

and

$$(1.20)\qquad \begin{array}{ccc}
\lim\limits_{n\to\infty} \mathrm{Map}_{S^1}(S^{2n-1}, S^{2n-1}) & \longrightarrow & \lim\limits_{n\to\infty} \mathrm{Map}_{S^1}(S^{2n-3}, S^{2n-1}) \\
\uparrow & & \uparrow \\
Q(S^1) & & Q(S^3)
\end{array}$$

$$\begin{array}{ccccc}
\to \cdots \to & \lim\limits_{n\to\infty} \mathrm{Map}_{S^1}(S^{2(n-i)-1}, S^{2n-1}) & \to \cdots \\
& \uparrow & \\
& Q(S^{2i+1}) &
\end{array}$$

We may simplify 1.19 and 1.20 using a theorem of Becker and Schultz [2].

THEOREM 1.21. *The* $\mathbf{Z}/2$ *Schultz filtration of* $G(\mathbf{Z}/2)$ *is equivalent to the following filtration of* $Q(\mathbf{R}P(\infty)_+)$ *which is induced by the cofibre sequence* $\mathbf{R}P(n) \supset \mathbf{R}P(n-1) \supset \cdots \supset \mathbf{R}P(1) \supset \mathbf{R}P(0)$:

(1.22)

$$\begin{array}{ccccccc}
Q(\mathbf{R}P(\infty)_+) \to & Q(\mathbf{R}P(\infty)_+/\mathbf{R}P(1)) & \to \cdots \to & Q(\mathbf{R}P(\infty)_+/\mathbf{R}P(n)) & \to \cdots \\
\uparrow & \uparrow & & \uparrow & \\
Q(S^0) & Q(S^1) & & Q(S^n) &
\end{array}$$

THEOREM 1.23. *The* S^1 *Schultz filtration of* $G(S^1)$ *is equivalent to the following filtration of* $Q(S^1 \wedge CP(\infty)_+)$ *which is induced by the cofibre sequence* $S^{2n+1}/S^1 \supset S^{2n-1}/S^1 \supset \cdots \supset S^1/S^1$:

$$(1.24)\qquad \begin{array}{ccc}
Q(S^1 \wedge CP(\infty)_+) & \longrightarrow & Q(S^1 \wedge CP(\infty)_+/S^1 \wedge S^0) \\
\uparrow & & \uparrow \\
Q(S^1) & & Q(S^3)
\end{array}$$

$$\begin{array}{ccc}
\to \cdots \to & Q(S^1 \wedge CP(\infty)_+/S^1 \wedge CP(n)) & \to \cdots \\
& \uparrow & \\
& Q(S^{2n+1}) &
\end{array}$$

Proof of 1.21 *and* 1.23. Apply [2 : 6.14] to 1.19 and 1.20.

Section 2

In this section we compute

$$H_*(SG(\mathbb{Z}/2,n)),\ H_*(SG(S^1,2n)),\ H_*(BSG(\mathbf{Z}/2,n)) \text{ and } H_*(BSG(S^1,2n)).$$

We begin with $G(\mathbf{Z}/2,n)$. For each n the filtration 1.14 induces a sequence of Serre spectral sequences $\{E_r(\mathrm{Map}_{\mathbf{Z}/2}(S^{n-1-i},S^{n-1}))\}$ for $i=0,1,\ldots,n-2$ with associated E_2 term additively isomorphic to

$$H_*(F_i)\otimes E_\infty\big(\mathrm{Map}_{\mathbf{Z}/2}(S^{n-2-i},S^{n-1})\big)$$

converging to a filtration of $H_*(\mathrm{Map}_{\mathbf{Z}/2}(S^{n-1-i},S^{n-1}))$. Recall by 1.15 that $F_i\simeq\Omega^{n-1-i}S^{n-1}$. Similarly the filtration 1.22 induces a sequence of Serre spectral sequences

$$\Big\{E_r\Big(\lim_{n\to\infty}\mathrm{Map}_{\mathbf{Z}/2}(S^{n-1-i},S^{n-1})\Big)\Big\}$$

for all non-negative i with associated E_2 term additively isomorphic to

$$H_*(Q(S^i))\otimes E_\infty\Big(\lim_{n\to\infty}\mathrm{Map}_{\mathbf{Z}/2}(S^{n-2-i},S^{n-1})\Big)$$

converging to a filtration of

$$H_*\Big(\lim_{n\to\infty}\mathrm{Map}_{\mathbf{Z}/2}(S^{n-1-i},S^{n-1})\Big).$$

Now the suspension map

$$\sigma\colon \mathrm{Map}_{\mathbf{Z}/2}(S^{n-1-i},S^{n-1})\to\mathrm{Map}_{\mathbf{Z}/2}(S^{n-1},S^{n})\to\cdots\to\lim_{n\to\infty}\mathrm{Map}_{\mathbf{Z}/2}(S^{n-1-i},S^{n-1})$$

induces a map of spectral sequences

$$E_r\big(\mathrm{Map}_{\mathbf{Z}/2}(S^{n-1-i},S^{n-1})\big)\to E_r\Big(\lim_{n\to\infty}\mathrm{Map}_{\mathbf{Z}/2}(S^{n-1-i},S^{n-1})\Big)$$

for each i. However [9] shows that $H_*(SG(\mathbf{Z}/2))$ is additively isomorphic to $\otimes_{i=0}^{\infty}H_*(Q(S^i))$ and hence there can not be any non-trivial differentials in any of the Serre spectral sequences in the stable sequence. As $H_*(\Omega^{n-1-i}S^{n-1})\simeq H_*(F_i)$ maps monomorphically into $H_*(Q(S^i))$ under the natural inclusion of spaces (recall we are at the prime 2) it follows by induction on $n - i - 1$ that

$$E_2\big(\mathrm{Map}_{\mathbf{Z}/2}(S^{n-1-i}, S^{n-1})\big)$$

maps monomorphically into

$$E_2\Big(\lim_{n\to\infty} \mathrm{Map}_{\mathbf{Z}/2}(S^{n-1-i}, S^{n-1})\Big)$$

for every i. Thus there can not be any non-trivial differentials in any of the Serre spectral sequences $\{E_r\mathrm{Map}_{\mathbf{Z}/2}(S^{n-1-i}, S^{n-1})\}$. When $i = 0$ this implies

$$E_2(G(\mathbf{Z}/2, n)) \cong E_\infty(G(\mathbf{Z}/2, n))$$

injects into

$$E_2(G(\mathbf{Z}/2)) \cong E_\infty(G(\mathbf{Z}/2)).$$

Summarizing we have:

LEMMA 2.1. *$H_*(G(\mathbf{Z}/2, n))$ is additively isomorphic to $\otimes_{i=0}^{n-1} H_*(\Omega^{n-1-i}S^{n-1})$ and the natural inclusion i: $G(\mathbf{Z}/2, n) \to G(\mathbf{Z}/2)$ induces a monomorphism in homology.*

We may obtain in an identical manner the following lemma for $G(S^1, 2n)$.

LEMMA 2.2. *$H_*(G(S^1, 2n))$ is additively isomorphic to $\otimes_{i=0}^{n-1} H_*(\Omega^{2(n-1-i)}S^{2n-1})$ and the natural inclusion i: $G(S^1, 2n) \to G(S^1)$ induces a monomorphism in homology.*

Proof. Results of [12] imply the relevant sequences of spectral sequences collapse.

Restriction to the degree 1 component yields H-space maps

$$i: SG(\mathbf{Z}/2, n) \to SG(\mathbf{Z}/2) \tag{2.3}$$

and

$$i: SG(S^1, 2n) \to SG(S^1) \tag{2.4}$$

which, by 2.1 and 2.2, induce monomorphisms in homology. To state the main results of this section we must first fix our notation. We assume the reader is familiar with the definitions, structure and properties of the Dyer-Lashof algebra (for details see [4]). Recall $H_*(SO) \cong E(e_1, e_2, \dots, e_r, \dots)$ with $\dim e_r = r$. Let $\bar{e}_t = \chi(e_t)$.

DEFINITION 2.5. Let x_r and $y_{I,t}$ be defined by the following formulae:
(a) $x_r = J_{\mathbf{Z}/2}(e_r) * J_{\mathbf{Z}/2}(e_r) * [-1]$ for r a positive integer,
(b) $y_{I,t} = Q^I((\bar{e}_t)) * ((e_0))^{2l(I)} * [1]$ where $I = (i, \dots, i_k)$ indexes a Dyer-Lashof operation of length k and (I, t) is an admissible sequence of positive length and positive excess. Here t is a non-negative integer.

THEOREM 2.6 [9]. *As a Hopf algebra under the composition product*

$$H_*(SG(\mathbf{Z}/2)) \cong H_*(SO) \otimes P[x_r, y_{I,t}]$$

where x_r *and* $y_{I,t}$ *are given by* 2.5.
We may now prove:

THEOREM 2.7. *As a Hopf algebra under the composition product*

$$H_*(SG(\mathbf{Z}/2, n)) \cong H_*(SO(n)) \otimes P[x_r, y_{I,t}]$$

where x_r *and* $y_{I,t}$ *are given by* 2.5 *under the additional restrictions*
(a) $1 \le r \le n-2, 0 \le t \le n-2$,
(b) $0 < excess(I) < n - 2 - t,\ t < i_k$.

Proof. $J_{\mathbf{Z}/2}$ is a homology monomorphism into $H_*(SG(\mathbf{Z}/2))$ [9] and thus must be so into $H_*(SG(Z/2, n))$. Thus by 2.1,

$$i_*: H_*(SG(\mathbf{Z}/2, n))//H_*(SO(n)) \to H_*(SG(\mathbf{Z}/2))//H_*(SO)$$

is a ring monomorphism into a polynomial algebra. The theorem then follows from a simple counting argument using 2.1.
Recall $H_*(U) \cong P[a_0, a_1, \dots, a_r, \dots]$ with $\dim a_r = 2r + 1$.

DEFINITION 2.8. Let z_r and $w_{I,t}$ be defined by the following formulae:
(a) $z_r = J_{S^1}(a_r) * J_{S^1}(a_r)$ for r a non-negative integer,
(b) $w_{I,t} = Q^I(J_{S^1}(a_t))$ where t is a non-negative integer and (I, t) is an admissible sequence of positive length and positive excess.

THEOREM 2.9 [12]. *As a Hopf algebra under the composition product*

$$H_*(SG(S^1)) \cong H_*(U) \otimes P[z_r, w_{I,t}]$$

where z_r *and* $w_{I,t}$ *are given by* 2.8.
Thus we may prove:

THEOREM 2.10. *As a Hopf algebra under the composition product,*

$$H_*(SG(S^1, 2n)) \cong H_*(U(n)) \otimes P[z_r, w_{I,t}]$$

where z_r *and* $w_{I,t}$ *are given by* 2.8 *under the additional restrictions*

(a) $0 \le r \le n-1, 0 \le t \le n-1$,

(b) $0 < excess(I) < n-2-t,\ t < i_k$.

Proof. Identical to that for 2.7 after replacing $J_{\mathbf{Z}/2}$ by J_{S^1} and 2.1 by 2.2. Here [12] shows J_{S^1} is a homology monomorphism.

We may extend our computations to the classifying space level.

THEOREM 2.11. *$H_*(BSG(\mathbf{Z}/2, n))$ is additively isomorphic to*

$$H_*(BSO(n)) \otimes \Omega(\mathbf{Z}/2, n)$$

where $\Omega(\mathbf{Z}/2, n)$ is a primitively generated exterior algebra tensored with a primitively generated polynomial algebra.

Proof. The Eilenberg-Moore spectral sequence converging to $H_*(BSG(Z/2, n))$ maps naturally to the Eilenberg-Moore spectral sequence converging to $H_*(BSG(Z/2))$. Now, 2.7 implies that $E_2(BSG(Z/2, n))$ injects into $E_2(BSG(Z/2))$ and [9] shows that $E_2(BSG(Z/2)) \cong E_\infty(BSG(Z/2))$. The theorem then follows from the known structure of $H_*(BSG(Z/2))$ [9].

THEOREM 2.12. *$H_*(BSG(S^1, 2n)$ is additively isomorphic to $H_*(BU(n)) \otimes \Omega(S^1, 2n)$ where $\Omega(S^1, 2n)$ is a primitively generated exterior algebra tensored with a primitively generated polynomial algebra.*

Proof. Identical to 2.11 using results of [12].

We conclude this section with an observation: Unfortunately 2.7 and 2.10 are not quite as satisfactory as they first appear. More precisely consider the commutative diagram mentioned in the introduction:

$$\text{(2.13)} \qquad \begin{array}{ccc} H_*(SG(\mathbf{Z}/2, n)) & \xrightarrow{i(\mathbf{Z}/2, n)_*} & H_*(SG(\mathbf{Z}/2)) \\ \big\downarrow{\scriptstyle f(n)_*} & & \big\downarrow{\scriptstyle f_*} \\ H_*(SG(n)) & \longrightarrow & H_*(SG) \end{array}$$

As [9] computes f_* and both $i(n)_*$ and $i(\mathbf{Z}/2, n)_*$ are monomorphisms it appears that $f(n)_*$ should be easy to compute. This is not the case for, although $i(n)_*$ has known image by Milgram's computation [14], $i(Z/2, n)_*$ is computed in this section only up to filtration. For example recall, in the notation [9], that

$$J_{\mathbf{Z}/2}(e_r) = \sum_{i+j=r} (\bar{e}_i) * Q^j(e_0) * (\bar{e}_0) * [1]$$

is in the image of $i(\mathbf{Z}/2, n)_*$ for $n \geq r + 1$. But when $n = r + 1$ this formula sum does *not* appear as an element of $H_*(SG(\mathbf{Z}/2, n))$ in Theorem 2.7. We will explain this phenomenon in sections 4, 5 and 6.

Section 3

Calculations in [9] show that the stable forgetful map f: $SG(\mathbf{Z}/2) \to SG$ and its classifying map Bf: $BSG(\mathbf{Z}/2) \to BSG$ are onto in homology. Actually more is true as pointed out to us by Jim Becker.

THEOREM 3.1. *There is a section* $\mathscr{S}$: $BSG \to BSG(\mathbf{Z}/2)$ *so that* $Bf \circ \mathscr{S} = \mathrm{id}$.
There is also a similar section in the PL category [8]. In fact the following diagram commutes:

(3.2)
$$\begin{array}{ccccc} BSPL & \xrightarrow{\mathscr{S}} & BSPL(\mathbf{Z}/2) & \xrightarrow{Bf} & BSPL \\ \downarrow i & & \downarrow i(\mathbf{Z}/2) & & \downarrow i \\ BSG & \xrightarrow{\mathscr{S}} & BSG(\mathbf{Z}/2) & \xrightarrow{Bf} & BSG \end{array}$$

Proof of 3.1 (Becker, private communication, see also [5] and [20].) Let

$$S\hat{G}(\mathbf{Z}/2) = \lim_{\substack{n \to \infty \\ m \to \infty}} \mathrm{Map}^{+1}_{\mathbf{Z}/2}(S(nV \otimes m\mathbf{R}), S(nV \otimes m\mathbf{R}))$$

where V has the free $\mathbf{Z}/2$ action, $m\mathbf{R}$ has the trivial $\mathbf{Z}/2$ action, and we consider degree 1 maps which are also degree 1 when restricted to $m\mathbf{R}$. Restriction to the fixed point set gives a map ρ: $S\hat{G}(\mathbf{Z}/2) \to SG$ and the fibre F of ρ is the space of $\mathbf{Z}/2$-homotopy equivalences whose restriction to the fixed point set is the identity. Thus, by the Freudenthal suspension theorem [20], the natural inclusion i: $SG(\mathbf{Z}/2) \to SF$ is a homotopy equivalence and one has the fibration

$$SG(\mathbf{Z}/2) \xrightarrow{i} S\hat{G}(\mathbf{Z}/2) \xrightarrow{\rho} SG$$

which may be classified to give

$$BSG(\mathbf{Z}/2) \xrightarrow{Bi} BS\hat{G}(\mathbf{Z}/2) \xrightarrow{B\rho} BSG.$$

It suffices to show that

$$[X, BSG(\mathbf{Z}/2)] \overset{\text{forget}}{\to} [X, BSG]$$

is onto for all finite complexes X. Let α be a spherical fibration over X and let β be a negative of α. The spherical fibration $\omega = \alpha * \alpha * \beta$ has an involution induced by $x * y * z \to y * x * z$. The classifying map

$$\omega \colon X \to BS\hat{G}(\mathbf{Z}/2)$$

has the property that $B\rho(\omega) = 0$ because the restriction to the fixed point set classifies $\alpha * \beta$. Thus there exists a lift $\gamma \colon X \to BSG(\mathbf{Z}/2)$ such that $Bi(\gamma) = \omega$. Clearly $Bf(\gamma) = \alpha$.

We now return to the unstable case. Not only does the unstable analog of 3.1 fail but, in fact, the forgetful map $Bf(n) \colon BSG(\mathbf{Z}/2, n) \to BSG(n)$ is *not* onto in homology. This was first noticed, in dimensions $n = 2^j - 1$, by Hodgson [6].

THEOREM 3.3 [6]. *The map*

$$Bf(2^j - 1)_* \colon H_*\big(BSG(\mathbf{Z}/2, 2^j - 1)\big) \to H_*\big(BSG(2^j - 1)\big)$$

is not onto. In fact the class

$$\sigma_*\big(Q^{2^j - 1}(1) * Q^{2^j - 1}(1) * [-3]\big) \varepsilon H_*\big(BSG(2^j - 1)\big)$$

is a homological obstruction to giving a $2^j - 2$ *dimensional spherical fibration a free fibre-preserving involution.*

We will extend 3.3 to arbitrary values of n in the next section.

Section 4

In this section we begin to analyze the homological image of $i(\mathbf{Z}/2, n)$ in more detail. We begin by recalling that given t disjoint pairs of antipodal points $\{x_1, -x_1\}, \{x_2, -x_2)\ldots,\{x_t, -x_t\}$ in S^{n-1} there is a canonical self map of S^{n-1} of degree $1 - 2t$. One first chooses disjoint ε neighborhoods of the $\{x_i, -x_i\}$. The self map is then defined to be the identity on the complement of the ε neighborhoods and to map each ε neighborhood radially onto its own complement. Regarding $\mathbf{R}P(k)$ for $k \le n - 1$ as pairs of antipodal points in S^k and hence in S^{n-1} this construction defines a map $i \colon \mathbf{R}P(k) \to \mathrm{Map}^{-1}_{\mathbf{Z}/2}(S^{n-1}, S^{n-1})$. By taking ε to be the circumference of

S^{n-1} we see that i is homotopic to the map which associates to a line l in S^k the reflection R_l in the plane perpendicular to l in $\mathbf{R}^n$. The composite

$$(4.1) \qquad \mathbf{R}P(k) \to \mathrm{Map}_{\mathbf{Z}/2}^{-1}(S^{n-1}, S^{n-1}) \to Q^{-1}(\mathbf{R}P(\infty)_+)$$

is computed in [12] and the following result is proved.

PROPOSITION 4.2. $i_*([\mathbf{R}P(k)]) = \bar{e}_k \in H_k(Q^{-1}(\mathbf{R}P(\infty)_+))$.

This construction easily generalizes to t disjoint spheres $S^{k_1}, \ldots, S^{k_t}$ in S^{n-1} which are invariant in S^{n-1} under the antipodal map to give the map

(4.3)

$$\begin{array}{ccc} \mathbf{R}P(k_1) \times \mathbf{R}P(k_2) \times \cdots \times \mathbf{R}P(k_t) & \xrightarrow{\quad i \quad} & Q^{-t}(\mathbf{R}P(\infty)_+) \\ & \searrow \qquad \nearrow & \\ & \mathrm{Map}_{\mathbf{Z}/2}^{1-2t}(S^{n-1}, S^{n-1}) & \end{array}$$

for $\Sigma_{i=1}^t k_i \le n-1$ such that

$$i_*([\mathbf{R}P(k_1) \times \cdots \times \mathbf{R}P(k_t)]) = \bar{e}_{k_1} * \cdots * \bar{e}_{k_t}$$

in $H_*(Q^{-t}(\mathbf{R}P(\infty)_+))$. More importantly we have:

THEOREM 4.4. *The image of* $i(\mathbf{Z}/2, n)_*$: $H_*(SG(\mathbf{Z}/2, n)) \to H_*(SG(\mathbf{Z}/2))$ *contains the elements* $y_{r+k,k} = Q^{r+k}(\bar{e}_K)*(e_0)^2*1$ *for* $k < n-1$ *where the normal bundle* $\nu(\mathbf{R}P(k) \to \mathbf{R}P(n-1))$ *of the standard embedding* $\mathbf{R}P(k) \to \mathbf{R}P(n-1)$ *has* $r+1$ *linearly independent sections.*

Proof. Suppose there exists a continuous S^r parameter family of disjointly embedded k-spheres that are invariant under the antipodal map. That is, for each $x \in S^r$ we have a $\mathbf{Z}/2$ invariant $S^k(x)$ embedded in S^{n-1} with $S^k(x) \cap S^k(y) = \phi$ for $x \ne y$. We may then construct the map

$$(4.5) \quad f\colon E_p^r(\mathbf{R}P(k)) = S^r \times_{\mathbf{Z}/2} \mathbf{R}P(k) \times \mathbf{R}P(k) \to Q^{-2}(\mathbf{R}P(\infty)_+)$$

which represents $Q_r(\bar{e}_k) = Q^{r+k}(\bar{e}_k) \in H_{r+2k}(Q^{-2}(\mathbf{R}P(\infty)_+))$ as follows: $f(s, l_1, l_2)$ is the self map of S^{n-1} determined by $\{x_1, -x_1\}$, $\{x_2, -x_2\}$ where $\pm x_1, \pm x_2$ are determined by $l_1 \in S^k(s) \subset S^{n-1}$ and $l_2 \in S^k(-s) \subset S^{n-1}$. Here $E_p^r(\mathbf{R}P(k))$ is the well known quadratic construction [7], [11].

Of course for the map $f\colon Ep^r(\mathbf{R}P(k)) \to Q^{-2}(\mathbf{R}P(\infty)_+)$ to actually factor through $\mathrm{Map}_{\mathbf{Z}/2}(S^{n-1}, S^{n-1})$ we must have the S^r parameterized family of

$\mathbf{Z}/2$ free disjoint embedded k-spheres contained in S^{n-1}. If

$$\nu(\mathbf{R}P(k) \to \mathbf{R}P(n-1))$$

has $r+1$ linearly independent sections the required family

$$S^r \times \mathbf{R}P(k) \subset S(\mathbf{R}^r) \subset \nu(\mathbf{R}P(k) \to \mathbf{R}P(n-1))$$

clearly exists.

Note that as

$$\nu(\mathbf{R}P(k) \to \mathbf{R}P(n-1)) \cong (n-k-1)H$$

where H is the canonical line bundle it follows that for r less than $\dim \nu - \dim \mathbf{R}P(k) = n-k-1-k = n-2k-1$ such a family

$$S^r \times \mathbf{R}P(k) \subset \nu(\mathbf{R}P(k) \to \mathbf{R}P(n-1))$$

always exist. We may iterate this construction to obtain:

THEOREM 4.6. *$y_{I,k}$ is in the image*

$$i(\mathbf{Z}/2, n)_*\colon H_*(SG(\mathbf{Z}/2, n)) \to H_*(SG(\mathbf{Z}/2))$$

for $2^{l(I)-1}(n-1) \geq |y_{I,k}| + 1$.

Proof. When $l(I) = 1 |y_{r+k,k}| = r + 2k$ and we have seen that $\nu(\mathbf{R}P(k) \to \mathbf{R}P(n-1))$ always has $r+1$ linearly independent sections for $n-2k-1 \geq r+1$ or equivalently $n-1 \geq r+2k+1$. In general write

$$y_{I,k} = Q_{r_e} Q_{r_{e-1}} \cdots Q_{r_1}(\bar{e}_k) * (e_0)^{2^{l(I)-1}} * 1.$$

As above there is an embedding

$$f_1\colon M_1 = Ep^{r_1}(\mathbf{R}P(k)) \to \mathbf{R}P(n-1) \times \mathbf{R}P(n-1) - \Delta/\mathbf{Z}/2$$

sending points of M_1 to unordered disjoint pairs of antipodal points in S^{n-1}, equivalently to unordered disjoint points in $\mathbf{R}P(n-1)$. If the normal bundle of f_1 has $r_2 + 1$ linearly independent sections then there is a map

$$f_2\colon M_2 = Ep^{r_2}(M_1) \to (\mathbf{R}P(n-1))^4 - \Delta/\mathbf{Z}/2 \wr \mathbf{Z}/2$$

sending points of M_2 to unordered four tuples of points in $\mathbf{R}P(n-1)$. This will always occur if

$$r_2 < \dim \nu(f_1) - \dim(M_1) = 2(n-1) - (r_1 + 2k) - (r_1 + 2k)$$

or equivalently

$$2(n-1) \geq r_2 + 2r_1 + 4k + 1 = \dim\left(Q_{r_2}Q_{r_1}(e_k)\right) + 1.$$

Iterating this construction we obtain maps

$$f_s\colon M_s = Ep^{r_s}(M_{s-1}) \to (\mathbf{R}P(n-1))^{2^s-1} - \Delta/^s \wr \mathbf{Z}/2$$

of the iterated quadratic construction into 2^{s-1} unordered disjoint points if $\mathbf{R}P(n-1)$ whenever $2^{s-1}(n-1) - 2\dim(M_{s-1}) \geq r_s + 1$ or equivalently

$$2^{s-1}(n-1) \geq \dim(M_s) + 1.$$

As the inequality $2^{s-1} \geq \dim(M_s) + 1$ implies $2^{t-1}(n-1) \geq \dim(M_t) + 1$ for $s > t$ it follows that if $2^{l(I)+1}(n-1) \geq |y_{I,k}| + 1$ all the requisite normal bundles will have sufficiently many linearly independent sections to make the construction at every stage. The theorem follows.

The following theorem, which is proved in the next section, is critical to our analysis.

THEOREM 4.7. *Suppose $0 \leq i, j < n$ and $i + j \geq n$. Then*

$$\left[\bar{e}_i * \bar{e}_j + \Sigma(i+j-n-k, n-k)Q^{i+j-k}(\bar{e}_k)\right]^*(e_0)^{2*1}$$

is in the image of

$$i(\mathbf{Z}/2, 2n+1)_*\colon H_*(SG(\mathbf{Z}/2, 2n+1)) \to H_*(SG(\mathbf{Z}/2)).$$

Here (a, b) is the binomial coefficient $C(a+b, a)$. Theorem 4.7 implies that the Hodgson "phenomena" of 3.4 occurs for all values of n.

COROLLARY 4.8. $\sigma_*(Q^{n+k}(-1) * Q^{n+k}(-1) * [5])$ *is **not** in the image of the forgetful map f: $H_*(BSG(\mathbf{Z}/2, 2n+1)) \to H_*(BSG(\mathbf{Z}/2))$. Thus these classes are homology obstructions to endowing a $2n$ dimensional spherical fibration with a free fibre preserving involution.*

Corollary 4.8 will be proved in Section 6.

Section 5

In this section we prove Theorem 4.7. Let $i, j \leq n$. Recall that when $i + j < n$ we can construct a homology class in $H_*(\mathrm{Map}_{\mathbf{Z}/2}(S^n, S^n))$ which represents $\bar{e}_i * \bar{e}_j$ in $H_*(Q(\mathbf{R}P_+^\infty)) \cong H_*(G(\mathbf{Z}/2))$ under the natural inclusion i_* as follows:

Let α: $S^i \to S^n$ and β: $S^j \to S^n$ be disjoint $\mathbf{Z}/2$ equivariant geodesic embeddings. Then the map

$$(5.1)\quad \omega: \mathbf{R}P(i) \times \mathbf{R}P(j) \xrightarrow{\alpha\times\beta} A_{2,n} \xrightarrow{\mathcal{S}} \mathrm{Map}_{\mathbf{Z}/2}(S^n, S^n) \to Q(\mathbf{R}P_+^\infty)$$

sends the fundamental class of $\mathbf{R}P(i) \times \mathbf{R}P(j)$ to $\bar{e}_i * \bar{e}_j$. Here $A_{2,n}$ is the space of two pairs of disjoint antipodal points in S^n and $\mathcal{S}$ is the canonical map defined in Section 4.

Now suppose $i + j = n$. Then it is impossible to find two $\mathbf{Z}/2$-equivariant geodesic embeddings α: $S^i \to S^n$ and β: $S^j \to S_n$ that are *disjoint*. We can however choose α and β so they intersect transversally at $S^0 = \{p, -p\}$. Thus restricting the map ω to $\mathbf{R}P(i) \times \mathbf{R}P(j) - D_\varepsilon^i[\bar{p}] \times D_\varepsilon^j[\bar{\bar{p}}]$ where $\bar{p} = \alpha^{-1}(p)$, $\bar{\bar{p}} \in \beta^{-1}(p)$ and $D_\varepsilon(x)$ is an ε neighborhood of the point x we obtain a map

$$(5.2)\quad \psi: \mathbf{R}P(i) \times \mathbf{R}P(j) - D_\varepsilon^i[\bar{p}] \times D_\varepsilon^j[\bar{\bar{p}}] \xrightarrow{\alpha\times\beta} A_{2,n} \xrightarrow{\mathcal{S}} \mathrm{Map}_{\mathbf{Z}/2}(S^n, S^n) \to Q(\mathbf{R}P_+^\infty).$$

PROPOSITION 5.3. *ψ extends to a map*

$$\phi: \mathbf{R}P(i) \times \mathbf{R}P(j) \# \mathbf{R}P(i+j) \xrightarrow{\overline{\alpha\times\beta}} A_{2,n} \xrightarrow{\mathcal{S}} \mathrm{Map}_{\mathbf{Z}/2}(S^n, S^n) \to Q(\mathbf{R}P_+^\infty)$$

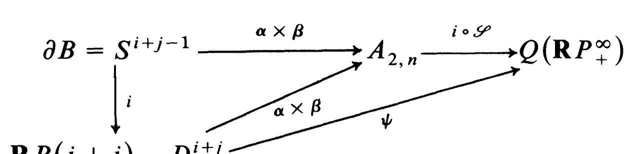
such that

$$\phi_*([\mathbf{R}P(i) \times \mathbf{R}P(j) \# \mathbf{R}P(i+j)]) = \bar{e}_i * \bar{e}_j + Q^{i+j}(\bar{e}_0)$$

in $H_*(G(\mathbf{Z}/2))$.

Proof. Let B denote $D_\varepsilon^i[\bar{p}] \times D_\varepsilon^j[\bar{\bar{p}}]$. Clearly $B \simeq D^{i+j}$ and ψ restricted to $\partial B = S^{i+j-1}$ is well defined. We can also regard ∂B as the boundary of $\mathbf{R}P(i+j) - \mathring{D}^{i+j}$. We wish therefore to extend $\alpha \times \beta$ to obtain the following commutative diagram:

(5.4)



Then $\psi \# \bar{\psi} = \phi$ will be our desired map.

As $\alpha \times \beta$ restricted to ∂B factors through $\hat{A}_{p,n}$, the space of two distinct points near p in S^n, it suffices to define $\alpha \times \beta$: $\mathbf{R}P(i+j) - \mathring{D}^{i+j} \to \hat{A}_{p,n}$.

Recall $\mathbf{R}P(i+j) - \{p\}$ can be thought of as $\mathbf{R}^{i+j}$ "blown up at the origin"; that is, the space of pairs (x, l) where l is a line in $\mathbf{R}^{i+j}$ and x is a vector in l. Thus we have the following model for $\mathbf{R}P(i+j) - \mathring{D}^{i+j}$:

$$(5.5) \quad \mathbf{R}P(i+j) - \mathring{D}^{i+j} = \{(x,l) | x \in l,\ l \text{ a line in } \mathbf{R}^{i+j}, \text{ and } |x| \leq 1\}.$$

Let ρ: $D^{i+j} \to B = D^i \times D^j$ be the homeomorphism

$$\rho(u,v) = \frac{1}{r}(u,v)$$

where $(u,v) \in \mathbf{R}^i \times \mathbf{R}^j$, $\|u\|^2 + \|v\|^2 \leq 1$, and $r = \max(\|u\|, \|v\|)$. Notice that $\alpha \times \beta$ is well defined on $B - \{\bar{p} \times \bar{\bar{p}}\} = \rho(D^{i+j} - \{\vec{0}\})$.

When $\|u\|^2 + \|v\|^2 = 1$ the formula $\alpha \times \beta(\rho(u,v))$ defines $\alpha \times \beta$ on ∂D^{i+j}. We extend $\alpha \times \beta$ to all of $\mathbf{R}P(i+j) - \mathring{D}^{i+j}$ in the following way.

We may assume, possibly after a diffeomorphism of S^n that is the identity off $\alpha \times \beta(B)$, that α and β are the transverse coordinate embeddings

$$\alpha: D^i \times 0 \to D^i \times D^j \simeq U \text{ and } \beta: 0 \times D^j \to D^i \times D^j \simeq U$$

where we have identified a neighborhood U of p containing $\alpha \times \beta(B)$ with $D^i \times D^j$. For any $\rho(u,v) \in \partial B$ if $\alpha \times \beta(\rho(u,v)) = (a,b)$ then

$$\alpha \times \beta(\rho(-u,-v)) = \alpha \times \beta(-\rho(u,v)) = (-a,-b).$$

Thus there is a well-defined line, $l_{u,v}$, through the origin in $D^i \times D^j = U$ that is parallel to the vector $b - a$ and the vector $-b - (-a)$. Denote by $\pm y_{u,v}$ the two points where $l_{u,v}$ intersects ∂U. There is a homotopy

$$H: \partial(D^i \times D^j) \times [0, 1/2] \to \hat{A}_{P,n}$$

such that

$$(5.6) \qquad H_0 = \alpha \times \beta \text{ and } H_{1/2}(\rho(u,v)) = \{y_{u,v}, -y_{u,v}\} \in \hat{A}_{P,n}.$$

We then define $\overline{\alpha \times \beta}$ on $\mathbf{R}P(i+j) - \mathring{D}^{i+j}$ to be

$$(5.7) \quad \overline{\alpha \times \beta}(x,l) = \begin{cases} H_{1-\|x\|}(\rho(x)) & \text{if } 1 \geq \|x\| \geq 1/2 \\ H_{1/2}(\rho(u_0,v_0)) = \{\pm y_{u_0,v_0}\} & \text{if } \|x\| \leq 1/2. \end{cases}$$

Here (u_0, v_0) is a vector of length 1 in l (by construction, $H_{1/2}(\rho(u_0,v_0)) = H_{1/2}(\rho(-u_0,-v_0)))$. It is an easy verification to show that $\overline{\alpha \times \beta}$ is a well-defined extension of $\alpha \times \beta$. Thus $\psi \# \bar{\psi} = \phi$ is a map from

$$\mathbf{R}P(i) \times \mathbf{R}P(j) \# \mathbf{R}P(i+j) \to Q(\mathbf{R}P_+^\infty)$$

that factors through $\mathrm{Map}_{\mathbf{Z}/2}(S^n, S^n)$. To complete the proof of 5.3 (and hence 4.7 when $i+j=n$) we must check what ϕ_* does to the fundamental class of $\mathbf{R}P(i) \times \mathbf{R}P(j) \# \mathbf{R}P(i+j)$.

Consider the commutative diagram

$$\begin{array}{ccc} \mathbf{R}P(i) \times \mathbf{R}P(j) \# \mathbf{R}P(i+j) & \xrightarrow{\phi} & A_{2,i+j} \\ & \searrow^{i \circ \phi} & \downarrow i \\ & & A_{2,i+j+1} \end{array} \tag{5.8}$$

where i is induced by the equatorial inclusion.

In S^{i+j+1} we may equivariantly push $\alpha(S^i)$ and $\beta(S^j)$ off each other. Of course we also have the map $f\colon \mathbf{R}P(i) \times \mathbf{R}P(j) \to A_{2,i+j+1}$ defined as in 5.1.

LEMMA 5.9. *There is a cobordism between the maps $i \circ \phi$ and $f \amalg g$ where g: $\mathbf{R}P(i+j) \to A_{2,i+j+1}$ factors through*

$$\{\textit{pairs of points}\, \{x, y\} \mid x \neq y,\, x, y \in D_\epsilon^{i+j+1}(p)\} \subset \hat{A}_{P,i+j+1}.$$

Proof. $i \circ \phi$ is clearly deformable to f when restricted to

$$\begin{aligned} &\mathbf{R}P(i) \times \mathbf{R}P(j) \# \mathbf{R}P(i+j) - (\mathbf{R}P(i+j) - \mathring{D}^{i+j}) \\ &\quad \simeq \mathbf{R}P(i) \times \mathbf{R}P(j) - \mathring{D}^{i+j}. \end{aligned}$$

Let H be such a homotopy. Then

$$(i \circ \phi \times \mathrm{id}_I) \cup H \cup (f \times \mathrm{id}_I) \tag{5.10}$$

with $(i \circ \phi \times \mathrm{id}_1)$ restricted to $\mathbf{R}P(i) \times \mathbf{R}P(j) - \mathring{D}^{i+j}$ identified with H_0 and $(f \times \mathrm{id}_0)$ restricted to $\mathbf{R}P(i) \times \mathbf{R}P(j) - \mathring{D}^{i+j}$ identified with H_1 is the required cobordism.

LEMMA 5.11. *The map g given in Lemma 5.9 is homotopic to*

$$\bar{g}\colon \mathbf{R}P(i+j) \to A_{2,i+j+1}$$

where $\bar{g}$ factors through $\{$pairs $\{x, -x\} \mid x, -x \in D_\varepsilon^{i+j+1}(p)\}$.

Proof. Trivial.

As $(i \circ \mathscr{S} \circ g)_*[\mathbf{R}P(i+j)] = Q^{i+j}(\bar{e}_0)$, 5.9 and 5.10 imply

$$\phi_*[\mathbf{R}P(i) \times \mathbf{R}P(j) \# \mathbf{R}P(i+j)] = \bar{e}_i * \bar{e}_j + Q^{i+j}(\bar{e}_0).$$

This finishes the proof of 5.3.

Proof of 4.7. When $i + j = n$, 4.7 is just 5.3. Assume $i + j > n$. Then an analogous construction to 5.2 gives a map

$$\tau\colon \mathbf{R}P(i) \times \mathbf{R}P(j) - N \to A_{2,n} \tag{5.12}$$

where N is a neighborhood of the diagonal embedding of $\mathbf{R}P(i+j-n) \to \mathbf{R}P(i) \times \mathbf{R}P(j)$ (recall $i, j < n$).

Just as in 5.3, the map τ can be extended to $\bar{\tau}$: ($\mathbf{R}P(i) \times \mathbf{R}P(j)$ blown up along

$$\left.\mathbf{R}P(i+j-n) \xrightarrow{\Delta} \mathbf{R}P(i) \times \mathbf{R}P(j)\right) \to A_{2,n}$$

such that $\bar{\tau}$ is cobordant to $f \amalg g$ where

$$f\colon \mathbf{R}P(i) \times \mathbf{R}P(j) \to Q(\mathbf{R}P^{\infty}_{+})$$

with

$$f_*[\mathbf{R}P(i) \times \mathbf{R}P(j)] = \bar{e}_i * \bar{e}_j$$

and g is homotopic to

$$\bar{g}\colon (\text{Blow-up of } \mathbf{R}P(i+j-n) \to \mathbf{R}P(i) \times \mathbf{R}P(j)) \to A_{2,n}$$

which factors through

$$\{\text{pairs of antipodal points } \pm(x, y),\ \pm(x, -y),\ x \in S^{i+j-n},\ y \in S^n\}.$$

But this space is the product of $\mathbf{Z}/2$ sets

$$\{(x, -x) | x \in S^{i+j-n}\} \times \{(y, -y) | y \in S^n\} = A \times B$$

regarded as a $\mathbf{Z}/2$ set via the diagonal embedding. As A represents $\bar{e}_{i+j-n}$ and B represents $\bar{e}_n$ in homology, [10] implies g_* sends the fundamental class to

$$\mathrm{tr}_\Delta(\bar{e}_{i+j-n} \otimes \bar{e}_n) = \Sigma(i+j-n-k, n-k) Q^{i+j-k}(\bar{e}_k),$$

where tr_Δ is the diagonal transfer induced by $\Delta\colon \mathbf{Z}/2 \to \mathbf{Z}/2 \times \mathbf{Z}/2$.

Section 6

We wish to examine the forgetful maps

$$f(n)_*\colon H_*(SG(\mathbf{Z}/2, n) \to H_*(SG(n))$$

and

$$Bf(n)_*\colon H_*(BSG(\mathbf{Z}/2, n)) \to H_*(BSG(n))$$

in more detail. Results of [9] and Section 2 imply it suffices to analyze $f(n)_*$. We will do this by using the Schultz filtration and results of [9].

We begin by recalling the following result:

THEOREM 6.1.

$$\begin{aligned} H_*(SG(\mathbf{Z}/2)) &\cong H_*(SO) \otimes P[x_{k,k}, y_{I,k}]_{k>0} \otimes P[y_{I,0}] \\ &\cong H_*(SO) \otimes A \otimes B. \end{aligned}$$

Furthermore,

(a) $f_*\colon H_*(SG(\mathbf{Z}/2)) \to H_*(SG)$ *is an isomorphism on* $H_*(SO) \otimes A$ *and*

(b) f_* *is determined by the formula*

$$(y_{I,0}) = \begin{cases} (y_{I',0})^{*2} & \text{is } I = 2I' \\ 0 & \text{otherwise.} \end{cases}$$

Remark. The formula in (b) forces a "twisting" of the kernel of f_* (for example $f_*(Q^n(\bar{e}_n) + Q^{2n}(\bar{e}_0)) = 0$) which becomes important in what follows.

Now, as promised in Section 4, we may prove 4.8.

Proof of 4.8. Theorem 6.1 shows the only monomials in $H_*(BSG(\mathbf{Z}/2))$ that transfer to

$$\sigma_*\left(Q^{n+k}(-1) * Q^{n+k}(-1) * [5]\right)$$

in $H_*(BSG)$ are

$$x = \sigma_*\left(Q^{2(n+k)}(\bar{e}_0) * (e_0)^2 * 1\right) \quad \text{and} \quad y = \sigma_*\left(Q^{n+k}(\bar{e}_{n+k}) * (e_0)^2 * 1\right).$$

But 2.7 and 4.7 show that while $x + y$ is in $H_*(BSG(\mathbf{Z}/2, 2n+1))$ neither x nor y itself is in $H_*(BSG(\mathbf{Z}/2, 2n+1))$. The results follows.

Now consider $SG(\mathbf{Z}/2, n)$. Set

$$E_{i,n} = \{ f \in SG(\mathbf{Z}/2, n) | f|_{S^{n-1-i}} = \text{id} \} \quad \text{for } i < n$$

and

$$E_{n,n} = SG(\mathbf{Z}/2, n).$$

Note that $E_{0,n} = \{\text{id}\}$, $E_{i,n} \subset E_{i+1,n}$, and $\{E_{i,n}\}$ is a filtration of $SG(\mathbf{Z}/2, n)$ by associative H-spaces.

Now grade $H_*(SG(\mathbf{Z}/2, n))$ via the images of $H_*(E_{i,n})$. There is a natural map

$$L: E_{i,n} \to \Omega^{n-i} S^{n-1}$$

defined by associating to $f \in E_{i,n}$ the map f restricted to the upper hemisphere of S^{n-i}. As Schultz observed, the fibre of L is $E_{i-1,n}$. Thus the commutative diagrams

(6.2)
$$\begin{array}{ccccc} E_{i-1,n} & \longrightarrow & E_{i,n} & \xrightarrow{L} & \Omega^{n-i}S^{n-1} \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ E_{i-1,n+1} & \longrightarrow & E_{i,n+1} & \xrightarrow{L} & \Omega^{n+1-i}S^{n-1} \end{array}$$

in conjunction with Lemma 2.1 prove:

LEMMA 6.3. *The following diagram of inclusions commutes*

$$\begin{array}{ccccc} \operatorname{gr} H_*(SG(\mathbf{Z}/2, n)) & \xrightarrow{i_*} & \operatorname{gr} H_*(SG(\mathbf{Z}/2)) & \xrightarrow{\text{id}} & \operatorname{gr} H_*(SG(\mathbf{Z}/2)) \\ \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\ \bigotimes_{i=0}^{n-1} H_*(F_i) & \xrightarrow{i_*} & \bigotimes_{i=0}^{\infty} H_*(Q(S^i)) & \longrightarrow & H_*(SO) \otimes A \otimes B. \end{array}$$

Thus, given

$$y_{I,k}^{(n)} \in \operatorname{gr} H_*(SG(\mathbf{Z}/2, n)),$$

any representative has image in $H_*(SG(\mathbf{Z}/2))$ of the form $y_{I,k} + z$ where z is decomposable in terms of the $e_{k'}$, $x_{k',k'}$ and $y_{I',k'}$ with $k' < k$. Notice that $z \in H_*(SG(\mathbf{Z}/2))$ may not correspond to a graded class in $H_*(SG(\mathbf{Z}/2, n))$.

For example as mentioned at the end of Section 2,

$$J_{\mathbf{Z}/2}(e_{n-1}) = \sum_{i+j=n-1} (\bar{e}_i) * Q^j(e_0) * (\bar{e}_0) * 1 = \bar{e}_{n-1} * (e_0) * 1 + z$$

in $H_*(SG(\mathbf{Z}/2))$ and z contains the monomial $Q^{n-1}(e_0) * (\bar{e}_0)^2 * 1$ which does not correspond to a class in $\operatorname{gr} H_*(SG(\mathbf{Z}/2, n))$. Of course Corollary 4.8 is another manifestation of this phenomena.

Since 6.3 is an isomorphism only up to filtration care must be taken in using 6.1 to compute $f(n)_*$. Fortunately 6.1 and 6.3 do imply that $f(n)$ behaves very well modulo filtration.

We conclude with:

THEOREM 6.4. *$f(2n+1)_* : H_*(SG(\mathbf{Z}/2, 2n+1)) \to H_*(SG(2n+1))$ is completely described as follows:*

(a) *$f(2n+1)_* = \mathrm{id}$ on $H_*(SO(2n+1))$.*

(b) *$f(2n+1)_*(x_{k,k}(n)) = Q^k(1) * Q^k(1) * (-3)$ for $k < n$.*

(c) *$f(2n+1)_*(x_{n+i,n+i}(n)) = 0$ for $i \geq 0$.*

(d) *$f(2n+1)_*(y_{I,k}(n)) = Q^I Q^k(-1) * [1 + 2^{l(I)+1}]$ modulo F_{k-1} for $k > 0$*

(e) *$f(2n+1)_*(y_{I,0}(n))$*

$$= \begin{cases} Q^{I'}(-1) * Q^{I'}(-1) * [1 + 2^{l(I)+1}] & \text{if } I = 2I' \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Parts (a), (b), (d) and (e) follow from 6.1 and 6.3 while (c) follows from the proof of 4.8.

COROLLARY 6.5. *$f(2n+1)_*$ is onto*

$$H_*(SG(2n+1))//\{Q^{n+i}(-1) * Q^{n+i}(-1) * [5]\}_{i \geq 0}.$$

Section 7

We conclude this paper by using results of Section 4 to verify a conjecture made in [8] concerning the representation of Poincaré duality spaces by projective bundles in unoriented bordism. The reader is referred to [8] for relevant definitions, notation, calculational formulae, and theorems quoted in what follows. Let V denote the maximal polynomial subalgebra in $\eta^{\mathrm{PL}}//\eta^{\mathrm{Diff}}$ and recall V injects into $\eta^{\mathrm{PD}}//\eta^{\mathrm{Diff}}$ [3]. The generators of V are indexed by certain Dyer-Lashof operations and it is natural to ask for geometric representatives for these generators. Recall from section one that $BSG(\mathbf{Z}/2, n)$ classifies $\mathbf{Z}/2$ equivariant n-dimensional fibrations. Let (E, τ) be such a fibration. The $\mathbf{P}(E)$, the projective bundle of (E, τ) is defined to be $E/x \sim \tau x$. In [8] we give a method for computing $[\mathbf{P}(E)] \in \eta^{\mathrm{PD}}$ (see [11] for related results). Using Theorem 4.6 we are able to strengthen Theorem C of [8] and prove:

THEOREM 7.1. *Every element of V may be represented by a spherical projective bundle in η^{PD}.*

Proof. (Using the notation of [8]). We must compute the Hurewicz homomorphism $\bar{h}([\mathbf{P}(E)])$ for $(E, \tau) \to BSG(\mathbf{Z}/2, p)$ where $p \leq \dim(E)$ and show we obtain all of V. By induction on dimension it suffices to prove the result modulo decomposables. Thus we are reduced to computing $\bar{A}_{G^*}(e_j, x)$ in $H_*(BSG)$ where $x \in H_*(BSG(\mathbf{Z}/2, p))$ with $j \geq p$. Recall $\bar{A}_{G^*}(e_j, \cdot)$ increases length by 1. Lexicographically order both V and $H_*(BSG)$ by (a) length, (b) dimension, (c) excess, and finally (d) by the natural lexicographic order on $I = (i_1, \ldots, i_k)$ starting on the left. We now work modulo higher weight. Let

$$\sigma_*(S) = \sigma_*(s_1, \ldots, s_{k+1}) = \sigma_*\left(Q^{s_1}Q^{s_2} \cdots Q^{s_{k+1}}(1) * [1 - 2^{k+1}]\right)$$

be an admissible sequence of positive excess in the image of V in $H_*(BSG)$.

The only admissible sequence of positive excess with $s_{k+1} = 1$ is

$$x_1 = \sigma_*(2^k, 2^{k-1}, \ldots, 2, 1).$$

But

$$y_1 = \sigma_*\left(Q^{2^{k-1}}Q^{2^{k-2}} \cdots Q^1(e_0) * (\bar{e}_0)^{2^{k-1}} * 1\right)$$

in $H_*(BSG(\mathbf{Z}/2))$ is actually in $H_*(BSG(\mathbf{Z}/2, 3))$ by 4.6 and direct computation shows $\bar{A}_{G_*}(e_{2^k}, y_1) = x_1$.

Now suppose $s_{k+1} > 1$. Consider the sequence

$$T_n = (n \cdot 2^k, n \cdot 2^{k-1}, \ldots, n \cdot 4, n \cdot 2, n \cdot 2).$$

Choose n to be the largest integer such that $2n \leq s_{k+1}$ (recall $s_{k+1} > 1$). As S is admissible it follows that the sequence $S - T_n$ is also admissible. Furthermore excess(S) = excess$(S - T_n)$. Now 4.6 implies that $\sigma_*(y_{S-T_n})$ is in $H_*(BSG(\mathbf{Z}/2, p))$ for $p = \sum_{i=1}^{k+1}(s_i - t_i)$. Furthermore, by our choice of n and the fact that S is admissible of positive excess, it follows that

$$\sum_{i=1}^{k+1} t_i > \sum_{i=1}^{k+1} (s_i - t_i) = p.$$

However $\Sigma_{i=1}^{k+1} t_i = n \cdot 2^{k+1}$ and thus

$$(7.2) \qquad \bar{A}_{G_*}\left(e_{2^{k+1}n}, \sigma_*(y_{S-T_n})\right) = \sigma_*(S) + \Sigma x_I$$

where the sum Σx_I runs over certain terms of excess strictly less than excess(S). The theorem follows.

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