ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 4, Winter 1985

THE CARDINALITY OF THE SET OF LEFT INVARIANT MEANS ON A LEFT AMENABLE SEMIGROUP

BY

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1. Introduction

The study of the cardinality of the set of invariant means on a (discrete) group essentially goes back to Banach ([1], [2]) who showed that there exist at least two invariant means on the circle group. In his famous paper [5] of 1957, Day continued the study, showing that many infinite amenable groups admit more than one left invariant mean. Following progress made by Granirer, C. Chou ([4]) obtained in 1976 the following definitive result: *the cardinality of the set* $\mathfrak{L}(G)$ *of left invariant means on an infinite amenable group* G *is*

$$2^{2^{|G|}} (= |l_\infty(G)'|!)$$

where $|E|$ is the cardinality of a set E. The method used by Chou has become canonical. The idea (expressed precisely in (3.1)) is to construct a "large" disjoint family $\mathscr{A}$ of subsets of G, each of which supports a left invariant mean, then to close up these subsets and their complements in the Stone-Cěch compactification βG, and then to find an even larger family of left invariant means supported on intersections of translates of these sets. This procedure is also followed in the present paper which deals with the result corresponding to Chou's for a left amenable semigroup S.

The semigroup case is substantially more difficult than the group case owing to former's more complicated multiplication structure.

Luthar [16] obtained the first positive result for the semigroup case: *if* S *is abelian, then* S *has more than one invariant mean if and only if it does not contain a finite ideal.* Granier [8]–[10] showed, along with other results, that $|\mathfrak{L}(S)|$ is infinite if S is infinite, left amenable and left cancellative. Chou [3] showed that if S is infinite and cancellative, then $|\mathfrak{L}(S)| \geq 2^{\aleph_0}|S|$. The work of Granirer [8], [9] and Klawe [14] led to the following definitive result dealing with the case where the span $\mathfrak{J}_l(S)$ of the set of left invariant means on S is

Received November 24, 1982.

[1] The author gratefully acknowledges financial support from the National Science and Engineering Research Council of Canada, the Royal Society and the Carnegie Trust for the Universities of Scotland.

finite dimensional: *the space* $\mathfrak{F}_l(S)$ *is n-dimensional* $(n < \infty)$ *if and only if* S *contains exactly n finite, left ideal groups*.

This leaves the case where $\dim \mathfrak{F}_l(S)$ is infinite dimensional to be dealt with. It is convenient, for this case, to deal with the cardinal $|\mathfrak{L}(S)|$. In her paper [15], Maria Klawe introduced the cardinal $\kappa l(S)$, where

$$\kappa l(S) = \min\{|B| : B \subset S, \mathbf{m}(B) = 1 \text{ for all } \mathbf{m} \in \mathfrak{L}(S)\}.$$

Theorem 2.6 of [15] asserts that $\dim \mathfrak{F}_l(S)$ is infinite if and only if $\kappa l(S)$ is infinite, and that if $\kappa l(S)$ is infinite, then $|\mathfrak{L}(S)| = 2^{2^{\kappa l(S)}}$ $(= \dim \mathfrak{F}_l(S))$. However, there is a set theoretic difficulty in the proof (on p. 238), and this seems to be irredeemable. (We will show in (3.9) that the above Theorem 2.6 is true in many cases.)

An intrinsic difficulty with the cardinal $\kappa l(S)$ is that in order to calculate it in general, we need to have detailed information about every (!) left invariant mean $\mathbf{m}$ in order to know which B's will give $\mathbf{m}(B) = 1$. (As Klawe shows, however, the situation is much better when S is amenable.) It is thus natural to look for another cardinal, defined in terms of the algebraic structure of the semigroup, to replace $\kappa l(S)$.

Such a cardinal $\mathfrak{m}$ is introduced in §2: we define

$$\mathfrak{m} = \min\left\{\left|\bigcup_{i=1}^{n} s_i S_i\right| : n \geq 1, \{S_1, \ldots, S_n\} \text{ is a partition of } S,\ s_1, \ldots s_n \in S\right\}.$$

In many cases ((2.6)), $\mathfrak{m}$ equals the simpler cardinal $\mathfrak{p}$, where $\mathfrak{p} = \min\{|sS| : s \in S\}$. Our main theorem ((3.6)) asserts that *if* $\mathfrak{m}$ *is finite, then* $\dim \mathfrak{F}_l(S) < \infty$, *while if* $\mathfrak{m}$ *is infinite, then* $|\mathfrak{L}(S)| = 2^{2^{\mathfrak{m}}}$ $(= \dim \mathfrak{F}_l(S))$.

The problem of determining $|\mathfrak{L}(S)|$ is a special case of the following more general, naturally occurring problem. Suppose that the left amenable semigroup S has a left action on a set X. A simple application of Day's fixed-point theorem shows that the set $\mathfrak{L}(X)$ of S-invariant means on X is not empty. *What is the cardinality of* $\mathfrak{L}(X)$? We obtain partial progress with this problem, introducing the cardinal $\mathfrak{m}(S, X)$ which is defined along the same lines as $\mathfrak{m}$ above, the partition $\{S_1, \ldots, S_n\}$ of S being replaced by a partition $\{X_1, \ldots, X_n\}$ of X. We show in (3.3) that *if* $\mathfrak{m}(S, X)$ *is infinite and* $|S| \leq \mathfrak{m}(S, X)$, *then* $|\mathfrak{L}(X)| = 2^{2^{\mathfrak{m}(S,X)}}$. We have been unable to remove the requirement $|S| \leq \mathfrak{m}(S, X)$ from the result. (However, as (3.6) shows, when $X = S$, then this requirement is unnecessary.)

The main technical proof of the paper is that of (3.2). We claim that this argument is paradigmatic for constructing large sets of invariant means, and justify this claim by showing how other known cardinality results for sets of invariant means can be derived using such a proof.

2. Left invariant means and the a.l.c. condition

Let S be a semigroup and X be a left S-set, i.e. there is a left action $(s, x) \to sx$ of S on X. A *mean* on X is an element $\mathbf{m} \in l_\infty(X)'$ with $\mathbf{m}(1) = 1 = \|\mathbf{m}\|$. There is a right action $(\phi, s) \to \phi s$ of S on $l_\infty(X)$ given by $\phi s(x) = \phi(sx)$ $(x \in X)$. The dual left action of S on $l_\infty(X)'$ preserves the set of means on X, and such a mean is called (*left*) *invariant* if $s\mathbf{m} = \mathbf{m}$ $(s \in S)$. Thus a mean $\mathbf{m}$ on X is invariant if and only if $\mathbf{m}(\phi s) = \mathbf{m}(\phi)$ for all $\phi \in l_\infty(X)$, $s \in S$. The set of left invariant means on X is denoted by $\mathfrak{L}(S, X)$, or simply, by $\mathfrak{L}(X)$. The space $\{\mathbf{p} \in l_\infty(X)': s\mathbf{p} = \mathbf{p}$ for all $s \in S\}$ of left invariant functionals in $l_\infty(X)'$ is denoted by $\mathfrak{I}_l(S, X)$.

Of particular interest is the case where $X = S$ and S acts by left multiplication. The semigroup S is said to be *left amenable* if $\mathfrak{L}(S) \neq \varnothing$. *Right amenability* and (*two-sided*) *amenability* for S are defined in the obvious ways.

Now let X be as in the first paragraph above. The well-known fixed-point theorem of Day yields that if S is left amenable, then $\mathfrak{L}(X) \neq \varnothing$.

Every mean $\mathbf{m}$ on X can be realised as a finitely additive, positive measure of total mass 1 on X: simply write $\mathbf{m}(E) = \mathbf{m}(\chi_E)$ $(E \subset X)$. The mean $\mathbf{m}$ is left invariant if and only if

$$\mathbf{m}(E) = \mathbf{m}(s^{-1}E) \text{ for all } E \subset X,\ s \in S,$$

where $s^{-1}E = \{x \in X: sx \in E\}$.

The subsets of X which support a left invariant mean admit a well-known, elegant, algebraic characterisation which we now state. A proof can be given along the lines of [6, Theorem 7.4].

(2.1) PROPOSITION. *Let S be left amenable and $E \subset X$. Then there exists $\mathbf{m} \in \mathfrak{L}(X)$ with $\mathbf{m}(E) = 1$ if and only if E is left thick, i.e. given $F \in \mathscr{F}(S)$, the family of finite subsets of S, there exists $x \in X$ such that $Fx \subset E$.*

(2.2) DEFINITION. The set X is said to be *almost left cancellative* (a.l.c.) (for S) if, whenever $n \geq 1$, $\{X_1, \dots, X_n\}$ is a partition of X and $s_1, \dots, s_n \in S$, then

$$\left|\bigcup_{i=1}^{n} s_i X_i\right| = |X|. \tag{1}$$

(Here, $|E|$ is the cardinal of a set E.)

To justify this nomenclature, suppose that X is a.l.c. and let $s \in S$. Applying (1) with $\{X_1, \dots, X_n\} = \{X\}$, we have $|sX| = |X|$. So the action of S on X is "close" to the left cancellative case in the sense that multiplication of X by s does not "collapse" X too much.

Note that if the action of S on X is left cancellative (in the sense that $s = t$ if $sx = tx$ for some $x \in X$) and X is infinite, then, obviously, X is a.l.c..

The semigroup S is said to be a.l.c. if it is a.l.c. with respect to the left multiplication action of S on itself. We will return to the a.l.c. condition in this case later in the section. For the present, we introduce an important cardinal associated with the condition.

(2.3) DEFINITION. The cardinal $\mathfrak{m}(S, X)$, or simply $\mathfrak{m}$, is defined by

$$\mathfrak{m}(s, X) = \min\left\{ \left| \bigcup_{i=1}^{n} s_i X_i \right| : n \geq 1, \{ X_1, \dots, X_n \} \right.$$
$$\left. \times \text{ is a partition of } X, s_1, \dots, s_n \in S \right\}.$$

The significance of the cardinal $\mathfrak{m}$ is that if $\mathfrak{m}$ is infinite, then, in certain circumstances, $|\mathfrak{L}(X)| = 2^{2^{\mathfrak{m}}}$ (See (3.3).) If X is a.l.c., then $\mathfrak{m} = |X|$.

If $Y \subset X$, let $S_Y = \{s \in S\colon sY \subset Y\}$. Obviously, S_Y (if non-empty) is a subsemigroup of S, and Y is a left S_Y-set. If $S_Y = \varnothing$, then we take $\mathfrak{L}(S_Y, Y)$ to be the set of means on Y.

The next result enables us to reduce to the a.l.c. case.

(2.4) PROPOSITION. *Let $n \geq 1$, $\{ X_1, \dots, X_n\}$ be a partition of X and $s_1, \dots s_n \in S$ be such that $|A| = \mathfrak{m}$, where $A = \bigcup_{i=1}^{n} s_i X_i$. Let Y be such that $A \subset Y \subset X$ and $|Y| = \mathfrak{m}$.*

(i) *If $S_Y \neq \varnothing$, then Y is a.l.c. for S_Y, and $\mathfrak{m}(S_Y, Y) = \mathfrak{m}$.*

(ii) *$\mathbf{m}(Y) > 0$ for all $\mathbf{m} \in \mathfrak{L}(X)$, and the map $\mathbf{m} \to (\mathbf{m}_{|Y})/\mathbf{m}(Y)$ is one-to-one from $\mathfrak{L}(X)$ into $\mathfrak{L}(S_Y, Y)$; further, the map $\mathbf{p} \to \mathbf{p}_{|Y}$ is a one-to-one, linear map from $\mathfrak{J}_l(S, X)$ into $\mathfrak{J}_l(S_Y, Y)$.*

(iii) $|\mathfrak{L}(X)| \leq |\mathfrak{L}(S_Y, Y)|$.

Proof. (i) Let $\{Y_1, \dots, Y_m\}$ be a partition of Y, and $t_1, \dots, t_m \in S_Y$. For $1 \leq i \leq n$, $1 \leq j \leq m$, let $A_{ij} = X_i \cap s_i^{-1} Y_j$. Suppose that $A_{ij} \cap A_{kl} \neq \varnothing$. Since $\{X_1, \dots, X_n\}$ is disjoint, we have $i = k$, and since $s_j^{-1} Y_j \cap s_i^{-1} Y_l = s_i^{-1}(Y_j \cap Y_l)$ and $\{Y_1, \dots, Y_m\}$ is disjoint, we have $j = l$. Further, if $x \in X$, then $x \in X_{i'}$ for some i', and $s_{i'}x \in Y_{j'}$ for some j' (since $s_{i'}X_{i'} \subset A \subset Y$), so that $s \in A_{i'j'}$. It follows that

$$\{ A_{ij}\colon 1 \leq i \leq n, 1 \leq j \leq m \}$$

is a partition of X, and so

$$\mathfrak{m} \leq \left| \bigcup_{i,j} t_j s_i A_{ij} \right| \leq \left| \bigcup_{j=1}^{m} t_j Y_j \right| \leq |Y|.$$

It follows that $\mathfrak{m}(S_Y, Y) = |Y| = \mathfrak{m}$ as required.

(ii) Let $\mathbf{m} \in \mathfrak{L}(X)$. For each i,

$$\mathbf{m}(Y) \geq \mathbf{m}(A) \geq \mathbf{m}(s_i X_i) = \mathbf{m}\big(s_i^{-1}(s_i X_i)\big) \geq \mathbf{m}(X_i).$$

Since $\sum_{i=1}^{n} \mathbf{m}(X_i) = \mathbf{m}(X) = 1$, we must have $\mathbf{m}(Y) > 0$. Let $\mathbf{m}_Y = \mathbf{m}|_Y/\mathbf{m}(Y)$. As in the semigroup case [5, pp. 518–519], $\mathbf{m}_Y \in \mathfrak{L}(S_Y, Y)$. We now show that the map $\mathbf{m} \to \mathbf{m}_Y$ is one-to-one.

Suppose that $\mathbf{m}, \mathbf{n} \in \mathfrak{L}(X)$ are such that $\mathbf{m}_Y = \mathbf{n}_Y$. Now $\mathfrak{I}_l(S, X)$ is an abstract L-space (cf. [6, p. 9]) under the canonical ordering. Let $\mathbf{p} = \mathbf{n}(Y)\mathbf{m} - \mathbf{m}(Y)\mathbf{n}$. Then $\mathbf{p} \in \mathfrak{I}_l(S, X)$ and $\mathbf{p}$ vanishes on $l_\infty(Y)$. So for $\phi \geq 0$ in $l_\infty(Y)$,

$$|\mathbf{p}|(\phi) = \sup\{|\mathbf{p}(\psi)| : \psi \in l_\infty(Y), 0 \leq |\psi| \leq \phi\} = 0. \tag{2}$$

If $|\mathbf{p}| \neq 0$, then $|\mathbf{p}| = k\mathbf{r}$ for some $\mathbf{r} \in \mathfrak{L}(X)$, $k > 0$, and since $\mathbf{r}(Y) > 0$, we would have $|\mathbf{p}|(Y) > 0$, contradicting (2). So $|\mathbf{p}| = 0$, and hence $\mathbf{p} = 0$. Thus $\mathbf{n}(Y)\mathbf{m} = \mathbf{m}(Y)\mathbf{n}$, and evaluating at X gives $\mathbf{m} = \mathbf{n}$ as required. The proof of the second assertion of (ii) is similar.

(iii) This is an immediate consequence of (ii). □

The next result shows that when $\mathfrak{m}$ is infinite, the set Y above can be taken to be left thick. This result, for $X = S$, follows from [14].

(2.5) PROPOSITION. *Let $\mathfrak{m}$ be infinite and S be left amenable. Then there exists a left thick subset Y of X such that $A \subset Y$ and $|Y| = \mathfrak{m}$.*

Proof. Let $\mathbf{m} \in \mathfrak{L}(X)$, and

$$k = \sup\{\mathbf{m}(RA) : R \text{ is a countable subset of } S\}.$$

For each $n \geq 1$, we can find a countable subset R_n of S such that $\mathbf{m}(R_n A) \geq k - n^{-1}$. Let $R = \bigcup_{n=1}^{\infty} R_n$. Then R is countable, and since, for each n,

$$k - n^{-1} \leq \mathbf{m}(R_n A) \leq \mathbf{m}(RA) \leq k,$$

we have $\mathbf{m}(RA) = k$. Let $Y = A \cup RA$. Since $\mathfrak{m}$ is infinite, we have $|Y| = \mathfrak{m}$. It remains to show that Y is left thick in X. By (2.1), it is sufficient to show that $\mathbf{m}(Y) = 1$.

Suppose, on the contrary, that $\mathbf{m}(Y) < 1$, and let $Z = X \sim Y$. Then $\mathbf{m}(Z) > 0$. Define $\mathbf{n} \in l_\infty(X)'$ by $\mathbf{n}(E) = \mathbf{m}(E \cap Z)$. It is sufficient to show that $\mathbf{n} \in \mathfrak{I}_l(S, X)$; for then $\mathbf{n}/\mathbf{m}(Z) \in \mathfrak{L}(X)$ and vanishes on Y, contradicting (2.4 (ii)).

To this end, let $E \subset X$, $s \in S$. Then

$$\begin{aligned}\mathbf{n}(s^{-1}E) &= \mathbf{m}(s^{-1}E \cap Z) = \mathbf{m}(s^{-1}E) - \mathbf{m}(s^{-1}E \cap Y)\\ &= \mathbf{m}(E) - \mathbf{m}(s^{-1}E \cap Y)\\ &= \mathbf{n}(E) + \left[\mathbf{m}(E \cap Y) - \mathbf{m}(s^{-1}E \cap Y)\right].\end{aligned}$$

We therefore have to show that

$$\mathbf{m}(E \cap Y) = \mathbf{m}(s^{-1}E \cap Y) \tag{3}$$

Now

$$\begin{aligned}s^{-1}E \cap s^{-1}Y &\subset (s^{-1}E \cap Y) \cup (s^{-1}Y \sim Y),\\ s^{-1}E \cap Y &\subset (s^{-1}E \cap s^{-1}Y) \cup (Y \sim s^{-1}Y),\end{aligned}$$

and applying $\mathbf{m}$ to both of the preceding inclusions and noting that

$$\mathbf{m}(s^{-1}E \cap s^{-1}Y) = \mathbf{m}\big(s^{-1}(E \cap Y)\big) = \mathbf{m}(E \cap Y),$$

we see that (3) will follow once we have shown

$$\mathbf{m}(s^{-1}Y \triangle Y) = 0. \tag{4}$$

To prove (4), observe first that $\mathbf{m}(Y) = k = \mathbf{m}(sY)$, since

$$k = \mathbf{m}(RA) \leq \mathbf{m}(Y) \leq \mathbf{m}(sY) = \mathbf{m}((\{s\} \cup sR)A) \leq k.$$

Similarly, $\mathbf{m}(sY \cup Y) = \mathbf{m}(Y)$, and it follows that $\mathbf{m}(Y \triangle sY) = 0$. Thus

$$\begin{aligned}0 &= \mathbf{m}(sY \triangle Y) = \mathbf{m}\big(s^{-1}(sY) \triangle s^{-1}Y\big) \leq \mathbf{m}\big(s^{-1}(sY) \triangle Y\big) + \mathbf{m}(Y \triangle s^{-1}Y)\\ &= \mathbf{m}\big(s^{-1}(sY)\big) - \mathbf{m}(Y) + \mathbf{m}(s^{-1}Y \triangle Y) = \mathbf{m}(s^{-1}Y \triangle Y),\end{aligned}$$

and (4) is established. □

We now turn to the case in which $X = S$. In many cases, the cardinal $\mathfrak{m}$ ($= \mathfrak{m}(S, S)$) is equal to a more easily calculated cardinal $\mathfrak{p}(S)$ (or simply, $\mathfrak{p}$).

The cardinal $\mathfrak{p}$ is defined by:

$$\mathfrak{p} = \min\{|sS| : s \in S\}.$$

Thus $\mathfrak{p}$ is the smallest possible cardinality of a right ideal of S. Since $\{S\}$ is a partition of S, we always have $\mathfrak{m} \leq \mathfrak{p}$. In general $\mathfrak{m} < \mathfrak{p}$. For example, if S is a finite group $\{x_1, \dots, x_n\}$ and we take $S_i = \{x_i^{-1}\}$ and $s_i = x_i$ in the definition of $\mathfrak{m}$ (see (2.3)), then we obtain $\mathfrak{m} = 1$, while, clearly, $\mathfrak{p} = |S|$.

Our next result shows that for many left amenable semigroups S, $\mathfrak{m} = \mathfrak{p}$. (Characterise the class of left amenable semigroups S for which $\mathfrak{m} = \mathfrak{p}$?)

Recall that a semigroup S is *left* [*right*] *reversible* if the family of right [left] ideals of S has the finite intersection property. Every left [right] amenable semigroup is right [left] reversible.

A semigroup S is *extremely left amenable* or *ELA* if it admits a left invariant mean which is multiplicative on $l_\infty(S)$. A remarkable result of Granirer [10] asserts that S is ELA if and only if wherever $F \in \mathscr{F}(S)$, there exists $s \in S$ such that $Fs = \{s\}$.

(2.6) PROPOSITION. *Let S be left amenable. Consider the following*:

(i) *S is right reversible*;

(ii) *S is amenable*;

(iii) *S is left cancellative*;

(iv) *S is right cancellative and left amenable*;

(v) *S is ELA.*

If S satisfies (v) *or if $\mathfrak{p}$ is infinite and S satisfies either of the conditions* (i), (ii), *then* $\mathfrak{m} = \mathfrak{p}$. *If s is infinite and satisfies either of the conditions* (iii), (iv), *then S is a.l.c.* (*so that* $\mathfrak{m} = \mathfrak{p} = |S|$).

Proof. Suppose that (i) holds and that $\mathfrak{p}$ is infinite. Let $\{S_1, \dots, S_n\}$ be a partition of S and $s_1, \dots, s_n \in S$. Since S is right reversible, we can find $u \in \bigcap_{i=1}^n S s_i$. Let $t_i \in S$ be such that $t_i s_i = u$. Then $|uS_i| = |t_i s_i S_i| \leq |s_i S_i|$, and using the infinitude of $\mathfrak{p}$,

$$\mathfrak{p} \leq |uS| \leq \max_{1 \leq i \leq n} |s_i S_i| = \left| \bigcup_{i=1}^n s_i S_i \right|.$$

It follows that $\mathfrak{p} \leq \mathfrak{m}$, and since the reverse inequality is always true, we have $\mathfrak{p} = \mathfrak{m}$.

If (ii) holds, then so does (i), and so $\mathfrak{m} = \mathfrak{p}$ if $\mathfrak{p}$ is infinite.

Now suppose that (v) holds. Let S_i, s_i be as above, and let

$$Z = \{s \in S \colon s_i s = s \text{ for } 1 \leq i \leq n\}.$$

Then $Z \neq \varnothing$, and is clearly a right ideal of S. Now for each i,

$$s_i S_i \supset s_i(S_i \cap Z) = S_i \cap Z,$$

so that for $z \in Z$,

$$\mathfrak{p} \leq |zS| \leq |Z| = \left| \bigcup_{i=1}^n (S_i \cap Z) \right| \leq \left| \bigcup_{i=1}^n s_i S_i \right|.$$

Thus $\mathfrak{m} \geq \mathfrak{p}$, and so $\mathfrak{m} = \mathfrak{p}$.

This proves the first assertion of the proposition, and we turn to the second. Suppose that S is infinite. If (iii) holds, then ((2.2)) S is a.l.c..

Suppose, then, that (iv) holds. Let S_i, s_i be as above and $B = \bigcup_{i=1}^{n} s_i S_i$. We need to show that $|B| = |S|$. If $|B| < |S|$, then an argument of Klawe [14] shows that there exists a sequence $\{y_n\}$ in S such that the family $\{y_n B: n \geq 1\}$ is disjoint: but then $\mathbf{m}(B) = 0$ for any $\mathbf{m} \in \mathfrak{L}(S)$, and we have contradicted (2.4. (ii)) (which obviously applies with $B = Y$).

This completes the proof of the second assertion of the proposition. □

In connection with (iv) above, Klawe has shown that there exists a right cancellative, left amenable semigroup S which is not left cancellative. (In fact, an example is provided by a semidirect product $F \times_\rho \mathbf{P}$ where F is the free commutative semigroup on an infinite countable set.) Such a semigroup belongs to the class $\mathscr{A}$ of a.l.c. but not left cancellative semigroups. Here are some more examples of semigroups in the latter class.

Let S be a left amenable, left cancellative, infinite semigroup and F be a left amenable semigroup which is not left cancellative and is such that $|F| < |S|$. Then $S \times F \in \mathscr{A}$. Now let $\{S_n\}$ be a sequence of finite, left amenable semigroups, not all left cancellative, such that S_n is a subsemigroup of S_{n+1} and $|K_n| \to \infty$ where K_n is the kernel of S_n. Then $T = \bigcup_{n=1}^{\infty} S_n$ is a.l.c. but not left cancellative. Finally, if V, W are semigroups with V left amenable and a.l.c. and such that there exists an epimorphism $Q: V \to W$ and a cardinal $\mathfrak{m}$ such that $|Q^{-1}(\{w\})| \leq \mathfrak{m} < |V|$ for all $w \in W$, then W is also a.l.c..

3. Cardinalities of sets of invariant means

The following result, relating left thick subsets to sets of left invariant means, is well known in one form or another, e.g., Chou [3], [4], Rosenblatt [19], [20], Paterson [17], and Klawe [15]. For completeness, we briefly sketch the proof. Let X be a left S-set as in §2.

(3.1) PROPOSITION. *Let S be left amenable, A be an infinite set and $\{\theta_\varepsilon: \varepsilon \in A\}$ be a disjoint family of left thick subsets of X. Then there exists a subset Ψ of $\mathfrak{L}(X)$ such that:*

(i) *Ψ is contained in the set* Ext $\mathfrak{L}(X)$ *of extreme points of* $\mathfrak{L}(X)$;

(ii) $|\Psi| = 2^{2^{|A|}}$.

In particular, $|\mathfrak{L}(X)| \geq 2^{2^{|A|}}$.

Proof. A result from set theory (cf. [12], (16.8)) yields a family $\{N_\gamma: \gamma \in \Gamma\}$ of subsets of A such that $|\Gamma| = 2^{|A|}$ and, whenever $\gamma_1, \ldots, \gamma_m$ are distinct elements of Γ and $\varepsilon_i \in \{1, c\}$, where, for $B \subset A$, $B^1 = B$ and $B^c = A \sim B$, then

$$\bigcap_{i=1}^{m} N_{\gamma_i}^{\varepsilon_i} \neq \varnothing .$$

Let $P = \{1, c\}^\Gamma$, so that $|P| = 2^{2^{|A|}}$. For $\gamma \in \Gamma$, let $E_\gamma = \cup\{\theta_\varepsilon \colon \varepsilon \in N_\gamma\}$. Let $p \in P$, $\gamma_1, \ldots, \gamma_n$ be distinct elements of Γ and $s_1, \ldots, s_n \in S$. Then we can find

$$\eta \in \bigcap_{i=1}^{n} N_{\gamma_i}^{p(\gamma_i)}.$$

For $E \subset X$, set $E^1 = X$, $E^c = X \sim E$. Using the disjointness of the family $\{\theta_\varepsilon\colon \varepsilon \in A]$, we have

$$\bigcap_{i=1}^{n} s_i^{-1}\left(E_{\gamma_i}^{p(\gamma_i)}\right) \supset \bigcap_{i=1}^{n} s_i^{-1}\theta_\eta, \tag{5}$$

and the latter intersection is non-empty since the left thickness of θ_n entails ((2.1)) the existence of $x \in X$ such that $\{s_1, \ldots, s_n\}x \subset \theta_\eta$.

The left action of S on X extends, in the natural way, to give a left action of S on the Stone-Cech compactification βX of X. Each map $\xi \to s\xi$ is continuous on βX. The Gelfand transform enables us to identify $l_\infty(X)$ with $C(\beta X)$ and, dually, $l_\infty(X)'$ with the space $M(\beta X)$ of complex, regular Borel measures on βX. For $B \subset X$, let B^- be the closure of B in βX.

Let p be as above. Then by (5), the family $\{(s^{-1}(E_\gamma^{p(\gamma)}))^-\colon s \in S,\ \gamma \in \Gamma\}$ of compact subsets of βX has the finite intersection property, and hence

$$C_p = \bigcap\left\{\left(s^{-1}\left(E_\gamma^{p(\gamma)}\right)\right)^-\colon s \in S,\ \gamma \in \Gamma\right\}$$

is non-empty and compact. If $q \in P \sim \{p\}$, then, for some $\gamma_0 \in \Gamma$, $q(\gamma_0) \neq p(\gamma_0)$, and for any $s \in S$,

$$C_p \subset \left(s^{-1}\left(E_{\gamma_0}^{p(\gamma_0)}\right)\right)^-, \ C_q = \left(s^{-1}\left(E_{\gamma_0}^{q(\gamma_0)}\right)\right)^-,$$

and since $s^{-1}E_{\gamma_0}^{p(\gamma_0)} \cap s^{-1}(E_{\gamma_0}^{q(\gamma_0)}) = \varnothing$, we have $C_p \cap C_q = \varnothing$. Further, if $s, t \in S$ and $\gamma \in \Gamma$, then

$$tC_p \subset t\left\{\left[(st)^{-1}\left(E_\gamma^{p(\gamma)}\right)\right]^-\right\} \subset \left[s^{-1}\left(E_\gamma^{p(\gamma)}\right)\right]^-$$

so that $tC_p \subset C_p$. Thus C_p is S-invariant, and by applying Day's fixed-point theorem to the natural action of S on the set of probability measures in $M(\beta X)$ vanishing outside C_p, we obtain a mean $\mathbf{m}_p \in \operatorname{Ext} \mathfrak{L}(X)$ such that $\mathbf{m}_p$, regarded as a probability measure on βX, vanishes outside C_p. Now take $\Psi = \{\mathbf{m}_p,\ p \in P\}$.

We now come to the fundamental proposition required to establish our theorems. The use of transfinite induction to establish the existence of invariant means goes back to Banach [1]. More recently, Chou, Rosenblatt and Klawe have used such arguments to calculate or estimate the cardinalities of sets of invariant means.

The following proof is of this kind, but is rather more involved since it uses a "transfinite recursion within a transfinite recursion" argument. The author wishes to claim that this argument is paradigmatic: we will show later that, with minor modifications, the same argument applies to give the cardinalities of other sets of invariant means.

The family of finite subsets of a set Y is denoted by $\mathscr{F}(Y)$.

(3.2) PROPOSITION. *Let X be infinite and be a.l.c. for S. Let $|S| \leq |X|$. Let α be the smallest ordinal of cardinality $|X|$. Then there exists a disjoint family $\{\theta_\varepsilon: \varepsilon \in \alpha\}$ of left thick subsets of X.*

Proof. Clearly, $|\mathscr{F}(S)| \leq |\mathscr{F}(X)| = |X|$ since X is infinite. Thus $|\mathscr{F}(S) \times X| = |X| = |\alpha|$, and we can find a bijection Q: $\alpha \to \mathscr{F}(S) \times X$. Let F_β be the first coordinate of $Q(\beta)$ ($\beta \in \alpha$) and note that $\mathscr{F}(S) = \{F_\beta: \beta \in \alpha\}$ and that if $F \in \mathscr{F}(S)$, then $|\{\beta \in \alpha: F_\beta = F\}| = |\alpha|$. Using transfinite recursion, we will construct a family $\{\Delta_\varepsilon^\beta: \varepsilon, \beta \in \alpha, \varepsilon \leq \beta\}$ of subsets of X having the following properties:

(i) $\Delta_\varepsilon^\beta \cap \Delta_\eta^\beta = \varnothing$ if $\varepsilon \neq \eta$;
(ii) $|\Delta_\varepsilon^\beta| \leq |\beta|$ if β is infinite and Δ_ε^β is finite if β is finite;
(iii) $\Delta_\varepsilon^\beta \subset \Delta_\varepsilon^{\beta_1}$ whenever $\varepsilon \leq \beta < \beta_1$;
(iv) $\cap\, \{s^{-1}(\Delta_\varepsilon^\beta): s \in F_\beta\} \neq \varnothing$ whenever $\varepsilon \leq \beta$.

Suppose that $\gamma \in \alpha$ and that sets Δ_ε^β have been constructed so that (i)–(iv) are satisfied for $\beta, \beta_1 < \gamma$. Let

$$K = \bigcup\{\Delta_\varepsilon^\beta: \varepsilon \leq \beta < \gamma\}.$$

If γ is finite, then K is finite by (ii), so that $|K| < |X|$. If γ is infinite, then, again using (ii), $|K| \leq |\gamma|^3 = |\gamma| < |X|$. So in both cases, $|K| < |X|$. Let $F_\mathrm{G}\mathrm{g} = \{s_1, \ldots, s_n\}$.

We now construct, again by transfinite recursion, a family $\{\Gamma_\varepsilon^\gamma: \varepsilon \leq \gamma\}$ of subsets of $X \sim K$ such that:

(a) $\Gamma_\varepsilon^\gamma = \{x_{\varepsilon,1}^\gamma, \ldots, x_{\varepsilon,n}^\gamma\}$;
(b) $\Gamma_\varepsilon^\gamma \cap \Gamma_\eta^\gamma = \varnothing$ if $\varepsilon \neq \eta$;
(c) for each ε, there exists $x_\varepsilon^\gamma \in X$ such that $s_i x_\varepsilon^\gamma = x_{\varepsilon,i}^\gamma$ for $1 \leq i \leq n$.

To this end, suppose that $\delta \leq \gamma$ and that sets $\{\Gamma_\varepsilon^\gamma: \varepsilon < \delta\}$ have been constructed so that (a), (b) and (c) are valid for $\varepsilon, \eta < \delta$. Let $E = \cup\{\Gamma_\varepsilon^\gamma: \varepsilon < \delta\}$ and $L = K \cup E$. Then $|L| < |X|$ since X is infinite and both $|K|, |E| < |X|$.

Now define recursively a disjoint family $\{X_1, \ldots, X_n\}$ of subsets of X as follows: $X_1 = s_1^{-1}L$ and, for $i > 1$, $X_i = (s_i^{-1}L) \sim (\bigcup_{r=1}^{i-1} X_r)$. Note that $\bigcup_{i=1}^n s_i^{-1}L = \bigcup_{i=1}^n X_i$. Suppose that $\bigcup_{i=1}^n X_i = X$. Then $|\bigcup_{i=1}^n s_i X_i| \leq |L| < |X|$, and we contradict the fact that X is a.l.c.. So $\bigcup_{i=1}^{n} X_i \neq X$. Choose $x_\delta^\gamma \in X \sim (\bigcup_{i=1}^{n} X_i)$, and let $x_{\delta,i}^\gamma = s_i x_\delta^\gamma$. Clearly, $x_{\delta,i}^\gamma \notin L$, and (a), (b) and (c) are true for $\varepsilon, \eta \leq \delta$. This completes the construction of the sets $\Gamma_\varepsilon^\gamma$.

Now set

$$\Delta_\varepsilon^\gamma = \left(\bigcup\left\{\Delta_\varepsilon^\beta \colon \varepsilon \leq \beta < \gamma\right\}\right) \cup \Gamma_\varepsilon^\gamma \ (\varepsilon < \gamma), \qquad \Delta_\gamma^\gamma = \Gamma_\gamma^\gamma.$$

We now check that the conditions (i)–(iv) are satisfied for $\beta, \beta_1 \leq \gamma$. Obviously, (iii) holds. We need only check (i), (ii) and (iv) when $\beta = \gamma$. Suppose that $\varepsilon, \eta \leq \gamma$ with $\varepsilon \neq \eta$. We can assume that $\varepsilon < \eta$. If $\varepsilon \leq \beta_0 < \gamma$, $\eta \leq \beta_1 < \gamma$ and $\beta' = \max\{\beta_0, \beta_1\}$, then by (iii) and (i), $\Delta_\varepsilon^{\beta_0} \cap \Delta_\eta^{\beta_1} \subset \Delta_\varepsilon^{\beta'} \cap \Delta_\eta^{\beta'} = \varnothing$. Noting that $\Gamma_\varepsilon^\gamma \cup \Gamma_\eta^\gamma \subset X \sim K$, we have, using (b),

$$\begin{aligned}\Delta_\varepsilon^\gamma \cap \Delta_\eta^\gamma &= \left[\left(\bigcup\left\{\Delta_\varepsilon^{\beta_0} \colon \varepsilon \leq \beta_0 < \gamma\right\}\right) \cup \Gamma_\varepsilon^\gamma\right] \cap \left[\left(\bigcup\left\{\Delta_\eta^{\beta_1} \colon \eta \leq \beta_1 < \gamma\right\}\right) \cup \Gamma_\eta^\gamma\right] \\ &= \varnothing.\end{aligned}$$

Now (i) follows for $\beta = \gamma$. Noting that $\Gamma_\varepsilon^\gamma$ is finite and that (c) holds, it readily follows that (ii) and (iv) hold for $\beta = \gamma$. This completes the construction of the sets Δ_ε^β $(\varepsilon \leq \beta < \alpha)$.

For each $\varepsilon \in \alpha$, let

$$\theta_\varepsilon = \bigcup\left\{\Delta_\varepsilon^\beta \colon \varepsilon \leq \beta,\ \beta \in \alpha\right\}.$$

Since $\Delta_\varepsilon^\beta \cap \Delta_\eta^{\beta_1} = \varnothing$ if $\varepsilon \neq \eta$, it follows that the family $\{\theta_\varepsilon \colon \varepsilon \in \alpha\}$ is disjoint.

It remains to show that each θ_ε is left thick in X. Let $F \in \mathscr{F}(S)$ and

$$A = \left\{\beta \in \alpha \colon F = F_\beta\right\}.$$

By construction, $|A| = |\alpha| = |X|$. Since $|\varepsilon| < |X|$, there exists $\beta \in A$ with $\varepsilon \leq \beta$. Then by (iv),

$$\bigcap\left\{s^{-1}\theta_\varepsilon \colon s \in F_\beta\right\} \supset \bigcap\left\{s^{-1}\Delta_\varepsilon^\beta \colon s \in F_\beta\right\} \neq \varnothing. \tag{6}$$

Let $x \in \bigcap\{s^{-1}\theta_\varepsilon \colon s \in F\}$. Then $Fx \subset \theta_\varepsilon$, and so θ_ε is left thick in X. □

(3.3) THEOREM. *Let S be left amenable, $\mathfrak{m}$ $(= \mathfrak{m}(S, X))$ be infinite and $|S| \leq \mathfrak{m}$. Then*

$$|\mathfrak{L}(X)| = 2^{2^{\mathfrak{m}}}.$$

Proof. Let A be as in (2.4) and $Y = SA$. Since $|S| \leq \mathfrak{m}$ and $|A| = \mathfrak{m}$ is infinite, we have $|Y| = \mathfrak{m}$. Clearly, in the notation of (2.3), we have $S_Y = S$. By (2.4), Y is a.l.c. for S, with $\mathfrak{m}(S, Y) = \mathfrak{m}$. Then (3.2) applies to give the existence of a disjoint family $\{\theta_\varepsilon : \varepsilon \in \alpha\}$ of left thick subsets of Y, where $|\alpha| = \mathfrak{m}$. Each θ_ε is obviously left thick in X, and from (3.1), we have $|\mathfrak{L}(X)| \geq 2^{2^{\mathfrak{m}}}$. For the reverse equality, it follows, using (2.4 (iii)), that

$$|\mathfrak{L}(X)| \leq |\mathfrak{L}(Y)| \leq |l_\infty(Y)'| = 2^{2^{|Y|}} = 2^{2^{\mathfrak{m}}}. \qquad \square$$

(3.4) COROLLARY. *Let S be countable and left amenable and $\mathfrak{m}$ be infinite. Then $|\mathfrak{L}(X)| = 2^{2^{\mathfrak{m}}}$.*

(3.5) NOTES. An unsatisfactory aspect of (3.4) is the requirement that $|S| \leq \mathfrak{m}$. What can be said if $|S| > \mathfrak{m}$? We will show in (3.6) that if $X = S$, then the above requirement can be removed. Here is another instance in which the conclusion of (3.3) remains valid even though $|S| > \mathfrak{m}$. Let X be the set of finite sequences $(x_1, \ldots, x_n)$ $(n \geq 1)$ with $x_i \in \{-1, 1\}$. Let S be the Cartesian product group $\{-1, 1\}^{\mathbf{P}}$, where $\mathbf{P}$ is the set of positive integers. Then X is a left S-set, where, for $f \in S$, $(x_1, \ldots, x_n) \in X$, we define

$$f(x_1, \ldots, x_n) = (f(1)x_1, \ldots, f(n)x_n).$$

Note that S is amenable since it is abelian. Then $\mathfrak{m} = |X| = \aleph_0$ while $|S| = \mathfrak{c}$ $(= 2^{\aleph_0})$. For each n, let X_n be the set of sequences in X of length n. Then $\{X_n : n \geq 1\}$ is a disjoint family of left thick subsets of X, and it readily follows that $|\mathfrak{L}(X)| = 2^{2^{\mathfrak{m}}}$.

When S is a group and X is infinite, then $\mathfrak{m} = |X|$ and (3.3) in this case is noted by Rosenblatt and Talagrand [21]. They ask the question: *If X is infinite, can there ever exist exactly one G-invariant mean on X?* (Of course, in such a case, $|G| > |X|$.) The corresponding question for semigroups can be formulated in the obvious way.

Another question arising from (3.3) is the following. What happens if $\mathfrak{m}$ is finite (and the condition $|S| \leq \mathfrak{m}$ is no longer required)? In this case, (2.4. (ii)) yields that $\mathfrak{J}_l(S, X)$ is finite-dimensional.

We now turn to the case where $X = S$. The next result is the most important one of the paper.

(3.6) THEOREM. *Let S be a left amenable semigroup and $\mathfrak{m} = \mathfrak{m}(S, S)$.*

(i) *If $\mathfrak{m}$ is infinite, then $|\mathfrak{L}(S)| = 2^{2^{\mathfrak{m}}}$;*

(ii) *If $\mathfrak{m}$ is finite, then $\mathfrak{J}_l(S, X)$ is n-dimensional for some $n < \infty$, and n is the number of finite, left ideal groups in S.*

Proof. Suppose that $\mathfrak{m}$ is infinite. Let A and Y be as in (2.5) and let T be the subsemigroup of S generated by Y. Then $|T| = \mathfrak{m}$, and in the notation of (2.3), $T \subset S_T$. By (2.4. (i)), T is a.l.c.. Applying (3.2), there exists a disjoint family $\{\theta_\varepsilon : \varepsilon \in \alpha\}$ of left thick subsets of T, where $|\alpha| = \mathfrak{m}$. Following Klawe

[14], we show that each θ_ε is left thick in S. Indeed, let $F \in \mathscr{F}(S)$. Since T is left thick in S, we can find $s \in S$ such that $Fs \subset T$. Since $Fs \in \mathscr{F}(T)$ and θ_ε is left thick in T, we can find $t \in T$ such that $(Fs)t \subset \theta_\varepsilon$, i.e., $F(st) \subset \theta_\varepsilon$. So θ_ε is left thick in S as asserted. By (3.1), $|\mathfrak{L}(S)| \geq 2^{2^{\mathfrak{m}}}$. For the reverse inequality, using (2.4. (iii)) and (3.3) we have $|\mathfrak{L}(S)| \leq |\mathfrak{L}(T)| = 2^{2^{\mathfrak{m}}}$. Thus (i) is established.

Now suppose that $\mathfrak{m}$ is finite. Using (2.4. (ii)),

$$\dim \mathfrak{I}_l(S, X) \leq \dim \mathfrak{I}_l(S_Y, Y) \leq \dim l_\infty(Y)' < \infty .$$

So $\mathfrak{I}_l(S, X)$ has finite dimension n, and by [14], n is the number of finite, left ideal groups in S. (Using the finiteness of A, a straight-forward, direct proof of the preceding assertion can be given.) This establishes (ii). □

(3.7) Corollary. *Let S be left amenable and satisfy any of the conditions* (i)–(v) *of* (2.6). *Let* $\mathfrak{p} = \min\{|sS|: s \in S\}$. *If $\mathfrak{p}$ is infinite, then* $|\mathfrak{L}(S)| = 2^{2^{\mathfrak{p}}}$. *If $\mathfrak{p}$ is finite, then $\mathfrak{I}_l(S, X)$ is finite-dimensional.*

Proof. Use (2.6). (A version of the result when (2.6. (ii)) holds is given by Klawe [15].). □

(3.8) Corollary ([15], [4]). *Let S be an infinite, left amenable semigroup which is either left or right cancellative. Then* $|\mathfrak{L}(S)| = 2^{2^{|S|}}$.

Proof. From (2.6), $\mathfrak{p} = |S|$. □

In [15], Maria Klawe considers the cardinal $\kappa l(S)$, where

$$\kappa l(S) = \min\{|B|: B \subset S, \mathbf{m}(B) = 1 \text{ for all } m \in \mathfrak{L}(S)\}.$$

We now show that, under the assumption of the generalised continuum hypothesis (GCH), in many cases, $\kappa l(S) = \mathfrak{m} = \mathfrak{p}$.

(3.9) Proposition. *Let S be left amenable and assume GCH. If $\mathfrak{p}$ is infinite and S satisfies any of the conditions* (i)–(v) *of* (2.6), *then* $\kappa l(S) = \mathfrak{m} = \mathfrak{p}$.

Proof. Since $\mathbf{m}(B) = 1$ for every $\mathbf{m} \in \mathfrak{L}(S)$ and every right ideal B of S, we have $\kappa l(S) \leq \mathfrak{p}$. On the other hand, if $C \subset S$ is such that $|C| = \kappa l(S)$ and $\mathbf{m}(C) = 1$ for all $\mathbf{m} \in \mathfrak{L}(S)$, then $\mathfrak{L}(S)$ can be regarded as a subset of $l_\infty(C)'$, so that, using (3.7),

$$2^{2^{\mathfrak{p}}} = |\mathfrak{L}(S)| \leq |l_\infty(C)'| = 2^{2^{\kappa l(S)}}.$$

The GCH then gives $\mathfrak{p} \leq \kappa l(S)$, so that $\mathfrak{p} = \kappa l(S)$. □

We conclude by justifying the earlier claim that the argument of (3.2) is paradigmatic. We will outline proofs to show that other known cardinality results can be readily derived using arguments along the same lines, thus indicating that (3.2) provides a unified treatment of such results.

References for the results below are [15], [4], [7], [13], [17], [18] and [21].

(3.10) THEOREM. (i) *Let S be an amenable semigroup and*

$$\mathfrak{n} = \min\{|sSs| : s \in S\}.$$

Let $\mathfrak{I}(S)$ be the set of invariant means on S. If $\mathfrak{n}$ is infinite, then $|\mathfrak{I}(S)| = 2^{2^{\mathfrak{n}}}$. If $\mathfrak{n}$ is finite, then S contains a finite ideal and $\mathfrak{I}(S)$ has exactly one member.

(ii) *Let G be an infinite, amenable group and $\mathfrak{I}^*(G)$ be the set of inversion invariant means on G. Then $|\mathfrak{I}^*(G)| = 2^{2^{|G|}}$.*

(iii) *Let G be an infinite, amenable group. For each $F \in \mathscr{F}(G)$, let*

$$C_F = \left\{x \in G\colon FxF \cap (FxF)^{-1} = \varnothing\right\}.$$

Then the following statements are equivalent:

(a) $|\mathfrak{I}(G) \sim \mathfrak{I}^*(G)| = 2^{2^{|G|}}$;

(b) $\mathfrak{I}(G) \neq \mathfrak{I}^*(G)$;

(c) *for each $F \in \mathscr{F}(G)$, we have $C_F \neq \varnothing$.*

(iv) *If G is an infinite, abelian group, then $\mathfrak{I}(G) \neq \mathfrak{I}^*(G)$ if and only if the set $B = \{x^2\colon x \in G\}$ is infinite.*

Proof. (i) Suppose that $\mathfrak{n}$ is infinite. Let $s_0 \in S$ be such that $|s_0Ss_0| = \mathfrak{n}$, and let $T_0 = s_0Ss_0$. If $\mathbf{m} \in \mathfrak{I}(S)$, then the two-sided invariance of $\mathbf{m}$ easily gives $\mathbf{m}(T_0) = 1$ so that $|\mathfrak{I}(S)| \leq 2^{2^{\mathfrak{n}}}$.

Suppose that $\mathfrak{n}$ is infinite, and let α be the smallest ordinal of cardinality $\mathfrak{n}$. Well-order $\mathscr{F}(T_0)$: $\mathscr{F}(T_0) = \{F_\beta\colon \beta \in \alpha\}$. Construct subsets Δ_ε^β of T_0 such that (i), (ii) and (iii) of (3.2) are satisfied, and the following condition holds:

(iv)[1] $\bigcap\{x^{-1}\Delta_\varepsilon^\beta y^{-1}\colon x, y \in F_\beta\} \neq \varnothing$ whenever $\varepsilon \leq \beta$.

To achieve this, we let K, F_γ be as in (3.2) and construct sets $\Gamma_\varepsilon^\gamma$ in $T_0 \sim K$ such that (b) of (3.2) is satisfied, and the following conditions hold:

(a)[1] $\Gamma_\varepsilon^\gamma = \{x_{\varepsilon,i,j}^\gamma\colon 1 \leq i,\ j \leq n\}$;

(c)[1] for each ε, there exists $x_\varepsilon^\gamma \in T_0$ such that $s_i x_\varepsilon^\gamma s_j = x_{\varepsilon,i,j}^\gamma$ for $1 \leq i,\ j \leq n$.

Let δ, E be as in (3.2). Then find a disjoint family $\{T_{ij}\colon 1 \leq i,\ j \leq n\}$ of subsets of T_0 such that

$$\left(\bigcup_{i,j} s_i^{-1}(K \cup E)s_0^{-1}\right) \cap T_0 = \bigcup_{i,j} T_{ij}$$

and $s_i T_{ij} s_j \subset K \cup E$. Since every $xT_0y (x, y \in S)$ supports every $\mathbf{m} \in \mathfrak{I}(S)$, we can find $s \in \bigcap_{i,j}(s_i T_0 s_j)$. If $\bigcup_{i,j} T_{ij} = T_0$, then $|sT_0 s| < \mathfrak{n}$, a contradiction. So we can find $x_\delta^\gamma \in T_0 \sim (\bigcup_{i,j} T_{ij})$. The construction of the sets $\Gamma_\varepsilon^\gamma$, $\Delta_\varepsilon^\gamma$, θ_ε now follow as in (3.2), with $x_{\delta,i,j}^\gamma = s_i x_\delta^\gamma s_j$. Then show that if $F \in \mathscr{F}(T_0)$, then for each ε,

$$\bigcap\{a^{-1}\theta_\varepsilon b^{-1}: a, b \in F\} \neq \varnothing.$$

The preceding conclusion is true with T_0 replaced by S, since T_0 is both left and right thick in S.

Now modify the proof of (3.1): take C_p to be the set

$$\bigcap\left\{\left(\left(x^{-1}E_\gamma^{p(\gamma)}\right)y^{-1}\right)^-: x, y \in S\right\}.$$

Each C_p is compact, non-empty and is both left and right invariant for S. Use Day's fixed-point theorem to find $\mathbf{m}_p \in \mathfrak{F}(S)$ supported on C_p. The first assertion of (i) follows.

If $\mathfrak{n}$ is finite, then a straight-forward semigroup argument gives the second assertion of (i).

(ii) Follow the proof of (i) with $S = G = T_0$, $\mathfrak{n} = |G|$. We require that the sets Δ_ε^β satisfy the additional condition:

(v)^2 $(\Delta_\varepsilon^\beta)^{-1} = \Delta_\varepsilon^\beta$.

The sets $\Gamma_\varepsilon^\gamma$ have to satisfy (a)^1, (c)^1 and the condition:

(b)^2 $[\Gamma_\varepsilon^\gamma \cup (\Gamma_\varepsilon^\gamma)^{-1}] \cap [\Gamma_\eta^\gamma \cup (\Gamma_\eta^\gamma)^{-1}] = \varnothing$ if $\varepsilon \neq \eta$.

We select

$$x_\delta^\gamma \in G \sim \left[\left(\bigcup_{i,j} s_i^{-1}(K \cup E)s_j^{-1}\right) \cup \left(\bigcup_{i,j} s_i^{-1}(K \cup E)^{-1}s_j^{-1}\right)\right].$$

Now take

$$\Delta_\varepsilon^\gamma = \left(\bigcup\{\Delta_\varepsilon^\beta: \varepsilon \leq \beta < \gamma\}\right) \cup \Gamma_\varepsilon^\gamma \cup \left(\Gamma_\varepsilon^\gamma\right)^{-1},$$
$$\Delta_\gamma^\gamma = \Gamma_\gamma^\gamma \cup \left(\Gamma_\gamma^\gamma\right)^{-1}$$

Note that $\theta_\varepsilon^{-1} = \theta_\varepsilon$, and so $E_\gamma^{-1} = E_\gamma$.

Now form the sets C_p. If $\mathbf{m} \in \mathfrak{F}(S)$ vanishes outside C_p, then $\mathbf{m}^*$, where $\mathbf{m}^*(B) = \mathbf{m}(B^{-1})$ $(B \subset G)$, is also in $\mathfrak{F}(S)$ and vanishes outside C_p. Thus $\frac{1}{2}(\mathbf{m} + \mathbf{m}^*)$ is in $\mathfrak{F}(S)$ and vanishes outside C_p, and the desired result follows.

(iii) Trivially, (a) implies (b), so that it suffices to show that (b) implies (c) and (c) implies (a).

Suppose that (c) does not hold. Then we can find $F_0 \in \mathscr{F}(G)$ such that for all $x \in G$, $F_0 x F_0 \cap (F_0 x F_0)^{-1} \neq \varnothing$. For $a, b, c, d \in F_0$, let

$$A_{a,b,c,d} = \left\{x \in G: axb = (cxd)^{-1}\right\}.$$

Then G is the union of the sets $A_{a,b,c,d}$, and if $E \subset A_{a,b,c,d}$, $\mathbf{m} \in \mathfrak{J}(G)$, then $\mathbf{m}(E) = \mathbf{m}(E^{-1})$. Thus $\mathfrak{J}(G) = \mathfrak{J}^*(G)$, contradicting (b). So (b) implies (c).

Now suppose that (c) holds and let $F \in \mathscr{F}(G)$. We claim first that $|C_F| = |G|$. To prove this, it is sufficient (cf. the proof of (2.6. (iv))) to show that C_F is left thick in G. If $F_1 \in \mathscr{F}(G)$, then $F_1 x \subset C_F$, where x is chosen so that $(HxH) \cap (HxH)^{-1} = \varnothing$ with $H = (FF_1 \cup F) \in \mathscr{F}(G)$. Thus C_F is left thick as required. We now show that (a) holds.

We modify the proofs of (i) and (ii) as follows. The condition $(\mathrm{v})^2$ is replaced by

$$(\mathrm{v})^3 \quad \Delta_\varepsilon^\beta \cap \left(\Delta_\eta^{\beta_1}\right)^{-1} = \varnothing \quad \text{whenever } \varepsilon \leq \beta < \alpha,\ \eta \leq \beta_1 < \alpha.$$

To achieve this, we require the sets $\Gamma_\varepsilon^\gamma$ to be contained in $G \sim (K \cup K^{-1})$, and to satisfy (b) of (3.2), $(\mathrm{a})^1$, $(\mathrm{c})^1$ above and the condition:

$(\mathrm{d})^3 \quad \Gamma_\varepsilon^\gamma \cap (\Gamma_\eta^\gamma)^{-1} = \varnothing \quad$ whenever $\varepsilon \leq \gamma$, $\eta \leq \gamma$.

Let E be as in (3.2), and $B = \bigcup_{i,j}(s_i^{-1}(K \cup K^{-1} \cup E \cup E^{-1})s_j^{-1})$. Then $|B| < |G|$, and since $|C_F| = |G|$ where $F = F_\gamma$, we can find $z_\delta^\gamma \in G$ such that

$$\left(F_\gamma z_\delta^\gamma F_\gamma\right) \cap \left(F_\gamma z_\delta^\gamma F_\gamma\right)^{-1} = \varnothing.$$

The construction of Γ_δ^γ now proceeds as in (i), and we obtain the desired sets Δ_ε^β. Each E_γ is such that $E_\gamma \cap E_\gamma^{-1} = \varnothing$. If $\mathbf{m} \in \mathfrak{J}(S)$ vanishes outside C_p and $C_p \subset E_\gamma^-$, then $\mathbf{m}(E_\gamma) = 1$, $\mathbf{m}(E_\gamma^{-1}) = 0$, i.e., $\mathbf{m} \in \mathfrak{J}(G) \sim \mathfrak{J}^*(G)$. The desired result now follows.

(iv) Let G be infinite and abelian. Suppose that for some $F \in \mathscr{F}(G)$ we have $C_F = \varnothing$. Then for all $x \in G$, $FxF \cap (FxF)^{-1} \neq \varnothing$, so that $B \subset F^{-4}$ is finite. Conversely, if B is finite, then we can find $F_0 \in \mathscr{F}(G)$ with $B \subset F_0^{-4}$ and obviously, $C_{F_0} \neq \varnothing$. Thus B is finite if and only if (c) of (iii) holds, and the desired result follows from (iii). □

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