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A NON STRUCTURE THEOREM FOR AN INFINITARY THEORY WHICH HAS THE UNSUPERSTABILITY PROPERTY[1]

BY

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Dedicated to the Memory of W.W. Boone

Abstract

Let κ, λ be infinite cardinals, $\psi \in L_{\kappa^+, \omega}$. We say that the sentence ψ has the λ-*unsuperstability property* if there are $\{\varphi_n(\bar{x}, \bar{y})\colon n < \omega\}$ quantifier free first order formulas in L, a model M of ψ, and there exist $\{\bar{a}_\eta\colon \eta \in {}^{\omega \geq}\lambda\} \subseteq |M|$ such that for all $\eta \in {}^{\omega}\lambda$, and for every $\nu \in {}^{\omega >}\lambda$,

$$\nu < \eta \Leftrightarrow M \models \varphi_{l(\nu)}[\bar{a}_\nu, \bar{a}_\eta].$$

THEOREM. *Let $\psi \in L_{\kappa^+, \omega}$, λ a Ramsey cardinal. If ψ has the λ-unsuperstability property then for every cardinal χ, $\chi > |L| \cdot \aleph_0 \Rightarrow I(\chi, \psi) = 2^\chi$.*

We shall prove a more general theorem; the proof uses a new partition theorem for trees.

We present an application of the theorem to the theory of modules by deriving the following:

COROLLARY. *Assume there exists a Ramsey cardinal. Let R be an integral domain. If DT_R (the class of torsion divisible R-modules) has a structure theorem (i.e., there are few cardinal invariants such that every module can be characterized by the invariants) then R must be Notherian. For example, if every module from DT_R is a direct sum of countable generated modules then R is Notherian.*

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1. Introduction

This paper is in the tradition of classification theory for non-elementary classes. Namely given a non-elementary class of models K (i.e., there is no first order theory T such that $K = \mathrm{Mod}(T)$) we try to classify them (in the sense of [13]). This is not the proper place to explain in details the plan/targets/aims/the philosophy of Classification Theory for non-elementary classes; we just want to say here that its aim is to find structure/non structure theorems. To a reader who is not familiar with classification theory for elementary classes we recommend [14].

In [16] Shelah proved (Theorem 1.6) that any reasonable non-elementary class is a PC class of $L_{\kappa^+,\omega}$ for some κ. In the sequel we shall work with PC classes; everything proved here is true for PC classes but in order to clarify our explanations sometimes we prefer to deal with narrower classes, like a class of models of some $\psi \in L_{\kappa^+,\omega}$.

In [10] Shelah proved a non-structure theorem for infinitary sentences which has the infinitary order property. Namely he proved the following result.

THEOREM 1.1. *Let λ, κ be cardinals, and let $\psi \in L_{\kappa^+,\omega}$. There exists a cardinal*

$$\mu^*(\lambda, \kappa) < \beth_{(2^{\lambda+\kappa})^+}$$

satisfying: if there are $\varphi(\bar{\mathbf{x}} : \bar{\mathbf{y}}) \in L_{\lambda^+,\omega}$, $M \models \psi$, and a set

$$\{\bar{\mathbf{a}}_i : i < \mu^*(\lambda, \kappa)\} \subseteq |M|$$

such that $(\forall i, j < \mu^*(\lambda, \kappa))$ $[i < j \Leftrightarrow M \models \varphi[\bar{\mathbf{a}}_i; \bar{\mathbf{a}}_j]]$ then $(\forall \chi > \kappa)$ $I(\chi, \psi) = 2^\chi$.

In [6], [7] this work is continued. The observant reader may have noticed that Theorem 1.1 is the natural generalization of the corresponding theorem for elementary classes (which says that an unstable first order theory has many non isomorphic models). In [12] Shelah proved another non-structure theorem for first order theories. In Theorem VIII2.1 he proved that if T is an unsuperstable first order theory then T has 2^χ non isomorphic models of cardinality χ, for every $\chi > |T|$.

It seems natural to ask: Does there exist a theorem generalizing Theorem VIII 2.1 of [12] to infinitary logics in the same sense as Theorem 1.1 above generalizes the theorem on unstable first order theories? Namely, we want to find a property for non-elementary classes which will behave as unsuperstability for elementary classes.

This property (unsuperstability for non-elementary classes) is interesting because similarly to the first order case we can derive from it a non structure theorem (this is what we show in this paper). Another use of the non structure theorem which will be proved below is the following phenomena (to be studied in detail in [9]). Suppose K is a PC_κ (to be defined below) such that K has the amalgamation property; if there exists $\chi > \kappa$ such that $I(\chi, \kappa) < 2^\chi$ than there exists an appropriate substitute to rank which enables us to define stable amalgamation. This should help to derive some structure theorems (like categoricity) for PC_κ clases with the amalgamation property.

The main aim of this paper is to answer the above question affirmatively. Namely we define a notion of unsuperstability suitable for infinitary theories, and prove that infinitary theories which have the unsuperstability property has many non isomorphic models. Also an application to algebra is presented (in spite of the application to algebra to be presented here. It should be clear that the goal of this paper is not the specific application. This is not the reason for introducing the unsuperstability property. The motivation is to advance toward finding dichotomy properties for a classification theory for non elementary classes in general).

DEFINITION 1.2. Let κ, λ be infinite cardinals, and let $\psi \in L_{\kappa^+, \omega}$.

(1) ψ has the *λ-unsuperstability property* if there are $\{\varphi_n(\bar{x}, \bar{y})\colon n < \omega\}$ quantifier free first order formulas in L, a model M, $M \models \psi$, and

$$\left\{\bar{a}_\eta\colon \eta \in {}^{\omega \geq}\lambda\right\} \subseteq |M|$$

such that for every $\eta \in {}^{\omega}\lambda$ and every $\nu \in {}^{\omega >}\lambda$,

$$\nu < \eta \Leftrightarrow M \models \varphi_{l(\nu)}[\bar{a}_\nu, \bar{a}_\eta].$$

(2) Let μ be an infinite cardinal. We say ψ has the *$(\lambda, L_{\mu^+, \omega})$-unsuperstability property* if there are $\{\varphi_n(\bar{x}, \bar{y})\colon n < \omega\} \subseteq L_{\mu^+, \omega}$, a model M, and a set $\{\bar{a}_\eta\colon \eta \in {}^{\omega \geq}\lambda\}$ as in (1).

(3) Let χ be a cardinal, and K a class of models. We say that $IE(\chi, K) \geq \mu$ iff there exists $\{M_i \in K\colon i < \mu\}$ such that $i < \mu \Rightarrow \|M_i\| = \chi$, and there is no $\mathscr{L}$-embedding from M_i into M_j. $IE(\chi, K) = \mu$ iff $IE(\chi, K) \geq \mu$ and it is not true that $IE(\chi, K) \geq \mu^+$. When $\psi \in L_{\kappa^+, \omega}$, $IE(\chi, \psi) \geq \mu$ means $IE(\chi, K) \geq \mu$ when $K = \operatorname{Mod}(\psi)$, and the K-embeddings are $\mathscr{L}$-elementary when $\mathscr{L}$ (the preserving formulas from $\mathscr{L}$) is some fixed fragment of $L_{\kappa^+, \omega}$ of cardinality κ. Note that $\mathscr{L}$ always contains the formulas $\{\varphi_n(\bar{x}, \bar{y})\colon n < \omega\}$.

(4) Let K be a class of models all in a similarity type L, λ a cardinal. Then K has the λ-unsuperstability property if there exist $\{\varphi_n(\bar{x}; \bar{y})\colon n < \omega\} \subseteq L$ quantifier free first order, a model $M \in K$, and

$$\left\{\bar{a}_\eta\colon \eta \in {}^{\omega \geq}\lambda\right\} \subseteq |M|$$

such that for all $\eta \in {}^{\omega}\lambda$, and for all $\nu \in {}^{\omega >}\lambda$,

$$\nu < \eta \Leftrightarrow M \models \varphi_{l(\nu)}[\bar{\mathbf{a}}_\nu, \bar{\mathbf{a}}_\eta].$$

Similarly to part (2) we can define the $(\lambda, L_{\mu^+,\omega})$-unsuperstability property for K.

LEMMA 1.3. *Let λ, κ, μ be cardinals $\psi \in L_{\mu^+,\omega}$. If ψ has the λ-unsuperstability property then ψ has the $(\lambda, L_{\mu^+,\omega})$-unsuperstability property.*

Proof. Trivial from the definitions. □

Remark. Suppose K is an elementary class.

(1) λ-unsuperstability implies μ-unsuperstability for every $\mu \geq \aleph_0$. (see Lemma VII 3.5 (1) in [12]).

(2) Suppose T is first order complete theory such that $K = \mathrm{Mod}(T)$. Then for stable T, K is λ-unsuperstable $\Leftrightarrow$ T is not superstable (see Theorem II 3.14 and Lemma VII 3.5(5) in [12]).

(3) The last two remarks explain our choice of name for the property, and how it generalizes the first order notion of unsuperstability.

(4) The necessity for adding a cardinal as a parameter to the property is because in non-elementary classes the compactness theorem fails (this is the theorem used to show that for elementary classes λ-unsuperstability is equivalent to μ-unsuperstability for any (and some) μ (i.e., Remark (1) above)).

Clearly according to Definition 1.2 and Lemma 1.3 the next theorem implies the theorem stated in the abstract.

MAIN THEOREM 1.4. *Let κ, λ be infinite cardinals, and let $\psi \in L_{\kappa^+,\omega}$. If λ is a Ramsey cardinal such that $\lambda > \kappa + \mu$, and ψ has the $(\lambda, L_{\mu^+,\omega})$-unsuperstability property then for every $\chi > \kappa$, $I(\chi, \psi) = 2^\chi$. Moreover for $\chi > \kappa$,*

$$[\chi \text{ regular} \vee \chi^{\aleph_0} = \chi > 2^{\aleph_0} \vee 0^{\#} \notin V] \Rightarrow IE(\chi, \psi) = 2^\chi,$$

when we restrict the embeddings to embeddings which preserve the formulas $\{\varphi_n(\bar{\mathbf{x}}, \bar{\mathbf{y}}): n < \omega\}$ (the formulas which exemplify the unsuperstability property).

Remark. In the theorem the use of a Ramsey cardinal is not necessary, and a weaker assumption is sufficient. Later (in Definition 2.4 and Theorem 2.5) we shall define the exact notion of largness we will use. The property is much weaker than having a Ramsey cardinal, for example if there exists a measurable cardinal in **V** then the same ordinal which is measurable in **V** is large enough for our purposes in **L** (the constructible universe). Since the weaker assumption has a relatively complicated definition we postpone it (see also the remarks at the end of this section).

DEFINITION 1.5. Let K be a class of models all of them in the same similarity type L. K is called a PC class if there exists a similarity type $L_1 \supseteq L$ and a first order theory T_1 in L_1, a set of first order types Γ such that $K = PC(T_1, \Gamma, L)$, when

$$PC(T_1, \Gamma, L) = \{M|L \colon M \vDash T_1 \text{ and } (\forall p \in \Gamma)\,[M \text{ omits the type } p]\}.$$

If $|T_1| \cdot |L_1| \cdot |\Gamma| \leq \kappa$ we say that K is a PC_κ class.

Remark. Some authors use a related notation. A class is PC_δ (in their notation) if it is $PC_{\aleph_0}$ with an empty set of types Γ (in ours).

C.C. Chang [1] proved the following:

FACT 1.6. *If $\psi \in L_{\kappa^+, \omega}$ then* Mod(ψ) *is a PC_κ class.*

Hence clearly, if in Theorem 1.4 we replace the assumption $\psi \in L_{\kappa^+, \omega}$ by K a PC_κ class, and replace every instance of Mod(ψ) by K then we have a stronger statement. Namely instead proving Theorem 1.4 we shall prove:

THEOREM 1.4*. *Let κ, μ be infinite cardinals, and let K be a PC_κ class. If λ is a Ramsey cardinal such that $\lambda > \kappa + \mu$ and K has the $(\lambda, L_{\mu^+, \omega})$-unsuperstability property then for every $\chi > \kappa$, $I(\chi, \kappa) = 2^\chi$; if in addition χ satisfies [χ regular $\vee\, \chi^{\aleph_0} = \chi > 2^{\aleph_0} \vee 0^\# \notin V$] then we have $IE(\chi, \kappa) = 2^\chi$, by restricting ourselves to embeddings which preserve the formulas $\{\varphi_n(\bar{x}, \bar{y}) \colon n < \omega\}$ from the unsuperstability property.*

Remark. You may think that we have a misprint in the statement of Theorem 1.4* and that the correct version should be "For every $\chi > \kappa + \mu$" (in order to cover the case $\mu > \kappa$). But fortunately it is sufficient to require $\chi > \kappa$ (this is done using the methods of [6] to obtain $\{\varphi'_n(\bar{x}, \bar{y}) \colon n < \omega\} \subseteq L_{\kappa^+, \omega}$ which exemplify the $(\lambda, L_{\kappa^+, \omega})$-unsuperstability property).

Now let us explain the proof of Theorem 1.4*. The main known method to get many pairwise not embeddable models is to use Ehrenfeucht-Mostowski models over trees of elements of height $\omega + 1$ which are indiscernible with respect to the structure of the tree as in Theorem VIII 2.1 in [Sh3]. The combinatorial content of that proof is contained in [11]. In order to apply the machinery of [11], clearly it is enough to find a model $M \in K$, and an increasing sequence $\langle k(n) < \omega \colon n < \omega\rangle$, and a tree $T \subseteq {}^{\omega \geq}\lambda$ such that for all $n < \omega$, and for every $\eta \in {}^{k(n)}\lambda \cap T$ we have

$$|\{\eta\hat{\ }\langle i\rangle \in T \colon i < \lambda\}| \geq \kappa;$$

and there is $\{\bar{a}_\eta \colon \eta \in T\} \subseteq |M|$ such that for all $\eta \in {}^{\omega}\lambda \cap T$, and every

$\nu \in {}^{\omega>}\lambda$,

$$\eta < \nu \Leftrightarrow M \models \varphi_{l(\nu)}[\bar{\mathbf{a}}_\eta, \bar{\mathbf{a}}_\nu],$$

and $\{\bar{\mathbf{a}}_\eta: \eta \in T\}$ is indiscernible with respect to the structure of the tree ${}^{\omega\geq}\lambda$. More formally:

DEFINITION 1.7. Let n be a natural number and let κ be a cardinal, M a model from a PC_κ class K, $T \subseteq {}^{\omega\geq}\lambda$ a subtree of the full tree (a subset closed under initial segments), and let $\{\bar{\mathbf{a}}_\eta: \eta \in T\} \subseteq |M|$. We say that $\{\bar{\mathbf{a}}_\eta: \eta \in T\}$ is n-*indiscernible with respect to the structure of the tree* T (usually when the identity of the tree is clear we just say *n-indiscernible*) iff:

(i) For every $\nu, \eta \in T$ if $atp_{L(T)}(\eta, \varnothing, T) = atp_{L(T)}(\nu, \varnothing, T)$ then $l(\bar{\mathbf{a}}_\eta) = l(\bar{\mathbf{a}}_\nu)$.

(ii) For every $\bar{\eta} = \langle \eta(0), \dots, \eta(n-1)\rangle$, and every $\bar{\nu} = \langle \nu(0), \dots, \nu(n-1)\rangle$ from T if $atp_{L(T)}(\bar{\eta}, \varnothing, T) = atp_{L(T)}(\bar{\nu}, \varnothing, T)$ then

$$tp_{L(M)}\left(\bar{\mathbf{a}}_{\nu(0)}, \dots, \bar{\mathbf{a}}_{\nu(n-1)}, \varnothing, M\right) = tp_{L(M)}\left(\bar{\mathbf{a}}_{\eta(0)}, \dots, \bar{\mathbf{a}}_{\eta(n-1)}, \varnothing, M\right).$$

$L(T)$ is the language of trees; has $\omega + 1$ unary predicates (for each level), a binary predicate $<$ (for being an initial segment), a binary predicate $<_l$ (for lexicographic order), and a binary function symbol h ($h(\eta, \nu)$ is the largest common subsequence of η and ν).

Remark. Sometimes sequences satisfying the requirements of 1.7 will be called simply *tree-indiscernibles.*

To prove Theorem 1.4* we will construct a model $M \in K$ which has a subset $\{\bar{\mathbf{a}}_\eta: \eta \in T\}$ which is n-indiscernible for every $n < \omega$. And we use [11] to produce a family of many trees which will serve as skeletons to Ehrenfeucht Mostowski models, such that the reducts of the models are not embeddable one to the other (note that we can assume without loss of generality that the language of T_1 contains Skolem functions, and T_1 contains Skolem axioms).

Let's review the structure of the paper. In the next section we shall construct a model which has an indiscernible subtree as explained in the last paragraph. In the proof we try to imitate Morley number (i.e., Hanf number) computation as done in [10] or in [6]. The major step in the proof of Theorem 1.1 (or in any Morley number computation) there is a use of a partition theorem, the Erdös Rado theorem. Since here we want to get elements which are tree indiscernibles rather than indiscernibles as linearly ordered sets we need another partition theorem, one which takes into account a structure of a tree of height $\omega + 1$ (this is where the large cardinal property of λ is used). So in the next section we prove Theorem 1.4* modulo a partition theorem on trees. In Section 3 the partition theorem is proved (we consider Section 3 the heart of this paper, or at least the hardest part of it). In Section 4 we present an application of our non structure theory to solve a problem in algebra (asked by Fuchs and Salce [4]). The original question was: For a uniserial domain R (see Definition 4.1) does a specific class of modules DT_R over R have a structure theorem? This class (and some related class) is not an elementary class, but it is a class of models of some sentence $\psi \in L_{|R|^+,\omega}$. A negative answer will be presented. By showing that if R is an integral domain and has an infinite ascending chain of ideals (not Noetherian) then the class of modules in question has the unsuperstability property.

Let us conclude this section with remarks about some extensions of the main theorem, and discuss some related questions (not used in the forthcoming sections).

Remark. Using the notation of Theorem 1.4 it is natural to define $\lambda(\mu, \kappa) = \mathrm{Min}\{\lambda$: for every $\psi \in L_{\kappa^+,\omega}$ if ψ has the $(\lambda, L_{\mu^+,\omega})$-unsuperstability property then $[\chi > \aleph_0$ regular$] \Rightarrow IE(\chi, \psi) = 2^\chi\}$.

It is interesting to check the lower and upper bounds on $\lambda(\mu, \kappa)$:

(1) It is easy to show that $\lambda(\mu, \kappa) \geq \mu^*(\mu, \kappa)$ (for $\mu^*(\mu, \kappa)$ see [6] and [7]).

(2) The statement of Theorem 1.4 can be formulated using the above notation. If λ is a Ramsey cardinal such that $\lambda > \mu + \kappa$ then $\lambda(\mu, \kappa) \leq \lambda$. Since there are no Ramsey cardinals in $\mathbf{L}$ it is natural to ask: Is is consistent that $\mathbf{L} \models$ "$\lambda(\mu, \kappa)$ is a cardinal"? We can answer this question affirmatively by assuming Con(ZFC + $\exists\chi[\chi$ measurable]). So let $\mathbf{V} \models$ "λ in measurable", hence $\mathbf{L} \models$ "λ is regular and λ has enough reflection properties" (see Definition 2.3 and the assumption of Theorem 2.5). Hence $\mathbf{L} \models$ "$\lambda(\mu, \kappa) \leq \lambda$".

(3) By (1) clearly $\lambda(\aleph_0, \aleph_0) \geq \beth_{\omega_1}$. So the next natural question to ask is: Is it consistent that $\lambda(\aleph_0, \aleph_0) = \beth_{\omega_1}$? Assuming Con(ZFC + there are $\aleph_1$ supercompacts) by successive collapses of the supercompacts it is possible to obtain a model for $\lambda(\aleph_0, \aleph_0) = \beth_{\omega_1}$. So the answer to the last question is yes, it is consistent that $\lambda(\aleph_0, \aleph_0) = \beth_{\omega_1}$ holds.

(4) We extracted the definition of $\lambda(\mu, \kappa)$ from the statement of Theorem 1.4 and similarly we can define $\lambda(\kappa)$ based on the theorem in the abstract. You may ask what is the relation between $\lambda(\kappa)$ and $\lambda(\mu, \kappa)$. Clearly $\lambda(\kappa) \leq \lambda(\mu, \kappa)$. But when $\mu \leq \kappa$ then $\lambda(\kappa) = \lambda(\mu, \kappa)$.

(5) Since the aim of this paper is to present a model theoretic property and to present its model theoretic conclusions, we do not try to minimalize the set theoretic assumptions (or to weaken them). Also there are other applications to algebra which we don't present (proving a uniform way to obtain some results which otherwise are based on tricks, e.g., for separable reduced groups and Boolean powers).

(6) Since the way to prove Theorem 2.5 (which implies many models) is by finding an Ehrenfeucht-Mostowski model for a specific first order theory clearly any model of set theory which contains the definition of the class K

will contain many models for K. For example if K (the class of models) is $PC_{\aleph_0}$ definable in $\mathbf{L}$ and there is a Ramsey cardinal in $\mathbf{V}$ then also there are many models in $\mathbf{L}$ (use the proof of Theorem 2.5 to construct an E.M. model as required and use the Levy-Schonfield absoluteness theorem to find such an E.M. model in $\mathbf{L}$). By similar absoluteness argument you can start with a model of set theory which has a cardinal large enough to prove Theorem 2.5, collapse the large cardinal by a complete enough forcing notion. The generic extension you obtain will satisfy the conclusion of Theorem 2.5 without having in it any large cardinals.

Notations. Let $\lambda, \kappa, \mu, \chi$ stand for infinite cardinal numbers, $\alpha, \beta, \gamma, \delta, \zeta, \xi, i, j$ ordinals. m, n, l, k natural numbers. Trees will always be subsets of ${}^{\omega \geq}\lambda$ (the set of finite and ω sequences of ordinals less than λ) which are closed under taking initial segments (see the last sentence in Definition 1.7). A subtree of a tree is a subset which is closed under initial segments. When μ is an uncountable regular let D_μ be the filter generated by the closed unbounded subsets of μ; for $S \subseteq \mu$, $S \not\equiv 0 \bmod D_\mu$ stands for "S is stationary subset of μ", $S \not\equiv T \bmod D_\mu$ means there is a club (closed unbounded) $C \subseteq \mu$ such that $C \cap S = C \cap T$. When $T \subseteq {}^{\omega \geq}\lambda$ is a tree and $\eta \in {}^{\omega >}\lambda \cap T$ then define

$$T[\eta] = \{\nu \in T\colon \nu < \eta \vee \nu = \eta \vee \eta < \nu\}.$$

When $T \subseteq {}^{\omega \geq}\lambda$ is a tree and $\eta \in {}^{\omega >}\lambda \cap T$ then $\mathrm{Succ}_T(\eta)$ denotes the set of immediate successors of η in T, namely

$$\mathrm{Succ}_T(\eta) = \{\nu \in T\colon \eta < \nu,\ l(\nu) = l(\eta) + 1\}.$$

We have $S^\lambda_{\aleph_0} = \{\delta < \lambda\colon cf(\delta) = \aleph_0\}$. Let $T \subseteq {}^{\omega \geq}\lambda$ and $\alpha \leq \lambda$; then $T|\alpha = \{\eta \in {}^{\omega \geq}\alpha\colon \eta \in T\}$. When M is a model $CD(M)$ stands for its complete diagram. Our notation is quite standard, and we use the same terminology as [12]. The end of a proof is denoted by □.

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2. Proof of the main theorem

In order to avoid some technical complications and to clarify the proof, instead of proving Theorem 1.4* directly, we shall first prove a weaker theorem (but still a stronger theorem than the one stated in the abstract). After completing the proof we shall explain how to modify it to a proof of Theorem 1.4*. We hope that this will simplify the understanding of the proof.

In order to clarify the role of the Ramsey cardinal in the statement of the theorem, we define the precise large cardinal properties we need. But first we define a certain largeness property of subtrees of ${}^{\omega\geq}\lambda$ which will be part of the induction hypothesis in the theorem, and used as well in the definition of the large cardinal properties.

DEFINITION 2.1. Let λ, μ be cardinals, μ a regular cardinal, $\leq \lambda$, and $T \subseteq {}^{\omega\geq}\lambda$ a tree.

(1) The tree T is called *big with respect to* μ if there exists an increasing continuous chain of subtrees $\bar{\mathbf{T}} = \langle T_i\colon i < \mu\rangle$ where $|T_i| < \mu$ for all $i < \mu$, and $T = \bigcup\{T_i\colon i < \mu\}$ such that $S(\bar{\mathbf{T}}) \not\equiv 0 \bmod D_\mu$ where we define

$$S(\bar{\mathbf{T}}) = \{\delta < \mu\colon \exists\{\eta_n \in {}^n\lambda \cap T_\delta\colon n < \omega\}[k < n \Rightarrow \eta_k < \eta_n] \wedge (\forall \alpha < \delta)\{\eta_n\colon n < \omega\} \not\subset T_\alpha\}$$

(2) When $\bar{\mathbf{T}}$ is a sequence as above, we say the sequence exemplifies the bigness of T with respect to μ.

Claim 2.2. (1) Let μ, λ and T be as in Definition 2.1. If $\bar{\mathbf{T}}^*$, and $\bar{\mathbf{T}}$ exemplify the bigness of T with respect μ then $S(\bar{\mathbf{T}}) \equiv S(\bar{\mathbf{T}}^*) \bmod D_\mu$.

(2) Suppose that T has the following special form: There exists a stationary set $S \subseteq S^\mu_{\aleph_0}$. For every $\delta \in S$ let $\eta_\delta \in {}^\omega\mu$ be a fixed increasing sequence converging to δ such that

$$T = \{\eta_\delta\colon \delta \in S\} \cup \{\nu\colon (\exists \delta \in S)[\nu < \eta_\delta]\}.$$

Then $S \equiv S(\bar{\mathbf{T}}) \bmod D_\mu$.

Proof. (1) Easy.

(2) Using (1) and Definition 2.1(2) it is enough to define a club $C \subseteq \mu$ which will do the work. Choose regular χ large enough such that $H(\chi) \supseteq 2^\mu$ and $\langle H(\chi), \in\rangle$ reflects the facts that S and $S(\bar{\mathbf{T}})$ are stationary sets and their definition ($\chi = (2^\mu)^+$ is big enough). Let $\mathfrak{U} = \langle H(\chi), \in, Q\rangle$ where Q is a unary predicate whose interpretation is the set of ordinals less than μ. Choose an increasing continuous elementary chain $\langle N_i\colon i < \mu\rangle$ of elementary submodels of $\mathfrak{U}$, such that $\|N_i\| < \mu$, and $N_i \ni \langle N_j\colon\ j < i\rangle$. Let $C = \{\delta < \mu\colon Q^{N_\delta} = \delta\}$; it is easy to verify that C exemplifies what we want. □

Here comes the definition of the largeness of a cardinal which will appear in the statement of the main theorem on the next definition. The reader may find the following advice useful: Immediately after reading Definitions 2.3 and 2.4

read the statements of Theorem 2.6, Lemma 2.7, and Conclusion D (below). The property to be defined below is the property which is mentioned in the statement of Theorem 2.5.

DEFINITION 2.3. Let λ, μ be cardinals and let α be an ordinal. We say that μ *is suitable for* α iff μ is a regular cardinal such that $\mu > \kappa$ (κ from Theorem 1.4) and for every T as in Definition 2.1(1) where $\overline{\mathbf{T}}$ exemplifies the bigness of T with respect to μ, we have the following:

Case 1. $\alpha = 1$. $\forall \chi < \mu\ \exists \mu^* < \mu$ regular such that $\mu^* > \chi$, and there exists an increasing continuous sequence $\overline{\mathbf{T}}^* = \langle T_i^* \subseteq T\colon i < \mu^* \rangle$ of subtrees of the tree $T^* =^{\mathrm{def}} \bigcup\{T_i^*\colon i < \mu^*\}$ such that $i < \mu^*$ implies $|T_i^*| < \mu^*$, and $S(\overline{\mathbf{T}}^*) \not\equiv 0 \bmod D_{\mu^*}$.

Case 2. $\alpha > 1$. Let T be a tree which is big with respect to μ exemplified by $\overline{\mathbf{T}} = \langle T_i\colon i < \mu \rangle$. Then $(\forall \chi < \mu)(\forall \beta < \alpha)\ \exists \mu^* < \mu$ where $\mu^* > \chi$ and μ^* is suitable for β such that there exists an increasing continuous sequence $\overline{\mathbf{T}}^* = \langle T_i^* \subseteq T\colon i < \mu^* \rangle$ of subtrees of the tree $T^* =^{\mathrm{def}} \bigcup\{T_i^*\colon i < \mu^*\} \subseteq T$ such that $i < \mu^*$ implies $|T_i^*| < \mu^*$, $S(\overline{\mathbf{T}}^*) \not\equiv 0 \bmod D_{\mu^*}$, and $S(\overline{\mathbf{T}}^*) = S(\overline{\mathbf{T}}) \cap \mu^*$.

Remark. Note that when μ is weakly compact then it is suitable for 1. If μ is weakly compact so that below it there is a coherent sequence of weakly compact cardinals (each of them reflects the same stationary set) then μ is suitable for 2. So clearly if μ is a Ramsey cardinal then it is suitable for every $\alpha < \mu$.

The following definition introduces a notion of bigness which is used to formulate the induction hypothesis in the proof of Theorem 2.5 (we could of course avoid introduction of this concept, but then the proof will look messy).

DEFINITION 2.4. (1) Let $T \subseteq {}^{\omega \geq}\lambda$ be a tree, n a natural number, β an ordinal, $\overline{\mathbf{k}} = \langle k(m)\colon m < n \rangle$ a sequence of numbers, $\overline{\boldsymbol{\mu}} = \langle \mu_\eta\colon \eta \in {}^{k(n-1) \geq}\lambda \cap T \rangle$ uncountable regular cardinals all less than λ. We say that $\overline{T}$ is *big with respect to* $(n, \beta, \overline{\mathbf{k}}, \overline{\boldsymbol{\mu}})$ iff

(a) $m_1 < m_2 \Rightarrow k(m_1) < k(m_2)$,

(b) $\eta(1), \eta(2) \in {}^{k(n-1)>}\lambda \cap T \Rightarrow \mu_{\eta(1)} = \mu_{\eta(2)}$ (denote this cardinal by $\mu(T, \overline{\boldsymbol{\mu}}, \overline{\mathbf{k}}, n)$, or sometimes simply $\mu(T)$)

(c) $\eta \in {}^{k(n-1)+1}\lambda \cap T \Rightarrow T[\eta]$ is big with respect to μ_η, and μ_η is suitable for β and $\mu_\eta > \beth_\beta(\prod\{\mu_\nu\colon \nu <_l \eta \wedge l(\nu) = l(\eta)\})$.

(d) $\eta \in {}^{k(n-1)\geq}\lambda \cap T$ if $\exists m < n$ such that $l(\eta) = k(m)$ then $|\mathrm{Succ}_T(\eta)| \geq \mu_\eta$

(2) Let $\lambda, n, T, \overline{\mathbf{k}}, \overline{\boldsymbol{\mu}}, \beta$ be as above. T is called *normal* iff $(\forall m < \omega)$ $[m < n \wedge \eta \in ({}^{k(m)>}\lambda - {}^{k(m-1)\geq}\lambda) \cap T] \Rightarrow |\mathrm{Succ}_T(\eta)| = \mu_\eta = 1$, and $\eta \in {}^{k(m)}\lambda \Rightarrow |\mathrm{Succ}_T(\eta)| = \mu_\eta$.

Recall [12, Definition VII 5.1] that $\delta(\kappa)$ is the not well ordering ordinal, namely $\delta(\kappa)$ is the first ordinal such that if a PC_κ class has a model of order type $\delta(\kappa)$ then it has a not well ordered model. Now the statement of the theorem we want to prove can be presented:

THEOREM 2.5. *Let K be a PC_κ class, λ a suitable cardinal for order $\delta(\kappa)$. If K has the λ-unsuperstability property then for every $\chi > \kappa$, $I(\chi, K) = 2^\chi$. Moreover, for $\chi > \kappa$ satisfying $[\chi$ regular $\vee\ \chi^{\aleph_0} = \chi > 2^{\aleph_0} \vee 0^\# \notin \mathbf{V}]$ we have $IE(\chi, K) = 2^\chi$.*

As mentioned earlier in the proof there is a use of a partition theorem which replaces the Erdös Rado theorem; this seems to be a proper place to introduce it.

THEOREM 2.6. *Let $n < \omega$, λ a suitable cardinal for γ, $\bar{\mathbf{k}} = \langle k(m)\colon m < n\rangle$ an increasing sequence of natural numbers.*

(1) *For every $l < \omega$, $\kappa < \lambda$, $\chi < \lambda$ such that $\chi > \kappa$ there exists $\mu(\chi, \kappa) < \lambda$ satisfying the following. Let $T \subseteq {}^{\omega \geq}\lambda$ be a normal tree and $\bar{\mu} = \langle \mu_\eta\colon \eta \in {}^{k(m)\geq}\lambda \cap T,\ m < n\rangle$ such that T is big with respect to $(n, \gamma, \bar{\mathbf{k}}, \bar{\mu})$. If $F\colon [T]^l \to 2^\kappa$ then there are $T' \subseteq T$ and $\bar{\mu}'$ such that T' is normal and big with respect to $(n, \gamma, \bar{\mathbf{k}}, \bar{\mu}')$ when $\bar{\mu}' = \langle \mu'_\eta\colon \eta \in {}^{k(n-1)\geq}\lambda \cap T'\rangle$; for $m < n-1$, $\eta \in {}^{k(m)}\lambda \cap T' \Rightarrow \mu'_\eta = \chi$, for $\eta \in {}^{k(n-1)}\lambda \cap T'$, μ'_η is suitable for γ. Furthermore we have the following the requirements from T'.*

(a) *For all $T^* \subseteq T'$ such that*

$$T^* \cap {}^{k(n-1)\geq}\lambda = T' \cap {}^{k(n-1)\geq}\lambda,$$

$\eta \in {}^{k(n-1)}\lambda \cap T^*$ implies $|T^*[\eta] \cap {}^{\omega}\lambda| = 1$, and $T^* \cap ({}^{k(n-1)\geq}\lambda \cup {}^{\omega}\lambda)$ is homogeneous with respect to F (i.e., for any l tuples $\bar{\eta}, \bar{\nu}$ from

$$T^* \cap \left({}^{k(n-1)\geq}\lambda \cup {}^{\omega}\lambda\right)$$

if $atp_{L(T)}(\bar{\eta}, \varnothing, T) = atp_{L(T)}(\bar{\nu}, \varnothing, T)$ then $F(\bar{\eta}) = F(\bar{\nu})$).

(b) *For every, $\eta \in {}^{k(n-1)}\lambda \cap T'$,*

$$\nu_1, \nu_2 \in T'[\eta] \cap {}^{\omega}\lambda \quad \text{and} \quad \eta_1, \ldots, \eta_{l-1} \in T' \cap \left({}^{k(n-1)\geq}\lambda - T'[\eta]\right),$$

we have $F(\nu_1, \bar{\eta}) = F(\nu_2, \bar{\eta})$.

(2) *Moreover $\mu(\chi, \kappa) \leq \beth_{l\cdot k(n-1)-1}(\chi)^+$.*

As we said in the introduction we defer the proof of Theorem 2.6 to the next section. But to prove Theorem 2.5 we need another combinatorial lemma whose proof will appear in the next section.

CONTINUATION OF LEMMA 2.7. *Let $n, T, \beta, \bar{\mathbf{k}}, \bar{\mu}$ be such that $T \subseteq {}^{\omega}\lambda$, and for all $\delta \in S^{\lambda}_{\aleph_0}$ fix an increasing sequence $\eta_\delta \in {}^{\omega}\lambda$ converging to δ: suppose that $T \subseteq \{\eta_\delta: \delta \in S^{\lambda}_{\aleph_0}\} \cup \{\nu: (\exists\delta \in S^{\lambda}_{\aleph_0})[\nu < \eta_\delta]\}$ and is big with respect to $(n, \beta, \bar{\mu}, \bar{\mathbf{k}})$. For every $\gamma < \beta$ there exists a sequence $\bar{\mathbf{k}}'$ of length $n + 1$ such that $\bar{\mathbf{k}}'|(n-1) = \bar{\mathbf{k}}|(n-1)$, and there exists a subtree T' of T such that T' is big with respect to $(n+1, \gamma, \bar{\mu}', \bar{\mathbf{k}}')$, and $\mu(T) = \mu(T')$.*

Proof of Theorem 2.5. Let λ be a suitable cardinal for $\delta(\kappa) + 1$, let M be a model such that $M \in K$ and M contains $\{\bar{\mathbf{a}}_\eta: \eta \in {}^{\omega \geq}\lambda\}$ which exemplifies the λ-unsuperstability property. Without loss of generality we may assume that T_1 (the first order theorem which exemplifies the fact that K is a PC_κ class) has Skolem functions and also the relations of $L(T)$. Our strategy to prove the theorem will be the use of three facts due to Shelah which appear implicitly in [12] in the proof of Theorem VIII 2.1.

FACT A. *Let K be a PC_κ class. If there are $M \in K$ and $\{\bar{\mathbf{a}}_\eta: \eta \in {}^{\omega \geq}\kappa\} \subseteq |M|$ tree-indiscernible then for every cardinal μ there are $N \in K$, and $\{\bar{\mathbf{b}}_\eta: \eta \in {}^{\omega \geq}\mu\} \subseteq |N|$ tree indiscernible such that N is the reduct to L of the Skolem hull of $\{\bar{\mathbf{b}}_\eta: \eta \in {}^{\omega \geq}\mu\}$ (in $L(T_1)$).*

FACT B. *In order to prove that $IE(\chi, K) = 2^\chi$ for some uncountable cardinal χ it is enough to find trees $\mathbf{I}_i \subseteq {}^{\omega \geq}\chi$, $i < 2^\chi$, such that for $i \neq j$, $E.M^1(\mathbf{I}_i)|L$ is not embeddable into $E.M^1(\mathbf{I}_j)|L$, where $E.M^1(\mathbf{I}_i)$ is the Skolem hull of $\{\bar{\mathbf{a}}_\eta: \eta \in \mathbf{I}_i\}$ in the language of T_1.*

FACT C. *Assume the hypothesis of Fact A. For every $\chi > \kappa$ there exists a family of trees $\{\mathbf{I}_i \subseteq {}^{\omega \geq}\chi: i < 2^\chi\}$ such that if $i \neq j$ then $E.M^1(\mathbf{I}_i)|L \not\cong E.M^1(\mathbf{I}_j)|L$, and moreover if χ is regular or $\chi = \chi^{\aleph_0} > 2^{\aleph_0}$ or $0^\# \notin \mathbf{V}$ then there exist trees $\mathbf{I}_i \subseteq {}^{\omega \geq}\chi$, $i < 2^\chi$, such that for $i \neq j$, $E.M^1(\mathbf{I}_i)|L$ is not embeddable into $E.M^1(\mathbf{I}_j)|L$.*

From the last three facts easily we can derive the following result.

CONCLUSION D. *If $M \in K$, $T \subseteq {}^{\omega \geq}\kappa$, $\{\bar{\mathbf{a}}_\eta: \eta \in {}^{\omega \geq}\kappa\} \subseteq |M|$, $\bar{\mathbf{k}} = \langle k(m) < \omega: m < \omega\rangle$ such that $\bar{\mathbf{k}}$ is increasing, T is a subtree of ${}^{\omega \geq}\kappa$ such that*

$$(\forall m < \omega)(\forall \eta \in {}^{k(m)}\kappa)|\mathrm{Succ}_T(\eta)| \geq \kappa$$

and $|\{\nu \in {}^{\omega}\kappa \cap T: \nu > \eta\}| \geq \kappa$ and $\{\bar{\mathbf{a}}_\eta: \eta \in T\}$ is a tree of indiscernibles then for $\chi > \kappa$ we have $I(\chi, K) = 2^\chi$ and if in addition χ satisfy

$$[\chi \textit{ regular } \vee \chi^{\aleph_0} = \chi > 2^{\aleph_0} \vee 0^\# \notin \mathbf{V}]$$

we have $IE(\chi, K) = 2^\chi$.

So to complete the proof of our theorem it is enough to find N, $T \subseteq {}^{\omega \geq} \kappa$, $\bar{\mathbf{k}}$, and $\{\bar{\mathbf{b}}_\eta : \eta \in \mathrm{T}\}$ as in Conclusion D. Let $\Phi = \{\varphi_n(\bar{\mathbf{x}}, \bar{\mathbf{y}}) : n < \omega\}$, and $A = \{\bar{\mathbf{a}}_\eta : \eta \in T\} \subseteq |M|$ exemplifying the λ-unsuperstability of K.

Choose a regular χ big enough such that $H(\chi) \supseteq P(P(|M| \cup \Phi \cup A \cup \lambda))$. Define a model $\mathfrak{U}^* = \langle H(\chi), \in, >, M, L, T_1, T, \Gamma, P, Q, \models, \Phi, A, c_\varphi, d_p \rangle_{\varphi \in L, p \in \Gamma}$, where $\in$ is the usual membership relation, $<$ a well ordering of the set $H(\chi)$, M a predicate for the universe $|M|$ and relations and functions for each symbol in the similarity type of M, T_1 the first order theory T_1, T a predicate for the tree T and relations and the function of $L(T)$, Γ the set of types needed in the definition of K as a PC_κ class, P a unary predicate for the set of ordinals less than λ, Q a unary relation for the ordinal $\delta(\kappa) + 1$, $\models$ the satisfaction relation (of $L(T_1)$), Φ and A stand for predicates to the corresponding set, c_φ and d_p are constants standing for each formula (of $L(M)$) and type of Γ respectively.

Note that since χ is regular and large enough, $\mathfrak{U}^*$ reflects everything we know on its predicates. Namely it reflects facts like "$M \in PC(T_1, \Gamma, L)$, $PC(T_1, \Gamma, L)$ has the λ-unsuperstability property, P is a suitable cardinal for $Q(= \delta(\kappa))$, and $P(= \lambda)$ satisfies Theorem 2.6 and Lemma 2.7". The reflection of those properties and similar ones will be used below freely without an explicit reference.

Let $T^* = Th(\mathfrak{U}^*)$,

$$\Gamma^* = \{x \in L \wedge x \neq c_\varphi : \varphi \in L\} \cup \{y \in \Gamma \wedge y \neq d_p : p \in \Gamma\}$$
$$\cup \{(\exists \aleph \in |M|)(\forall \varphi \in d_\varphi) M \models \varphi(\aleph) : p \in \Gamma\}.$$

Let $K^* = EC(T^*, \Gamma^*)$. Clearly, $\mathfrak{U}^* \in K^*$ (note the following important property of K^*: $\mathfrak{U}^* \in K^* \Rightarrow M^{\mathfrak{U}^*} \in K$). Since o.tp$(Q^{\mathfrak{U}^*}, \in^{\mathfrak{U}^*}) \geq \delta(\kappa)$ and $|T^*| \cdot |\Gamma^*| \leq \kappa$, K^* is a PC_κ class. Hence by the definition of $\delta(\kappa)$ there exist a model $\mathfrak{U} \in K^*$, such that there are $\alpha_n \in Q^{\mathfrak{U}}$, $n < \omega$, such that $\forall n < \omega$, $\mathfrak{U} \models$ "$\alpha_n < \alpha_{n-1} \wedge \alpha_0$ is the last element in Q".

Our aim is to define a model $\mathfrak{B} \in K^*$ such that $N = M^{\mathfrak{B}}$ will be the model we are seeking (i.e., satisfies the hypothesis of Conclusion D). We will define $\mathfrak{B}$ as a Skolem hull of a countable set of elements. (Remember that the model $\mathfrak{U}^*$ has a predicate $<$ which well orders the universe. Hence $L(T^*)$ has built in Skolem functions and the theory T^* contains Skolem functions. So any model for T^* has Skolem functions and it is meaningful to talk about Skolem hull of sets of elements.

Without loss of generality we may assume that $n < \omega \Rightarrow \alpha_{n+1} + \omega < \alpha_n$. Define $\beta_n = \alpha_n + \omega$ (for $0 < n < \omega$) and $\beta_0 = \alpha_0$.

By induction on $n < \omega$ defines four sequences (inside $\mathfrak{U}$), cardinals $\{\lambda_n : n < \omega\}$, trees $\{T_n : n < \omega\}$, an increasing sequence $\bar{\mathbf{k}} = \langle k(n) : n < \omega \rangle$, trees of cardinals $\{\bar{\mu}^n = \langle \mu^n_\eta : \eta \in T_n \rangle : n < \omega\}$ such that for all $n < \omega$:

(1) $\lambda_0 = \lambda$, $\lambda_{n+1} < \lambda_n$, λ_n is suitable for β_n.

(2) If $T_0 = T^{\mathfrak{U}}$, T_{n+1} is a subtree of T_n which is big with respect to

$$(n+1, \beta_n, \langle k(m) : m < n+1 \rangle, \bar{\mu}^n)$$

and $\langle \bar{\mathbf{a}}^n_\eta : \eta \in T_n \rangle$ is n-indiscernible (inside $M^{\mathfrak{U}}$ in the language of T_1) with respect to the structure of the tree T_n, and for every $\eta \in {}^{k(n-1)}\lambda \cap T'$,

$$v_1, v_2 \in T'[\eta] \cap {}^{\omega}\lambda \quad \text{and} \quad \eta_1, \ldots, \eta_{l-1} \in T' \cap \left({}^{k(n-1) \geq}\lambda - T'[\eta]\right),$$

then $\operatorname{tp}(\bar{\mathbf{a}}_{v_1}, A) = \operatorname{tp}(\bar{\mathbf{a}}_{v_2}, A)$ where $A = \{\bar{\mathbf{a}}_{\eta_1}, \ldots, \bar{\mathbf{a}}_{\eta_{l-1}}\}$.

(3) $\mu(T_n) \geq \beth_{\beta_n}(\mu(T_{n+1}))$.

Now we will derive the theorem from the above sequences. Let ATP_n be the set of atomic types in $\{P_k, <, <_l, h, P_\omega : k < n\}$ (a sublanguage of $L(T)$, the language of trees–see Definition 1.7) where $P_\alpha = (\eta : l(\eta) = \alpha\}$ for $\alpha \leq n$ or $\alpha = \omega$. We will identify ATP_n with a subset of ATP_{n+1} in the natural way. For every $p \in ATP_n$ let q_p be the first order type in the language of M (over the empty set) realized by $\bar{\mathbf{a}}_{v(1)}, \ldots, \bar{\mathbf{a}}_{v(n)}$ for any $v(1), \ldots, v(n)$ realizing the type p. By the n-indiscernibility of $\langle \bar{\mathbf{a}}^n_\eta : \eta \in T_n \rangle$ the type q_p is uniquely defined by p (the correspondence $p \vdash q_p$ is a function). Since $T_{n+1} \subseteq T_n$ note that if $p \in ATP_n$, $p^* \in ATP_{n+1}$ such that $p^* \supseteq p$ then $q_{p^*} \supseteq q_p$. Let $AT = \bigcup\{ATP_n : n < \omega\}$. For any chain $\bar{\mathbf{p}}$ (we consider $\langle AT, \subseteq \rangle$ as a partially ordered set) there is a corresponding chain $q_{\bar{\mathbf{p}}}$ of first order types in the language of M. By the finite character of consistency, clearly $\bigcup q_{\bar{\mathbf{p}}}$ is a consistent first order type (with infinitely many variables); we will denote this type by $q_{\bar{\mathbf{p}}}$ also. By the compactness theorem there exists a model $\mathfrak{B}^* \models T^*$ which realizes the set of types $\{q_{\bar{\mathbf{p}}} : \bar{\mathbf{p}}$ is a chain in $AT\}$. For every $n < \omega$ and $v(1), \ldots, v(n) \in T_n$ let $\bar{\mathbf{b}}_{v(1)}, \ldots, \bar{\mathbf{b}}_{v(n)} \in |M^{\mathfrak{B}^*}|$ be sequences which realize $q_{atp(v(1), \ldots, v(n), \varnothing, T_n)}$. It is easy to verify that

$$B = \left\{\bar{\mathbf{b}}_{v(1)}, \ldots, \bar{\mathbf{b}}_{v(n)} : v(1), \ldots, v(n) \in T_n, n < \omega\right\}$$

is tree indiscernible as required. But we have not yet finished the proof because there is no reason why $M^{\mathfrak{B}^*}$ omits Γ. So we have to replace $\mathfrak{B}^*$ by another model $\mathfrak{B}$ which will be the final model. It will be defined as the Skolem hull of a certain set of elements, using the model $\mathfrak{B}^*$. Let $\mathfrak{B} =^{\text{def}} E.M(B)$. It is standard to verify that $\mathfrak{B}$ is as required by showing that the model omits Γ^*. (Suppose that it does not. Then there is a term $\tau(\bar{x}_1, \ldots, \bar{x}_n)$ and $\bar{\mathbf{b}}_{v(1)}, \ldots, \bar{\mathbf{b}}_{v(n)} \in |M^{\mathfrak{B}}|$ such that $\tau(\bar{\mathbf{b}}_{v(1)}, \ldots, \bar{\mathbf{b}}_{v(n)})$ realizes a type from Γ^* which is a contradiction.) We can replace $\bar{\mathbf{b}}_{v(1)}, \ldots, \bar{\mathbf{b}}_{v(n)}$ by $\bar{\mathbf{a}}_{v(1)}, \ldots, \bar{\mathbf{a}}_{v(n)}$ from $|M^{\mathfrak{U}}|$ which realize a type from Γ^* contradicting the choice of $\mathfrak{U}$ as an element of K^*.

The only thing left to do is to construct the 4 sequences with the above properties. Here, Theorem 2.6 comes to our help. For $n = 0$ there is nothing to do. When $n \neq 0$, there is a natural coloring of increasing (remember there is a well ordering by $<$ of the universe of $\mathfrak{U}^*$) n-tuples from $\{\bar{\mathbf{a}}_\eta \colon \eta \in T_{n-1}\}$ by 2^κ colours (for each tuple take the first order type [over the empty set in the language L_1] it satisfies in M). By Theorem 2.6 there exists a subtree T' of T_n and a tree of cardinals $\bar{\mu}'$ such that T' is big with respect to

$$(n, \beta_{n+1}, \langle k(m) \colon m < n\rangle, \bar{\mu}')$$

and is homogeneous with respect to the above colouring. Define $\mu(T') = \beth_{\beta_{n+1}}(\kappa)^+$. By Lemma 2.7 there are $k_{n+1} < \omega$ such that $k_{n+1} > k_{n'}$ and T_{n+1} a subtree of T' such that T_n is big with respect to

$$(n + 1, \beta_{n+1}, \langle k(m) \colon m < n + 1\rangle, \bar{\mu}^{n+1})$$

where $\bar{\mu}^{n+1}|^{k(n)}\lambda = \bar{\mu}'|^{k(n)}\lambda$. We have to define the cardinals μ_η^{n+1} for η of length $k(n + 1)$, and at this point we use the assumption that we can reflect our stationary sets so that the bigness of the tree is preserved (see Definition 2.4(1) part (c)). Clearly T_{n+1} satisfies requirements (1), (2), and (2). $\square$

3. Proof of the partition theorem

Before beginning the proof of Theorem 2.5 let us quote a result due to Shelah (Theorem 2.6 in the appendix of [12]), which is used in the proof of Theorem 2.6.

THEOREM 3.1. *For every $n, m < \omega$ there exists $k = k(n, m) < \omega$ such that whenever $\lambda = \beth_k(\chi)^+$ the following is true: For all $f \colon [{}^{n\geq}\lambda]^m \to \chi$ there exists $T \subseteq {}^{n\geq}\lambda$ such that*

(i) *T is a tree, and if $\eta \in {}^{n>}\lambda \cap T$ then $|\mathrm{Succ}_T(\eta)| = \chi$;*

(ii) *if $\bar{\eta}, \bar{\nu}$ are m-tuples from T such that*

$$atp_{L(T)}(\bar{\eta}, \varnothing T) = atp_{L(T)}(\bar{\nu}, \varnothing, T)$$

then $f(\bar{\eta}) = f(\bar{\nu})$.

For the convenience of the reader and another reason (explained below), we shall reprove this theorem in the appendix of this paper. We present a different proof from the original, using the Erdos Rado theorem directly instead the polarized partition theorem used in the original proof. This gives an improvement of the bound on $k(n, m)$ (the original proof gives the bound $2^n + n$ for $k(n, n)$; the proof to be presented here shows that $k(n, m) \leq n \cdot m - 1$).

Proof of Theorem 2.6. Assume that $T \subseteq {}^{\omega\geq}\lambda$ is a tree which is big with respect to $(n, \gamma, \bar{\mathbf{k}}, \bar{\mu})$ where $\bar{\mu} = \langle \mu_\eta : \eta \in {}^{k(n-1)}\lambda \cap T \rangle$ and that we are given an ordinal $\beta < \gamma$, an integer $l < \omega$, and a colouring $F: [T]^l \to 2^\kappa$.

LEMMA (end homogeneity). *Let $\langle \lambda_\xi : \xi < \theta \rangle$ be a sequence of regular cardinals such that*

$$\lambda_\xi > \chi^{\Sigma\{\lambda_\zeta + |\alpha| + \theta : \zeta < \xi\}} + 2^{(\Pi\{\lambda_\zeta : \zeta < \xi\})}.$$

Then given any χ and α, for every family of sets $\{A_\xi \subseteq \lambda_\xi : \xi < \theta,\ A_\xi \not\equiv 0 \bmod_{\lambda_\xi}\}$, and every family of colorings $\{F_i : [\bigcup\{A_\xi : \xi < \theta\}]^m \to \chi \mid i < \alpha\}$ there are

$$\{B_\xi \subseteq A_\xi : B_\xi \not\equiv 0 \bmod_{\lambda_\xi}, \xi < \theta\} \quad \text{and} \quad \{a^*_\xi : \xi < \theta\}$$

such that for every $\bar{\mathbf{a}} \in [\bigcup\{B_\zeta : \zeta < \xi]^m$, $b \in B_{\xi*}$ and every $i < \alpha$ we have $F_i(\bar{\mathbf{a}}, b) = F_i(\bar{\mathbf{a}}, a^*_\xi)$.

Proof. For every $B \subseteq A$ let $S(A, B) =^{\text{def}} \{tp(a, B) : a \in A\}$ in the following expansion of the language of set theory: $\{\chi, \in, F_i : i < \alpha\}$. Clearly $|S(A, B)| \leq \chi^{|B|}$. Define a^*_ξ by induction on $\xi < \theta$ such that

(#) if $B_\zeta \subseteq A_\zeta$ stationary then there are stationarily many elements of A_ζ realizing the type $tp(a_{\xi*}, \bigcup\{B_\zeta : \zeta < \xi\})$.

Given $\{B_\zeta \subseteq A_\zeta : \zeta < \xi\}$ as above clearly there exists $a^* \in A_\xi$ such that the type $tp(a^*, \bigcup\{B_\zeta : \zeta < \xi\})$ is realized by stationarily many elements of A_ξ. (Why? Since $|S(A_\xi, \bigcup\{B_\zeta : \zeta < \xi\})| \leq \lambda_\xi$, by the regularity of λ_ξ there exists an element a^* which is suitable for $\bigcup\{B_\zeta : \zeta < \xi\}$, but since there are only $\Pi\{2^{\lambda}\zeta : \zeta < \xi\}$ possible choices for $\{B_\zeta : \zeta < \xi\}$.)

Now choose the sets $\{B_\zeta : \zeta < \theta\}$ by induction so that they will realize a large type. □

Stage 1. Prove the following claim: For every

$$\eta \in {}^{k(n-1)}\lambda \cap T', \quad v_1, v_2 \in T'[\eta] \cap {}^{\omega}\lambda,$$

and

$$\eta_1, \dots, \eta_{l-1} \in T' \cap \left({}^{k(n-1)\geq}\lambda - T'[\eta]\right),$$

we have $F(v_1, \bar{\eta}) = F(v_2, \bar{\eta})$. The claim follows from the last lemma by considering $F_i(x)$ as the coloring $F(x; \bar{\rho})$ when $\bar{\rho}$ are the entries from bounded part of the tree of the length of $\bar{\eta}$.

Stage 2. Use Theorem 3.1 combined with end-homogeneity of the ω sequences (from stage 1).

ASSUMPTION. For all $\delta \in S^{\lambda}_{\aleph_0}$, fix $\eta_\delta \in {}^{\omega}\lambda$ such that η_δ is increasing and converges to δ. Let $T \subseteq \{\eta_\delta: \delta \in S^{\lambda}_{\aleph_0}\} \cup \{\nu: (\exists \delta \in S^{\lambda}_{\aleph_0})[\nu < \eta_\delta]\}$.

Proof of Lemma 2.7. Clearly it is enough to prove the next two claims.

CLAIM A. *For every $\eta \in {}^{k(n-1)}\lambda \cap T$ there are a natural number $k(\eta) > k(n-1)$ and a set $T' \subseteq \{\nu \in {}^{k(\eta)}\lambda \cap T[\eta]: \eta < \nu\}$ such that $|T'| = \mu_\eta$ and*

$$(\forall \nu \in T')\big[\eta < \nu \Rightarrow T[\nu] \text{ is big with respect to } \mu_\eta\big].$$

CLAIM B. *There exists a normal subtree T^* of ${}^{k(n-1)}\lambda \cap T$ such that for all $\eta_1, \eta_2 \in T^*$ if $l(\eta_1) = l(\eta_2) = k(n-1)$ then $k(\eta_1) = k(\eta_2)$, and*

$$(\forall \eta \in T^*)(\forall m < n)\big[l(\eta) = k(m) \Rightarrow |\mathrm{Succ}^*_T(\eta)| = \mu_\eta\big].$$

First we prove Claim A, and later Claim B.

Proof of Claim A. Let $\eta \in {}^{k(n-1)}\lambda \cap T$ be given. Define $T = T[\eta]$, $S = S(T)$, $T = T[\eta]$, and $\mu = \mu_\eta$. Assume (for the same of contradiction) that for all $k < \omega$,

$$k > k(n-1) \Rightarrow \big|\{\nu \in {}^{k}\lambda \cap T;\ \eta < \nu \to T[\nu] \text{ is big with respect to } \mu\}\big| < \mu.$$

Namely $\forall k > k_{n-1}$, $\exists X_k \subseteq {}^{k}\lambda \cap T$, $|X_k| < \mu$ such that $\forall \nu \in ({}^{k}\lambda \cap T - X_k)$ there exists $C(\nu) \subseteq \mu$ closed unbounded such that $[\eta < \nu \to C(\nu) \cap S = \varnothing]$; and for ν such that $\eta \nless \nu$ defined $C(\nu) = \mu$. Let χ be regular such that

$$H(\chi) \supseteq \{S, \lambda, T, \langle X_k: k < \omega\rangle, \langle C(\nu); \nu \in {}^{\omega \geq}\lambda \cap T\rangle\}$$

and such that the model $\langle H(\chi), \in \rangle$ reflects everything. Let $P = \mu$, $Q = \lambda$,

$$M = \langle H(\chi), \in, \langle X_k: k < \omega\rangle, T, P, Q, S, C\rangle.$$

Choose $N^* < M$, $\|N^*\| = \mu$, such that $T^{N^*} = T^M$, $P^{N^*} = P^M$, $S^{N^*} = S^M$, $C^{N^*} = C^M$; let $H: Q^{N^*} \to P^{N^*}$ be an order preserving function. Finally let $N = \langle N^*, H\rangle$.

Let $\{N_i < N: i < \mu\}$ be an increasing continuous chain of models such that
(i) $N = \cup\{N_i: i < \mu\}$,
(ii) $\|N_i\| < \mu$,
(iii) $|N_i| \in N_{i+1}$,

(iv) any atomic type in $L(T)$ over ${}^{\omega>}Q^{N_i} \cap T^{N_i}$ is realized in N_{i+1}.
Let

$$C^1 = \{i < \mu: P^{N_i} = i\}, \qquad C^2 = \{i < \mu: T^{N_i} \text{ subtree of } T\},$$
$$C^3 = \{i \in C^1: (\forall j < i)[\operatorname{Sup} P^{N_{j+\omega}} < i]\},$$
$$C = C^2 \cap C^3.$$

CLAIM A.1. *There exists $S^* \subseteq C \cap S$ stationary such that for all $\delta \in S^*$ and every $n < \omega$, $T[\eta_\delta|n]$ is big with respect to μ.*

Proof. For the sake of contradiction assume there exists $C' \subseteq C$ such that for all $\delta \in C' \cap S$ there exists $n(\delta) < \omega$ such that $T[\eta_\delta|n(\delta)]$ is not big with respect to μ. Hence there exists $S^\# \subseteq C' \cap S$ stationary, and there exists $n(*) < \omega$ such that $\delta \in S^\# \Rightarrow n(\delta) = n(*)$.

Let $\{g_l: S^\# \to \mu | l < n(*)\}$, $g_l(\delta) =^{\text{def}} \eta_\delta[l]$. Since η_δ converges to δ, g_l is a regressive function for all $l < n(*)$. Hence by Fodor's theorem there are $\{S_l \subseteq S^\#: l < n(*)\}$, and $\{\alpha_l: l < n(*)\}$, such that for all $l < n(*)$, S_l is stationary; for $k < l$, $S_k \supseteq S^l$; for all $\delta \in S_l$ we have $\eta_\delta[l] = \alpha_l$. Hence

$$\left[\delta \in S^\#_{n(*)-1} \wedge l < n(*)\right] \Rightarrow \eta_\delta[l] = \alpha_l.$$

Namely, there exists $\upsilon \in {}^{n(*)}\lambda \cap T$ where $\upsilon = \langle \alpha_0, \ldots, \alpha_{n(*)-1}\rangle$ such that $S(T[\upsilon]) \supseteq S_{n(*)-1}$, a contradiction. □

CLAIM A.2. *There are $i \in C$, $\delta \in C \cap S^*$, $n < \omega$, such that $\eta_\delta|n \in T^{N_i}$ but* $\operatorname{Sup} P^{N_{i+1}} < \eta_\delta[\eta]$.

Proof. Easy (see the proof of Theorem VIII 2.2(1) of [12]). □

Back to the proof of Claim A. For all $n < \omega$ there exists $i(n) < \mu$ such that $\eta_\delta|n \in T^{N_{i(n)}}$. We have $N_{i(n)} \prec N$,

$$N \vDash \text{“}T[\eta_\delta] \text{ is big with respect to } \mu\text{”},$$

by Claim 1A, so

$$N_{i(n)} \vDash \text{“}T[\eta_\delta|n] \text{ is big with respect to } P\text{”}.$$

Since $\langle X_k: k < \omega\rangle \in N_{i(n)}$ and $N_{i(n)} \vDash \mathbf{ZF}$, we have $N_{i(n)} \vDash$ “$\eta_\delta|n \in X_n$”; but $N_{i(n)} \vDash |X_n| < \mu$ so there exists $j_n \in P^{N_{i(n)}}$ such that

$$(\#) \qquad N_{i(n)} \vDash (\forall \eta \in X_n)[\eta(n-1) < j_n].$$

Since $\{N_i\colon i<\mu\}$ is increasing, for every $\xi > i(n)$ we have

$$N_\xi \models (\forall \eta \in X_n)[\eta(n-1) < j_n].$$

By Claim A.2, $\exists i, \delta, n$ such that

$$N_i \models \eta_\delta | n \in X_n \quad \text{and} \quad \eta_\delta[n] > \operatorname{Sup} P^{N_{i+1}}.$$

By requirements (iv), and (iii) in the choice of $\{N_i\colon i<\mu\}$ there exists $v \in {}^{n+1}\lambda \cap T^{N_{i+1}}$ where v has the form $v = \eta_\delta | n(\alpha)$, and $v[n+1] = \alpha$ with $\alpha \in P^{N_{i+1}} - j_{n+1}$ such that

$$atp\left(\eta_\delta|(n+1), {}^{\omega >}\left(Q^{N_{i+1}}\right) \cap T^{N_{i+1}}, T\right) = atp\left(v, {}^{\omega >}\left(Q^{N_{i+1}}\right) \cap T^{N_{i+1}}, T\right)$$

and $T[v]$ is big with respect to μ. Hence $N_{i+1} \models$ "$T[v]$ is big with respect to P". So by the definition of X_{n+1}, we have $v \in X_{n+1}$. Hence by (#), $v[n] < j_{n+1}$ contradicting the choice of α as satisfying $\alpha \in P^{N_{i+1}} - j_{n+1}$. □

So we are left with Claim B; namely, there is a subtree with the same $\bar{\mu}$ (which is big with respect to the same quadrupole) and for all $\eta_1, \eta_2 \in {}^{k(n-1)}\lambda \cap T$ we have $k_{\eta 1} = k_{\eta 2}$.

A stronger statement than Claim B will be proved. Its proof is based on the idea of the first induction step of in the proof of Theorem 3.1 (here we refer to the proof of Theorem 2.6 in the appendix to [12] as the original proof).

Now comes the lemma which implies Claim B.

LEMMA 3.2. *Let $T \subseteq {}^{\omega \geq}\lambda$ be as in Claim* B. *For every $l < \omega$ and every f: ${}^{l\geq}\lambda \cap T \to \omega$ there exists a subtree T^* of ${}^{l\geq}\lambda \cap T$ such that for all $\eta_1, \eta_2 \in T^*$, if $l(\eta_1) = l(\eta_2) = k(m)$ for some m then $f(\eta_1) = f(\eta_2)$, and for all $\eta \in T^*$, if $l(\eta) = k(m) < l$ for some m then $|\mathrm{Succ}_{T^*}(\eta)| = \mu(T^*)$.*

Now we use Lemma 3.2 to prove Claim B.

Proof of Claim B. Let $l = k_{n-1}$ and for $\eta \in {}^{k(n-1)\geq}\lambda \cap T$ define the value of $f(\eta)$ as follows:

$$f(\eta) = \begin{cases} k(\eta) & \text{if } l(\eta) = k(n-1) \\ 0 & \text{otherwise (i.e., if } l(n) < k(n-1)) \end{cases}$$

By Lemma 3.2 there exists $T^* \subseteq {}^{k(n-1)\geq}\lambda \cap T$ which has many splittings at points of height $k(m)$ and for every $\eta_1, \eta_2 \in {}^{k(n-1)}\lambda \cap T$ we have $k(\eta_1) = k(\eta_2)$. □

Proof of Lemma 3.2. Define families of trees $\{\langle T_\eta \subseteq \{v \in {}^{l\geq}\lambda \cap T\colon \eta \leq v\}\colon \eta \in {}^{j}\lambda \cap T\rangle\colon j \leq l\}$ and functions $\{n_j\colon {}^{l>}\lambda \cap T \to \omega \mid j \leq l\}$ such that for all $\eta \in {}^{l\geq}\lambda \cap T$:

$(1)_\eta$ If $v \in {}^{l>}\lambda \cap T$ and for some m, $l(v) = k(m)$, then $|\mathrm{Succ}_{T_\eta}(v)| = \mu(T)$

$(2)_\eta$ If $i \leq l$ and $v \in {}^{i}\lambda \cap T_\eta$ (for $\eta \in {}^{i>}\lambda \cap T_\eta$) then $f(v) = n_i[\eta]$. Existence of trees and functions as above completes the proof. We will show why it completes the proof and later will construct families with the required properties.

We have to define a tree T^* and do so by defining a sequence of trees $\{T_i^* \subseteq {}^{l\geq}\lambda \cap T\colon i \leq l\}$ by induction on $i < l$, and at the end take $T^* = T_l^*$. The sequence of trees $\{T_i^*\colon i \leq l\}$ has to satisfy:

(a) $i < j \leq l \Rightarrow T_j^* \subseteq T_i^*$ and $T_j^* \cap {}^{j\geq}\lambda = T_i^* \cap {}^{j\geq}\lambda$.

(b) For $i < l$ if $v \in {}^{i>}\lambda \cap T_i^*$ and for some m, $l(v) = k(m)$ then $|\mathrm{Succ}_{T_1^*}(v)| = \mu(T)$

(c) For $i \leq l$, if $\eta_1, \eta_2 \in {}^{i\geq}\lambda \cap T_i^*$ and $l(\eta_1) = l(\eta_2) = k(m)$ for some m then $f(\eta_1) = f(\eta_2)$.

Clearly T_l^* has the required property. Why does a set of $l + 1$ trees satisfying (a)–(c) exist? We can define one by induction on $i \leq l$.

Case 1. $i = 0$. Let $T_0^* = \{\langle\ \rangle\}$.

Case 2. $i > 0$. Assume $\{T_j^*\colon j < i\}$ is defined; we shall define T_i^*. If there exists m such that $k(m) = i - 1$ for every $\eta \in T_{i-1}^*$ choose $S(\eta)$, $n_i(v_1) = n_i(v_2)$; this is possible since $\mu(T)$ is uncountable regular. Now let

$$T_i^* = \left({}^{i-1>}\lambda \cap T_i^*\right) \cup \left\{v \in T_i^*\colon (\exists \eta \in T_{i-1}^*)\left[v | i \in \mathrm{Succ}_{T^*i}(\eta)\right]\right\}.$$

By properties (1) and (2), clearly T_i^* is as required.

Now we have to construct sequences of trees and functions so that (1) and (2) will hold. By induction on $j \leq n$ and for $\eta \in {}^{n-j}\lambda \cap T$ define

$$T_\eta \subseteq \left\{v \in {}^{l\geq}\lambda \cap T\colon \eta \leq v\right\}$$

and numbers

$$n_l(\eta) < \omega, \ldots, n_{l-j}(\eta) < \omega$$

such that $(1)_\eta$ holds and $(2)_\eta$ hold for i satisfying $l - j \leq i \leq l$.

For $j = 0$ let $T_\eta = \{\eta\}$. Consider $\{T_\eta\colon \eta \in {}^{l-j+1}\lambda\}$ and let $\eta \in {}^{l-j}\lambda$ for any β such that $\eta\hat{\ }\langle\beta\rangle \in \mathrm{Succ}_T(\eta)$ is a well defined j-tuple

$$\left\langle \eta_l(\eta\hat{\ }\langle\beta\rangle), \ldots, n_{l-j+1}(\eta\hat{\ }\langle\beta\rangle)\right\rangle$$

of natural numbers. Then there exists $S(\eta) \subseteq \mathrm{Succ}_T(\eta)$, and numbers $\eta_l, \ldots, \eta_{l-j+1}$ such that $|S(\eta)| = (T)$, and for every β such that $\eta\hat{\ }\langle\beta\rangle \in S(\eta)$,

$$n_l(\eta\hat{\ }\langle\beta\rangle) = n_l, \ldots, n_{l-j+1}(\eta\hat{\ }\langle\beta\rangle) = n_{l-j+1} \quad \text{and} \quad n_{l-j}(\eta) = f(\eta).$$

Now define

$$T_\eta = \{\eta\} \cup \left\{T_{\eta\hat{\ }\langle\beta\rangle} : \left\langle n_l(\eta\hat{\ }\langle\beta\rangle), \ldots, n_{l-j+1}(\eta\hat{\ }\langle\beta\rangle)\right\rangle = \left\langle n_l, \ldots, n_{l-j+1}\right\rangle\right\}.$$

Clearly we are done. □

4. An application to the theory of modules

In a manuscript of their book [4], Fuchs and Salce asked a question about the existence of a structure theorem for a certain class of modules over a fixed uniserial domain. We shall answer this question using our main theorem from Section 2 (Theorem 2.5). But at first we need a few definitions.

DEFINITION 4.1. (1) A ring R is called a *uniserial domain* iff it is an integral domain (i.e., is commutative, has a multiplicative identity, and no zero divisors), and the set $\{I \subseteq R : I \text{ an ideal in } R\}$ is linearly ordered by inclusion.

(2) Div_R is the class of all divisible left modules over R. Namely, Div_R is the class of all left R-modules such that

$$M \in \mathrm{Div}_R \Leftrightarrow (\forall m \in M)(\forall r \in R - \{0\})(\exists n \in M)[r \cdot n = m].$$

(3) Tor_R is the class of all torsion left R-modules. Namely, Tor_R is the class of all R-modules such that

$$M \in \mathrm{Tor}_R \Leftrightarrow (\forall m \in M)(\exists r \in R)[r \neq 0 \wedge r \cdot m = 0].$$

(4) Let $K_R = \mathrm{Div}_R \cap \mathrm{Tor}_R$.

The following is known (see [4]).

FACT 4.2. *Let R be a uniserial domain. There exists a fixed number of cardinal invariants such that every torsion free (i.e.,* $(\forall m \in M)(\forall r \in R)[r \neq 0 \rightarrow r \cdot m \neq 0]$*) module from* Div_R *can be characterized by a cardinal invariant.*

In light of the above fact it is natural to ask the following:

Question 4.3 (see [4]). Can every member of K_R be characterized by few cardinal invariants?

We will answer this question negatively by proving a general theorem about classes of modules over integral domains which are not Notherian (has an infinite increasing chain of ideals). So we are not restricting ourselves to uniserial rings.

THEOREM 4.4. *Let R be an integral domain and let K_R be the class of torsion divisible modules over R. If R is not Notherian then K_R has the λ-unsuperstability property.*

COROLLARY 4.5. *Assume there exists a Ramsey cardinal λ. Let R be an integral domain such that $|R| < \lambda$. If R is not Notherian then for every $\chi > |R|$, $I(\chi, K_R) = 2^{\chi}$. Moreover, for $\chi > |R|$ satisfying*

$$\left[\chi \textit{ regular} \vee \chi^{\aleph_0} = \chi > 2^{\aleph_0} \vee 0^{\#} \notin \mathbf{V}\right]$$

we have $IE(\chi, K_R) = 2^{\chi}$.

Proof of Corollary 4.5. The corollary follows directly from Theorem 4.4 and Theorem 2.5. □

We also claim that the Corollary answers Question 4.3. Why? Suppose for example that every member of K_R can be characterized by $\aleph_0$ cardinal invariants. So given a cardinal $\lambda = \aleph_\alpha$, let's count the number of isomorphism types of modules of power λ. Since every module can be characterized by $\aleph_0$ cardinals all $\leq \aleph_\alpha$, certainly their number can not exceed $|\alpha|^{\aleph_0}$. So in order to refute the assumption that there are few cardinal invariants (namely a structure theorem) which determine the isomorphism types, it is enough to pick a cardinal $\aleph_\alpha$ such that $\aleph_\alpha \geq |\alpha|^{\aleph_0}$ and show that $I(\aleph_\alpha, K_R) > \aleph_\alpha$. Instead, dealing with the above example of a structure theorem which is given by $\aleph_0$ cardinals we can show that in general if a class has a structure theorem (every element in the class is determined by a family of families of families of families of...of cardinals) then the number of isomorphism types of structures of cardinality $\aleph_\alpha$ is bounded by a bounded amount (i.e., does not depend on α) of iterations of the power set operation to the cardinal $|\alpha|$. So by choosing a proper cardinal $\aleph_\alpha$ such that $\aleph_\alpha$ is greater than a bounded iteration of the power set operation to the cardinal $|\alpha|$. The statement $I(\aleph_\alpha, K) = 2^{\aleph_\alpha}$ (for arbitrary large α) certainly refutes the assumption that a structure theorem exists. For more on this point see [14].

Why did we include this result in this paper, and not in a paper about first order theories? In other words, why is K_R a non-elementary class? Certainly Div_R is an elementary class (see below) but the point is that Tor_R is not. But since K_R is our main concern here let's show the following.

LEMMA 4.6. *The class K_R is not an elementary class.*

Proof. Suppose $M \in K_R$. Find an elementary extension of M which has a non torsion element. Use the compactness theorem to show that $CD(M)$ is consistent with the set of sentences $\{a \cdot c \neq 0\colon a \in R - \{0\}\}$ where a is a unary function symbol standing for multiplication by the element a, and c is a new constant (to be interpreted as an element of the module). Let $\{a_1, \dots, a_n\} \subseteq R - \{0\}$. Since R is an integral domain, $\prod\{a_k\colon 1 \leq k \leq n\} \neq 0$. Since M is divisible, if $b \in M - \{0\}$ then $M \models (\exists y)[a_1 \cdots a_n \cdot y = b]$. □

So K_R is a non-elementary class, but why it is a PC_κ class? There are many arguments to show this. Let L be a similarity type which contains the usual language of modules (function symbols for $+, \cdot$; and a constant for 0), and $|R|$ unary function symbols for every element of R. It is easy to define a $\psi \in L_{|R|^+,\omega}$ such that $K_R = \mathrm{Mod}(\psi)$. The divisibility can be expressed in a first order logic by a set of sentences of cardinality $|R|$: for each element $r \in R$ add the sentence $(\forall m)(\exists n)[r \cdot n = m]$. To force the elements of $\mathrm{Mod}(\psi)$ to be torsion add the $L_{|R|^+,\omega}$ sentence

$$(\forall m)\bigvee\{r \cdot m = 0\colon r \in R \wedge r \neq 0\}.$$

Proof of Theorem 4.4. Since R is not Noetherian let $\{I_n\colon n < \omega\}$ be a strictly increasing sequence of ideals of R. For every $n < \omega$ pick $r_n \in I_{n+1} - I_n$. For every cardinal λ define a module M_λ^0 which is freely generated by the elements $\{x_\eta\colon \eta \in {}^{\omega \geq}\lambda\}$ subject to the following relations:

(i) $r_0 \cdot \chi_\eta = 0$;

(ii) for every $\eta \in {}^\omega\lambda$ and every $m < \omega$,

$$r_m \cdot (x_\eta - x_{\eta|0} - x_{\eta|1} - x_{\eta|2} - \cdots - x_{\eta|m}) = 0.$$

CLAIM 4.7. (1) $M_\lambda^0 \in \mathrm{Tor}_R$.

(2) *If* $\eta^* \in {}^\omega\lambda$ *and* $i \neq \eta^*[n]$ *then*

$$M_\lambda^0 \models r_{n+1} \cdot (x_{\eta^*} - x_{\eta^*|0} - \cdots - x_{\eta^*|n} - x_{\eta^*|n\langle i\rangle}) \neq 0.$$

Proof of Claim 4.7. (1) Trivial; follows immediately from the definition of M_λ^0 (part (i)).

(2) We show this by showing a stronger statement, namely that there exists another R-module and a homomorphism g from M_λ^0 into the second module carrying the product in question to a non zero element. The other module to be

considered is R/I_{n+1} which is an R-module in a natural way. The definition of the homomorphism is by cases: Let $v \in {}^{\omega \geq}\lambda$. Then $g(x_v)$ is 0 if $\eta^* | (n+1)$ is not equal to v (where v is a finite sequence) or when v is an infinite sequence but $\eta^* | (n+1)$ not an initial segment of η, $g(x_v)$ is $1 + I_{n+1}$ whenever $\eta^* | (n+1) = v^*$ or when v is an infinite sequence then $\eta^* | (n+1) < v$. To show that g is a homomorphism it is enough to show that g preserves the relations (i), (ii) from the definition of M_λ^0.

Clearly (i) is preserved since $r_0 \in I_1 \subseteq I_{n+1}$ and clearly $R/I_{n+1} \vDash r_0 \cdot a = 0$ for any element of R/I_{n+1} (hence in particular $g(x_v)$). For a relation of type (ii), we distinguish between two possibilities according to the specific relation with which we are dealing. We are given a sequence η, and a natural number m: if $m \leq n$ then since the sequence of ideals is increasing any product which has r_m as a factor is zero. So we are left with the possibility $m > n$. Now check with the definition of the mapping g. Let $\eta \in {}^{\omega}\lambda$ be the sequence which is mentioned by the relation. There are two possibilities: $\eta^* | (n+1)$ is an initial segment of η or is not an initial segment. □

Since the last claim provides us with a divisible R-module with the λ-unsuperstability property (let $\varphi_n(x_\eta, \bar{y}_v) =^{\text{def}} r_n \cdot (x_\eta - y_{v|0} - \cdots - y_{v|n}) = 0$ for $\eta \in {}^{\omega}\lambda$, and $v \in {}^{n}\lambda$). In order to complete the proof of Theorem 4.4 it is enough to prove the following:

CLAIM 4.8. *There exists a module $M_\lambda^1 \in K_R$ such that $M_\lambda^1 \supseteq M_\lambda^0$.*

Proof. Let's show that Tor_R has the following two properties:

(a) Tor_R is closed under direct limits (obvious).

(b) Let $N \in \mathrm{Tor}_R$, $x \in N$, and $a \in R$. There exists $N' \in \mathrm{Tor}_R$ such that $N' \supseteq N$ and

$$N' \vDash (\exists y)[a \cdot y = x].$$

(Let $N^* = N \oplus yR$, I the ideal of N^* generated by $a \cdot y - x$. Now clearly $N' =^{\text{def}} N^*/I$ is as required.)

It is obvious that by iterating (a) and (b) the claim follows. □

5. Combinatorial appendix

Here we prove the combinatorial theorem we need, namely we reprove Theorem 2.6 of the appendix of [12], the one which stated in §3 as Theorem 3.1. In fact we prove a better theorem (we also hope that the reader will find the proof presented here easier than the original):

(i) We improve the bound to $k(n, n) < n^2$. (versus $k(n, n) < 2^n + n + 1$ in the original).

(ii) Instead of one colouring we deal simultaneously with many colourings.

We will prove the following theorem which clearly implies Theorem 3.1.

THEOREM 5.1. (1) *For every $c, n, m < \omega$ and every cardinal χ there exists $k = k(n, m) < \omega$ such that if $\lambda \geq \beth_k(\chi)^+$ then for any C_i: $[{}^{n\geq}\lambda]^{k_1} \to \chi$ $(i < c)$ where $m = \operatorname{Max}\{b_i : i < c\}$ there exists $T \subseteq {}^{n\geq}\lambda$ such that*

(i) *T is a tree and $\eta \in {}^{n\geq}\lambda \cap T \Rightarrow |\mathrm{Succ}_T(\eta)| = \chi$ and*

(ii) *for any $\bar{\eta}, \bar{\nu} \in [T]^{<\omega}$, $atp_{L(T)}(\bar{\eta}, \varnothing, T) = atp_{L(T)}(\bar{\nu}, \varnothing, T) \Rightarrow C_i(\bar{\nu}) = C_i(\bar{\eta})$ for every $i < c$.*

(2) *Moreover $K(n, m) \leq n \cdot m - 1$.*

Instead of proving Theorem 5.1 directly we prefer to prove a more general theorem (which is more natural and elegant from the combinatorial point of view) which clearly implies Theorem 5.1 (the only change is that χ^+ from part (ii) is replaced by a variable). Theorem 5.1 follows from the theorem to be presented below by an application of the following instance of the Erdos Rado theorem: $\beth_{k\cdot n-1}(\chi)^+ \to (\chi^+)_{\chi^{k\cdot n}}$.

THEOREM 5.2. *Assume $\lambda \to (\mu^+)_{\chi^{k\cdot n}}$. For every set of functions*

$$\left\{C_i : [{}^{n\geq}\lambda]^{k(i)} \to \chi \,|\, i \leq c < \omega\right\}$$

where $k \geq \operatorname{Max}\{k(i) : i \leq c\}$ there exists $T \subseteq {}^{n\geq}\lambda$ such that:

(i) *T is a tree and $\eta \in {}^{n>}\lambda \cap T \Rightarrow |\mathrm{Succ}_T(\eta)| = \mu$.*

(ii) *For every $i < c$ if $\bar{\eta}, \bar{\nu} \in [T]^{<\omega}$ such that*

$$atp_{L(T)}(\bar{\eta}, \varnothing, T) = atp_{L(T)}(\bar{\nu}, \varnothing, T)$$

then $l(\bar{\eta}) = k(i) \Rightarrow C_i(\bar{\eta}) = C_i(\bar{\nu})$.

Proof of Theorem 5.2. We code the structure of the coloring of the full tree by defining a $k \cdot n$-place function F from λ to χ. But we need some notation. For $\bar{\alpha} = \langle \alpha_0, \ldots, \alpha_{k\cdot n-1}\rangle$ a sequence of ordinals less than λ, and a sequence η of natural numbers all less than $k \cdot n$ define

$$F(\bar{\alpha}) = \left\{\left\langle i, \eta, C_i(h(\bar{\alpha}, \eta))\right\rangle : i < c, \eta \in [k \cdot n]^{k(i)}\right\}.$$

So F is an $k \cdot n$-place function from λ and its range is a set of cardinality $\leq \chi + \aleph_0$. By the assumption $\lambda \to (\mu^+)_{\chi^{k\cdot n}}$ there exists $A \subseteq \lambda$ of cardinality

μ^+ which is homogeneous with respect to F. Without loss of generality we may assume that $A = \{j\colon j < \mu^+\}$.

We will define a set $T \subseteq {}^{n\geq}(\mu^+)$ and prove that T has the required properties: Let

$$\begin{aligned} T = \{\langle\ \ \rangle\} &\cup \{\langle\alpha_1\rangle\colon \alpha_1 < \mu\} \\ &\cup \{\langle\alpha_1, \alpha_2\rangle\colon (1+\alpha_1)\cdot\mu \leq \alpha_2 < (1+\alpha_1+1)\cdot\mu\} \\ &\cup \cdots \cup \{\langle\alpha_1, \ldots, \alpha_n\rangle\colon (((1+\alpha_1)\cdot\mu + 1 + \alpha_2) \\ &\qquad\qquad \cdot\mu + \cdots + \alpha_{n-1})\cdot\mu \leq \alpha_n \\ &\qquad < (\cdots((1+\alpha_1)\cdot\mu+\alpha_2)\cdot\mu^+ \cdots + \alpha_{n-1}+1)\cdot\mu\}. \end{aligned}$$

From the definition of T it follows that part (i) of Theorem 5.2 holds (i.e., T is a tree, and each level $< n$ has μ splittings). Also it follow that if $\eta, v \in T$ such that $\eta | l = v | l$ and $\eta[l] < v[l]$ then for all $l \leq j \leq \operatorname{Min}\{l(v), l(\eta)\}$, $v[j] > \eta[j]$. Part (ii) of the theorem (homogeneity of T with respect to the coloring) follows from the homogeneity of μ^+ $(= A)$ with respect to the coloring F. Let $\bar{\boldsymbol{\eta}}, \bar{\boldsymbol{\nu}} \in [T]^{k(i)}$. Assume $atp_{L(T)}(\bar{\boldsymbol{\eta}}, \varnothing, T) = atp_{L(T)}(\bar{\boldsymbol{\nu}}, \varnothing, T)$ which means that

$$\bar{\boldsymbol{\nu}}^* = \langle v_l[j]\colon l < k(i),\, j < l(v_l)\rangle \quad \text{and} \quad \bar{\boldsymbol{\eta}}^* = \langle \eta_l[j]\colon l < k(i),\, j < l(\eta_l)\rangle$$

have the same order type, hence $F(\bar{\boldsymbol{\nu}}^*) = F(\bar{\boldsymbol{\eta}}^*)$. Since the first coordinates of $F(\bar{\boldsymbol{\nu}}^*)$, and $F(\bar{\boldsymbol{\eta}}^*)$ coincide, we have $C_i(\bar{\boldsymbol{\nu}}) = C_i(\bar{\boldsymbol{\eta}})$. □

REFERENCES

1. C.C. CHANG, "Some remarks on the model theory of infinitary theory of infinitary languages in *The syntax and semantics of infinitary languages*, edited by J. Barwise, Springer Verlag, New York, 1968, 36–63.
2. C. FAITH, *Algebra II: Ring Theory*, Springer Verlag, New York, 1976.
3. ______, *Big decompositions of modules*, Notices Amer. Math. Soc., vol. 18 (1971), pp. 400.
4. L. FUCHS and L. SALCE, *On modules over uniserial domain*, manuscript, April 1984.
5. P. GRIFFITH, *On the decomposition of modules and generalized left uniserial rings*, Math. Ann., vol. 184 (1970), pp. 300–308.
6. R. GROSSBERG and S. SHELAH, *On the number of non isomorphic models of an infinitary theory which has the infinitary order property Part A*, J. Symbolic Logic, to appear.
7. ______, *On the number of non isomorphic models of an infinitary theory which has the infinitary order property Part B*,
8. ______, *On the number of non isomorphic models of an infinitary theory which has the not super stability property*, Abstracts Amer. Math. Soc., #84T-03-371.
9. ______, In preparation.
10. S. SHELAH, *A combinatorial problem, stability and order for models and theories in infinitary languages*, Pacific J. Math., vol. 41 (1972), pp. 247–261.
11. ______, *Construction of many complicated uncountable structurers and Boolean algebras*, Israel J. Math., vol. 45 (1983), pp. 100–146.
12. ______, *Classification theory* North Holland, 1978.
13. ______, *Classification theory*, second Revised edition, North Holland, 19??

14. ______, *Classification of first order theories which have a structure theory*, Bull. Amer. Math. Soc., vol. 12 (1985), pp. 227–233.
15. ______, *Classification theory for non-elementary classes I: The number of uncountable models of* $\psi \in L_{\omega_1, \omega}$ *Parts A, B* Israel J. Math., vol. 46 (1983), pp. 212–273.
16. ______, *Classification theory for non-elementary classes II: Abstract elementary classes.*

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