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ON THE NOVIKOV AND BOONE-BORISOV GROUPS

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In Memoriam W.W. Boone

1. Introduction

In the history of word problems in group theory the fundamental role was played by pioneering works of P.S. Novikov [1] and W. Boone [2]. The construction by Novikov in [1] of the centrally-symmetric group $\mathfrak{A} = \mathfrak{A}_{pd\mu l\rho}$ has never been given any further analysis different from [1]. The construction of Boone's group $G(T, q)$ [2] was analysed by many authors who introduced a number of groups which may be called the modifications of Boone's construction (for example, see [3], [4], [5]). One of these modifications is the construction due to V.V. Borisov [6]. We call the group $\Gamma(\Pi, P)$ from Borisov's work the Boone-Borisov group.

Our aim in this note is to make a survey of the author's recent results on the groups $\mathfrak{A}$ and $\Gamma(\Pi, P)$. The group $\mathfrak{A}$ has the "big" subgroup $\mathfrak{A}_{d\mu l\rho}$.

THEOREM 1. *Novikov's group* $\mathfrak{A}_{d\mu l\rho}$ *has a standard basis.*

This theorem was announced by the author in [7]. Theorem 1 provides a comparatively short proof for the criterion of equality of words in $\mathfrak{A}_{d\mu l\rho}$ which is the main theorem of chapters I-IV of [1] (the remaining two chapters V, VI of [1] treat some nongroup combinatorial calculus).

THEOREM 2. *The Boone-Borisov group* $\Gamma(\Pi, P)$ *has a standard basis.*

From Theorem 2 it is comparatively easy to deduce that the word problem in $\Gamma(\Pi, P)$ is Turing (or even truth-table) equivalent to the problem of the equality to the word P in the initial semigroup Π. Since for any Turing (truth-table) degree of unsolvability α there exists a f.p. semigroup in which for example the problem of the equality to the empty word has just the given degree of unsolvability, it follows that the Boone-Borisov group may have arbitrary Turing (truth-table) degree of unsolvability. The existence of f.p.

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groups with arbitrary Turing degrees of unsolvability was independently established by many authors (A.A. Friedman, W. Boone, G.S. Ceitin, Clapham; for example, see [8] and its list of references; recently another proof based on the Aanderaa construction was given in [9]). For truth-table degrees the similar fact was established in the works of M.K. Valiev [10] and D. Collins [11]. Also note the following:

COROLLARY. *For any Turing degree of unsolvability α there exists a group with* 14 *defining relations in which the word problem has degree α.*

This corollary follows from Theorem 2 and the following theorem due to D. Collins [5] (which was proved with the use of certain constructions due to G.S. Ceitin and Yu.V. Matijasevic; cf. [5]): for any Turing degree α there exists a semigroup Π with 2 generators and 3 defining relations and a word P in Π such that the problem of equality to P in Π has the degree of unsolvability α. For such a semigroup the group $\Gamma(\Pi, P)$ has just 14 defining relations.

The concept of a group with a standard basis was introduced by the author in [4]. For its definition, also see [7], [8], [9]. Many well known group constructions such as Novikov's groups $\mathfrak{A}_{p_1 p_2}$ [12], Boone's groups $G(T, q)$ [2], Aanderaa's groups $G(M)$ [13] turned out to have standard bases.

We shall give the definition of Novikov's and Boone-Borisov's groups and restrict ourselves to brief sketches of the proofs of Theorems 1, 2 and their corollaries. The detailed proofs are to appear in the Siberian Mathematical Journal.

2. The construction of P.S. Novikov

Let us fix a finite alphabet $\Sigma = \{a_1, \ldots, a_n\}$ and a finite collection of pairs (A_i, B_i), $1 \leq i \leq m$, of positive (nonempty) words in this alphabet.

Consider the tower of groups (each time we write only the additional generators and relations; the distinguished letters which are involved in the definition of group with a standard basis are underlined):

$$G_0 = \langle \rho_i, \tilde{\rho}_i, 1 \leq i \leq m \rangle,$$

$$G_1\colon\ \Sigma,\ \underline{\rho}_i a = a\rho_i \underline{\rho}_i,\ \underline{\tilde{\rho}}_i a = a\tilde{\rho}_i \underline{\tilde{\rho}}_i,\ a \in \Sigma, 1 \leq i \leq m,$$

$$G_2\colon\ \{l_{ai}, a \in \Sigma, 1 \leq i \leq m\},\ \underline{b} l_{ai} = l_{ai} \underline{b},\ b \in \Sigma,$$

$$G_3\colon\ \{\mu_{ki}, \tilde{\mu}_{ki}, k = 1,2, 1 \leq i \leq m\},\ \underline{a}\hat{\mu}_{1i} = \hat{\mu}_{1i} a l_{ai}^{-1}$$

$$a\underline{l}_{ai}\hat{\mu}_{2i} = \hat{\mu}_{2i}\underline{a},\ a \in \Sigma, 1 \leq i \leq m,$$

where $\hat{}$ is $\varnothing$ or $\tilde{}$ (within each inequality the symbol $\hat{}$ has the same meaning).

Also

$$G_4 = \mathfrak{A}_{d\mu l\rho}: \{d_i, 1 \le i \le m\},$$
$$\rho_i^{-1}\underline{\mu}_{1i}^{-1}\tilde{\mu}_{1i}\tilde{\rho}_i d_i = d_i\underline{\mu}_{2i}Q_i\tilde{\mu}_{2i}^{-1},\ \underline{a}d_i = d_i\underline{a},$$

where

$$Q_i = A_i^{-1}B_i, 1 \le i \le m, a \in \Sigma.$$

Using the standard argument (cf. [4], [8], [9]) we can verify that $\mathfrak{A}_{d\mu l\rho}$ is a group with a standard basis.

Now let G_4^+ be the antiisomorphic copy of G_4 with respect to the antiisomorphism $x \to x^+$ (where x is a letter from the alphabet of G_4). To get the group $G_5 = \mathfrak{A}$ we enrich the free product $G_4 * G_4^+$ with one additional letter p and relations $EpE^+ = p$, where E is one of the following words:

(1) $$\hat{\mu}_{2i}^{-1}l_{ai}\hat{\mu}_{2i},\ \hat{\mu}_{2i}^{-1}d_i^{-1}l_{ai}d_i\hat{\mu}_{2i},$$
$$\hat{\mu}_{2i}^{-1}d_i^{-1}\hat{\rho}_i d_i\hat{\mu}_{2i},\ \tilde{\mu}_{2i}^{-1}d_i^{-1}\tilde{\mu}_{1i}^{-1}\mu_{1i}d_i\mu_{2i},$$

Again, $\hat{}$ is $\varnothing$ or $\tilde{}$. Consider an arbitrary word $\mathfrak{A}_p$ from the subgroup generated by the words in (1) and rewrite it as a word consisting of the expressions

(2) $$\hat{\mu}_{2i}^{-1}V(l_{ai})\hat{\mu}_{2i}, C^{-1}Q_iC,$$
$$\hat{\mu}_{2i}^{-1}V_1(l_{ai})d_i^{-1}W(\rho_i, \tilde{\rho}_i, l_{ai}, C^{-1}Q_iC, A^{-1}N_iA, A^{-1}Q_iM_iA)d_iV_2(l_{ai})\check{\mu}_{2i},$$

where $\hat{}$, $\check{}$ are $\varnothing$ or $\tilde{}$, A and C are reduced Σ-words, C is a stable word (that is $\rho_iC = C\rho_i^k$ for some k),

$$N_i = \tilde{\mu}_{1i}^{-1}\mu_{1i},\quad M_i = \tilde{\rho}_i^{-1}\tilde{\mu}_{1i}^{-1}\mu_{1i}\rho_i.$$

We call a word semicanonical if it is a word consisting of expressions in (2), it is reduced and doesn't contain any forbidden subwords with respect to the letters d_i (nor any subwords $d_i^\varepsilon d_i^{-\varepsilon}$).

MAIN LEMMA. *Any word $\mathfrak{A}_p$ may be (effectively) reduced to a semicanonical form.*

COROLLARY [1, Chapter IV, Theorem]. *Let X be a Σ-word and $X = \mathfrak{A}_p$. Then X is a word consisting of the expressions $C^{-1}Q_iC$, where C is a stable Σ-word.*

3. The Boone-Borisov construction

Let $\Pi = \langle S_\beta, 1 \le \beta \le n, F_i = E_i, 1 \le i \le m \rangle$ be an arbitrary f.p. semigroup with nonempty words F_i, E_i. Consider an arbitrary (possibly empty) word P from Π and define the group $\Gamma(\Pi, P)$ as follows:

$$\Gamma_0 = \langle d, e \rangle,$$

$$\Gamma_1\colon \{S_\beta\},\ \underline{d} \dots dS_\beta = S_\beta \underline{d},\ \underline{e}S_\beta = S_\beta e \dots \underline{e}, 1 \le \beta \le n$$

(letters d, e occur $m+1$ times).

$$\Gamma_2\colon \{c\},\ \underline{S}_\beta c = c\underline{S}_\beta,\ d^i F_i e^i c = cd^i E_i e^i, 1 \le i \le m$$

The first and the last letters in F_i, E_i respectively are distinguished. Also,

$$\Gamma_3\colon \{t\},\ \underline{c}t = t\underline{c},\ \underline{d}t = t\underline{d},$$

$$\Gamma_4 = \Gamma(\Pi, P)\colon \{k\},\ \underline{c}k = k\underline{c},\ \underline{e}k = k\underline{e},\ P^{-1}\underline{t}Pk = kP^{-1}\underline{t}P.$$

The last expression may be written in the more general form

$$P^{-1}\underline{t}PV(c, e)k = kP^{-1}\underline{t}PV(c, e).$$

Again by the standard argument (cf. [4], [8], [9]) we show that the group $\Gamma(\Pi, P)$ has a standard basis. From this it follows that the word problem in Γ_3 is algorithmically solvable (for comparison with the Boone group $G(T, q)$, see [4]).

For the group Γ_4 the same argument as in [4] reduces the word problem in Γ_4 to the following one: for a given positive word Q of the alphabet $\{S_\beta\}$ determine whether or not there exist words $V(c, d)$, $W(c, e)$ such that $V(c, d)QW(c, e) = P$ in Γ_3. Lemma 4 from [6] which is similar to Boone's lemma from [2] (cf. Lemma 5 [4]) implies that the above assertion is valid for Q if and only if $Q = P$ in the semigroup Π. Thus the word problem in Γ_4 is Turing reducible to the problem of the equality to the word P in the semigroup Π. The reverse reduction follows from Borisov's lemma [6] which is also similar to a lemma due to Boone [2]:

$$Q = P(\Pi) \Leftrightarrow Q^{-1}tQk = kQ^{-1}tQ(\Gamma_4).$$

The above reduction of the word problem in Γ_4 to the problem $Q = P(\Pi)$ may be done with a truth-table by the Cohen-Aanderaa "trick" (see [13]).

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