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TANGENTIAL LIMITS OF BLASCHKE PRODUCTS AND FUNCTIONS OF BOUNDED MEAN OSCILLATION

BY

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1. Introduction

Let Δ and C denote the disk $\{|z| < 1\}$ and its boundary $\{|z| = 1\}$. For $\{a_k\}$ a sequence in Δ satisfying the Blaschke condition $\Sigma(1 - |a_k|) < \infty$, let $B(z) = B(z, \{a_k\})$ denote the Blaschke product

$$\prod \frac{\bar{a}_k}{|a_k|} \frac{a_k - z}{1 - \bar{a}_k z}, \quad z \in \Delta,$$

where we set $\bar{a}_k/|a_k| = -1$ if $a_k = 0$. Let H^p, $0 < p \leq \infty$, denote the usual Hardy classes of analytic functions on Δ, and let BMOA be the space of analytic functions on Δ having bounded mean oscillation. Corresponding to each Blaschke product $B(z)$, let

$$K_*(B) = K_2(B) \cap \text{BMOA},$$

where

$$K_2(B) = H^2 \ominus BH^2$$

is the orthogonal complement in H^2 of the invariant subspace BH^2. Recall that every function in H^p, $0 < p \leq \infty$, has finite nontangential limits defined almost everywhere (a.e.) with respect to linear Lebesgue measure $(d\theta)$ in C. Also, every Blaschke product is contained in H^∞ and has nontangential limits of modulus 1 a.e. $[d\theta]$, and $H^\infty \subsetneqq$ BMOA $\subsetneqq \bigcap_{0<p<\infty} H^p$. (See [7] and [9] for background concerning the spaces of functions defined above.)

In this paper we give conditions on the zero sequence $\{a_k\}$ of the Blaschke product $B(z, \{a_k\})$ which insure the existence of certain nontangential and tangential limits for every one of its subproducts and for the functions in the class $K_*(B)$. The following notation will be used to state our main results and

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in the sequel. The mapping

$$\Gamma : [0, \pi] \to \bar{\Delta} = \{|z| \leq 1\}$$

is a Jordan arc such that $\operatorname{Arg} \Gamma(t) = t$ and

$$\psi_\Gamma(t) = 1 - |\Gamma(t)|$$

is strictly increasing on $[0, \pi]$ with $\psi_\Gamma(0) = 0$. Associated with Γ is the set

$$\Omega_\Gamma = \{z : 0 \leq \operatorname{Arg} z < \pi, 0 < |z| < |\Gamma(\operatorname{Arg} z)|\}.$$

When it is convenient and there is no chance of confusion, the subscript Γ in ψ_Γ and Ω_Γ will be suppressed.

In addition, for any subset E of $\bar{\Delta}$ and $\eta \in C$, $\bar{E}$ denotes the closure of E,

$$\tilde{E} = \{z : z \in E \text{ or } \bar{z} \in E\}$$

where $\bar{z}$ is the complex conjugate of z, and

$$\eta E = \{\eta z : z \in E\}.$$

When $1 \in \overline{E \cap \Delta}$ and $f(z)$ is a complex-valued function defined on Δ such that

$$\lim_{\substack{z \to \eta, \\ z \in \Delta \cap \eta E}} f(z)$$

exists, we say that $f(z)$ has an *ηE-limit*. If ψ_Γ is concave upward (respectively satisfies $\psi_\Gamma'(0) = 0$), then we shall call $\Gamma, \Omega_\Gamma, \tilde{\Gamma}$, and $\tilde{\Omega}_\Gamma$ *concave upward* (respectively *tangential*).

The following result proved in §2 establishes the fundamental relationship between $B(z)$ and $K_*(B)$ with respect to $\eta\Omega_\Gamma$-limits.

THEOREM 1.1. *Let $\eta \in C$ and suppose that ψ_Γ is C^1-smooth. The following conditions concerning the Blaschke product $B(z)$ are equivalent*:

(1) *Every subproduct $f(z)$ of $B(z)$ has an $\eta\tilde{\Omega}$-limit* (*of modulus* 1).

(2) *Every function $f \in K_*(B)$ has an $\eta\tilde{\Omega}$-limit.*

The proof of Theorem 1.1 utilizes a characterization of condition (1) by Leung and Linden [12] stated as Theorem B in §2, along with a second characterization provided by Theorem 2.1. Theorems 1.1 and 2.1 generalize results of the second author in [6].

In §3 we consider conditions of the type given by Frostman [8] and Cargo [4] that insure the existence of nontangential and tangential limits off of small exceptional subsets of C. Here, we consider more general regions and improve their results which give exceptional sets of capacity 0 by showing that they have Hausdorff measure 0. Recall that if ω is a modulus of continuity (for example, a continuous, increasing, concave-downward function vanishing at 0) defined on $[0, 2\pi]$, then ω is a determining function for a Hausdorff measure H_ω defined on Borel subsets of E of C by

$$H_\omega(E) = \lim_{t \to 0^+} \left[\inf \left\{ \sum_{A \in \mathcal{O}} \omega(|A|) \right\} \right],$$

where the infimum is taken over all countable covers $\mathcal{O}$ of E by open arcs A having linear measure $|A| \leq t$. (See [2] and [13] for background regarding moduli of continuity and Hausdorff measure.)

THEOREM 1.2. *Suppose that $\psi = \psi_\Gamma$ is C^1-smooth and $\omega \not\equiv 0$ is a continuous modulus of continuity that is C^1-smooth on $(0, 2\pi]$ and satisfies*

$$\liminf_{t \to 0^+} \frac{\omega(t)}{\omega'(t)} \frac{[\psi(Mt)]'}{\psi(Mt)} > 1 \tag{1.1}$$

for some $M \in (0,1)$. If $B(z) = B(z, \{a_k\})$ is a Blaschke product such that

$$\sum \omega \circ \psi^{-1}(1 - |a_k|) < \infty, \tag{1.2}$$

then there exists a Borel subset E of C with $H_\omega(E) = 0$ such that every subproduct $f(z)$ of $B(z)$ has an $\eta\tilde{\Omega}_\Gamma$-limit (of modulus 1) for each $\eta \in C \setminus E$.

It is elementary to check that (1.1) is satisfied when $\omega(t) = t^\beta, 0 < \beta < 1$, for *any* concave-upward ψ, or when $\psi(t) = ct^\alpha, c > 0, 1 < \alpha < \infty$, for *any* concave-downward ω.

Letting $\psi(t) = ct$ where $c > 0$, we get a sharpened and generalized form of Frostman's original result concerning nontangential limits.

COROLLARY 1.1. *If*

$$\liminf_{t \to 0} \frac{\omega(t)}{t\omega'(t)} > 1$$

and

$$\sum \omega^{-1}(1 - |a_k|) < \infty,$$

then there exists a Borel subset E of C with $H_\omega(E) = 0$ such that every subproduct of $B(z)$ has nontangential limits (of modulus 1) at each point of $C - E$.

Taking $\omega(t) = t$, we obtain the next corollary.

COROLLARY 1.2. *If*

$$\liminf_{s \to 0} \frac{s\psi'(s)}{\psi(s)} > 1 \tag{1.3}$$

and

$$\sum \psi^{-1}(1 - |a_k|) < \infty, \tag{1.4}$$

then every subproduct $f(z)$ *of* $B(z, \{a_k\})$ *has an* $\eta\tilde{\Omega}$*-limit* (*of modulus* 1) *for each* η *in a subset* E *of* C *with* $|E| = 2\pi$.

In §4 we construct Blaschke products which demonstrate the sharpness of Theorem 1.2 and Corollaries 1.1 and 1.2.

THEOREM 1.3. *Assume that* ω *is a continuous modulus of continuity and both* $\omega|_{(0,2\pi]}$ *and* $\psi = \psi_\Gamma$ *are* C^1*-smooth. If* $E \subseteq C$ *is a Borel set such that* $H_\omega(E) = 0$, *then there exists a Blaschke product* $B(z) = B(z, \{a_k\})$ *such that*

$$\sum \omega \circ \psi^{-1}(1 - |a_k|) < \infty \quad \textit{and} \quad \liminf_{\substack{z \to \eta, \\ z \in \eta\tilde{\Omega}}} |B(z)| = 0, \ \eta \in E.$$

Moreover, if E *is a compact set, then there exists a subproduct* $f(z)$ *of* $B(z)$ *that fails to have an* $\eta\tilde{\Omega}$*-limit at each point* $\eta \in E$.

Theorem 1.3 can be thought of as a converse to Theorem 1.2 (for allowed ω and ψ). While Theorem 1.2 insures that the exceptional set E_B of η where a Blaschke product $B(z, \{a_k\})$ satisfying (1.2) does not have an $\eta\Omega_\Gamma$-limit of modulus 1 is of H_ω-measure 0, Theorem 1.3 shows that each Borel set E having H_ω-measure 0 is contained in E_B for some Blaschke product $B(z, \{a_k\})$ satisfying (1.2). This shows that under the assumption (1.2) holds, H_ω-measure 0 is the "correct" description of the size of the exceptional sets.

The next theorem shows that the conclusion of Corollary 1.2 cannot be drawn if a weaker condition is placed on the sequence $\{1 - |a_k|\}$.

THEOREM 1.4. *If* $\psi = \psi_\Gamma$ *is* C^1*-smooth and* $\{t_k\}_1^\infty$ *is a sequence in* $[0, 1)$ *such that*

$$\sum (1 - t_k) < \infty \tag{1.5}$$

and

$$\sum \psi^{-1}(1 - t_k) = \infty, \tag{1.6}$$

then there exists a Blaschke product $B(z) = B(z, \{a_k\})$ *such that*

$$|a_k| = t_k, \quad k = 1,2,\ldots,$$

and

$$\liminf_{\substack{z\to\eta,\\ z\in\eta\tilde{\Omega}}} |B(z)| = 0, \quad \eta \in C. \tag{1.7}$$

Moreover, there is a subproduct $f(z)$ *of* $B(z)$ *for which* (1.7) *holds with* $f(z)$ *replacing* $B(z)$, *and*

$$\limsup_{\substack{z\to\eta,\\ z\in\eta\tilde{\Omega}}} |f(z)| = 1, \quad \eta \in C. \tag{1.8}$$

In particular, $f(z)$ *fails to have an* $\eta\tilde{\Omega}$*-limit for every* $\eta \in C$.

In connection with the sharpness of Corollary 1.2, we also note that a construction of Frostman [8; p. 176] shows that hypothesis (1.3) cannot be omitted. In fact, if $\psi'(0) = 0$ and $\lim_{t\to 0^+}[t\log(1/t)/\psi^{-1}(t)] = \infty$, then there exists a Blaschke product $B(z, \{a_k\}_1^\infty)$ with $\Sigma\psi^{-1}(1 - |a_k|) < \infty$, but

$$\sum(1 - |a_k|)\log 1/(1 - |a_k|) = \infty$$

and for each $\eta \in C$, there exists a subproduct $f(z)$ of $B(z)$ that fails to have a radial limit at η.

At the end of §4 we give two theorems related to Theorem 1.4 where we focus specifically on the behavior of the Blaschke products on the rotates of Γ, not just the rotates of $\tilde{\Omega}_\Gamma$. Theorem 4.1 is an analogue of Theorem 1.4 where $\eta\tilde{\Omega}$ is replaced by $\eta\Gamma$ but stronger assumptions are made on ψ_Γ. In Theorem 4.2 we drop all the assumptions on ψ_Γ except that it is tangential. We show that there exists a Blaschke product B such that (1.7) holds with $\eta\tilde{\Omega}$ replaced by $\eta\Gamma$. Theorems 4.1 and 4.2 were motivated by questions posed to the authors by Pamela Gorkin.

Finally, we state a corollary of the results of this paper for a case that seems to be of particular interest. Note that H_t-measure is linear measure.

COROLLARY 1.3. *Suppose that* $\psi_\Gamma(t) = ct^\alpha$, $c > 0$, $\alpha > 1$, *and* $\beta \in (0,1]$. *Let* E *be a Borel subset of* C. *Then there exists a Blaschke product* $B(z) = B(z, \{a_k\})$ *for which* $\Sigma(1 - |a_k|)^{\beta/\alpha} < \infty$ *and* $B(z)$ *fails to have* $\eta\tilde{\Omega}$*-limits of modulus* 1 *at each point of* E *if and only if* $H_{t^\beta}(E) = 0$.

Let $\{t_k\}_1^\infty$ *be a sequence in* $[0, 1)$ *satisfying* (1.5). *Then*

$$|a_k| = t_k, \quad k = 1,2,\ldots, \tag{1.9}$$

implies that $B(z, \{a_k\})$ *has an* $\eta\tilde{\Omega}$*-limit* ($\eta\Gamma$*-limit*) a.e. $[d\theta]$ *if and only if*

$$\sum (1 - t_k)^{1/\alpha} < \infty.$$

If

$$\sum (1 - t_k)^{1/\alpha} = \infty,$$

then there exists a Blaschke product $B(z) = B(z, \{a_k\})$ *satisfying* (1.9) *that fails to have an* $\eta\Gamma$*-limit of modulus* 1 *at each point* η *of* C; *in fact,*

$$\liminf_{t \to 0} |B[\eta\Gamma(t)]| = 0, \quad \eta \in C.$$

Throughout this paper we shall use the convention that c (possibly subscripted) denotes a positive constant, independent of certain indicated parameters, whose value may change in a sequence of inequalities.

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2. Local conditions for tangential limits

In this section we give necessary and sufficient conditions for Blaschke products and functions in $K_*(B)$ to have specified tangential limits at a point of C. We also give a simple sufficient condition that will be used in §3 to prove Theorem 1.2. For simplicity, we shall assume that $\eta = 1$ and consider only Ω_Γ-limits. The modifications necessary to extend our results to $\eta\Omega_\Gamma$ or $\eta\tilde{\Omega}$-limits will be apparent.

We start by stating a theorem concerning radial (and nontangential) limits of Blaschke products which includes results from Frostman [8] and Cargo [3; p. 425].

THEOREM A. *The following conditions on a Blaschke product* $B(z) = B(z, \{a_k\})$ *are equivalent*:

(i)
$$\sum \frac{1 - |a_k|}{|1 - a_k|} < \infty.$$

(ii) $B(z)$ *and each of its subproducts have radial limits of modulus* 1 *at* 1 (*Frostman*).

(iii)
$$\sum \int_\delta^1 \frac{1 - |a_k|}{|r - a_k|^2} dr < \infty$$

for $\delta \in (0, 1)$ *sufficiently close to* 1 (Cargo).

We shall also need the following result of Leung and Linden [12] which provides the analogue of Frostman's characterization for more general Ω_Γ-limits.

THEOREM B (Leung and Linden). *Suppose that ψ_Γ is C^1-smooth. Then $B(z,\{a_k\})$ satisfies* (1) *of Theorem 1.1* (*with $\eta = 1$*) *if and only if*

(a) $$\sum \frac{1-|a_k|}{|1-a_k|} < \infty$$

and

(b) $$\lim_{t\to 0^+}\left[\sum_{\frac{1}{2}t<|\mathrm{Arg}\, a_k|<2t} \frac{1-|a_k|}{1-|\Gamma(t)|+|t-|\mathrm{Arg}\, a_k||}\right] = 0.$$

In order to prove Theorem 1.1, we will show that condition (b) of Theorem B can be replaced by a condition involving the limiting behavior of certain integrals. The following notation will be used. For each $t \in (0, \pi/2)$, let L_t denote the line segment with one endpoint on the real axis and the other at $\Gamma(t)$ which subtends an angle of $\pi/4$ with the real axis.

THEOREM 2.1. *Suppose that ψ_Γ is C^1-smooth. The following conditions on a Blaschke product $B(z) = B(z,\{a_k\})$ satisfying* (*a*) *of Theorem B are equivalent*:

(b) $$\lim_{t\to 0^+}\left[\sum_{\frac{1}{2}t<\mathrm{Arg}\, a_k<2t} \frac{1-|a_k|}{1-|\Gamma(t)|+|t-\mathrm{Arg}\, a_k|}\right] = 0.$$

(b′) $$\lim_{t\to 0^+}\left[\int_{L_t} \frac{1-|B(\zeta)|^2}{1-|\zeta|^2}|d\zeta|\right] = 0.$$

Proof. We first show that (b′) implies (b). Let $f(z)$ be any subproduct of $B(z)$. Then by Theorem A we have

(2.1) $$\lim_{r\to 1} f(r) = \lambda$$

for some $\lambda \in C$. By the Schwarz-Pick theorem (see [10; p. 226]), it follows that

$$|f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2} \le \frac{1-|B(z)|^2}{1-|z|^2}, \quad z \in \Delta.$$

If $z \in L_t$ and r_t is the endpoint of L_t on the real axis, then

$$\begin{aligned} |f(z) - f(r_t)| &\le \int_{L_t} |f'(\zeta)||d\zeta| \\ &\le \int_{L_t} \frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} |d\zeta|. \end{aligned} \tag{2.2}$$

We conclude from (2.1), (2.2), and (b′) that $f(z)$ has an Ω-limit of λ. Condition (b) now follows from Theorem B (stated relative to Ω_Γ instead of $\tilde{\Omega}_\Gamma$) since $f(z)$ was an arbitrary subproduct of $B(z)$.

Suppose now that (b) holds. It is easy to verify that

$$\frac{1 - |B(z)|^2}{1 - |z|^2} \le \sum \frac{1 - |a_k|^2}{|z - a_k|^2}, \quad z \in \Delta.$$

(See also [1; p. 80].) Thus

$$\begin{aligned} \int_{L_t} \frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} |d\zeta| &\le \int_{L_t} \sum \frac{1 - |a_k|^2}{|\zeta - a_k|^2} |d\zeta| \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$I_j = \sum_{S_j} \int_{L_t} \frac{1 - |a_k|^2}{|\zeta - a_k|^2} |d\zeta|$$

and S_j is the set of all a_k such that

$$\begin{aligned} &0 < \operatorname{Arg} a_k \le t/2, && j = 1, \\ &t/2 < \operatorname{Arg} a_k < 2t, && j = 2, \\ &2t \le \operatorname{Arg} a_k < \pi, && j = 3, \\ &\operatorname{Im}(a_k) \le 0, && j = 4. \end{aligned}$$

We must show that $\lim_{t \to 0^+} I_j = 0$ for $j = 1, 2, 3, 4$. Consider first I_2; this is the only time (b) will be used. Let $Q(t)$ be the quadrilateral contained in $\{\operatorname{Im} z \ge 0\}$ defined by the following properties: two sides are parallel to the imaginary axis, one side is contained in the real axis, and the remaining side is contained in the radius passing through $\Gamma(t)$; the side on the real axis has midpoint r_t and right-hand endpoint $\operatorname{Re}[\Gamma(t)]$, while the side parallel to the imaginary axis with one endpoint $\operatorname{Re}[\Gamma(t)]$ has as its other endpoint $\Gamma(t)$.

By our assumptions and Theorem B (relative to Ω_Γ), $B(z)$ has an Ω_Γ-limit of modulus 1. Since $Q(t) \subseteq \overline{\Omega}_\Gamma$, it follows that for $t > 0$ sufficiently small, there is no a_k in $Q(t)$. Also, for t in this range, there is a constant $c > 0$ independent of k or t such that

$$|\Gamma(t) - a_k| \leq c \operatorname{dist}[a_k, L_t]$$

when $t/2 \leq \operatorname{Arg} a_k \leq 2t$. Here and in the sequel, dist denotes Euclidean distance in the plane. Thus, for $t > 0$ small, we have

$$\begin{aligned}\int_{L_t} \frac{1 - |a_k|^2}{|\zeta - a_k|^2} |d\zeta| &\leq c \frac{1 - |a_k|}{|\Gamma(t) - a_k|} \\ &\leq c \frac{1 - |a_k|}{1 - |\Gamma(t)| + |t - \operatorname{Arg} a_k|}.\end{aligned}$$

Using (b), it follows that $\lim_{t \to 0^+} I_2 = 0$.

For I_1, the assumption that $0 < \operatorname{Arg} a_k \leq t/2$ implies that for $t > 0$ sufficiently small, $\operatorname{dist}[a_k, L_t] \geq ct$. Hence, for such t we have

$$\begin{aligned} I_1 &\leq c \sum_{S_1} \frac{1 - |a_k|}{t^2} |L_t| \\ &\leq c \sum_{S_1} \frac{1 - |a_k|}{t} \\ &\leq c \sum_{S_1} \frac{1 - |a_k|}{\operatorname{Arg}(a_k)}, \end{aligned}$$

where $|L_t|$ denotes the length of L_t. Since $\psi'_\Gamma(0) \in [0, \infty)$, we have

$$\operatorname{Arg} a_k \geq c|1 - a_k|$$

for allowed a_k, and (a) of Theorem B yields $\lim_{t \to 0^+} I_1 = 0$.

For I_3, note first that there is a constant $c > 0$ independent of t and k for which $2t < \operatorname{Arg} a_k < \pi$ such that $\operatorname{dist}[a_k, L_t] \geq c \operatorname{Arg} a_k$. Thus,

$$I_3 \leq c \sum_{S_3} \frac{1 - |a_k|}{(\operatorname{Arg} a_k)^2} \cdot t$$

since $|L_t| \leq ct$. We conclude that

$$\begin{aligned} I_3 &\leq c \sum_{\frac{t}{\operatorname{Arg} a_k} \leq \sqrt{t}} \frac{1 - |a_k|}{\operatorname{Arg} a_k} \frac{t}{\operatorname{Arg} a_k} + c \sum_{\frac{t}{\operatorname{Arg} a_k} > \sqrt{t}} \frac{1 - |a_k|}{\operatorname{Arg} a_k} \\ &\leq c\sqrt{t} \sum_{2t < \operatorname{Arg} a_k < \pi} \frac{1 - |a_k|}{\operatorname{Arg} a_k} + c \sum_{2t < \operatorname{Arg} a_k < \sqrt{t}} \frac{1 - |a_k|}{\operatorname{Arg} a_k} \end{aligned}$$

and (a) of Theorem B implies the desired result.

Finally, consider I_4. Let R_t be the projection of L_t onto the real axis. Evidently

$$\operatorname{dist}[a_k, \operatorname{Re}\zeta] \le \operatorname{dist}[a_k, \zeta], \quad \zeta \in L_t,$$

if a_k satisfies $\operatorname{Im}(a_k) \le 0$. Thus

$$I_4 \le c \sum_{S_4} \int_{R_t} \frac{1 - |a_k|^2}{|r - a_k|^2}\, dr.$$

From the assumption (a) of Theorem B and Theorem A, it follows that

$$\lim_{t \to 0^+} \sum_{S_4} \int_{R_t} \frac{1 - |a_k|^2}{|r - a_k|^2}\, dr = 0,$$

and the proof is complete.

We turn now to $K_*(B)$. The following result was proved by the second author in [6].

THEOREM C. *The following conditions concerning $B(z) = B(z, \{a_k\})$ are equivalent*:

(i) $$\sum \frac{1 - |a_k|}{|1 - a_k|} < \infty.$$

(ii) $\lim_{r \to 1^-} f(r)$ *exists for all* $f \in K_*(B)$.

We now prove a generalization.

THEOREM 2.2. *Suppose that ψ_Γ is C^1-smooth. The following conditions concerning $B(z) = B(z, \{a_k\})$ are equivalent*:

(i) *Condition* (i) *of Theorem C holds and*

$$\lim_{t \to 0^+} \int_{L_t} \frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} |d\zeta| = 0.$$

(ii) *$f(z)$ has an Ω-limit for all $f \in K_*(B)$.*

Proof. We first show that (ii) implies (i). By the assumption (ii) and Theorem C, it follows that (i) of Theorem C holds. By Theorem A, if f is a subproduct of B, then $\lim_{r \to 1} f(r) = \lambda$ where $|\lambda| = 1$. This taken together with the fact that

$$\frac{1 - \overline{f(a)}\, f(z)}{1 - \bar{a}z}$$

belongs to $K_*(B)$ for all $a \in \Delta$, implies that f has an Ω-limit of λ. We conclude from Theorem B (relative to Ω) and Theorem 2.1, that (i) must hold.

To show that (i) implies (ii), it suffices to establish an inequality of the form

$$|f'(z)| \leq c\frac{1-|B(z)|^2}{1-|z|^2}, \quad z \in \Delta,$$

for each $f \in K_*(B)$. (Here, c will depend on f.) For then (2.2) holds (with a constant) for $f \in K_*$ and condition (i) combined with Theorem C gives the result. We need the integral representation for $f \in K_*(B)$ obtained in [6]; that is,

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_0}\left[\overline{\frac{\psi(\zeta)}{B(\zeta)}}\right]\frac{d\bar{\zeta}}{1-z\bar{\zeta}}, \quad z \in \Delta,$$

where $\psi \in H^\infty$ (see [6; Lemma 3.2]) and $\Gamma_0 = \cup \gamma_m$, $\gamma_m = [\alpha_m, \beta_m] \subseteq \Delta$ is the (rectifiable) Carleson curve described in [6; §2] such that

(I) $$\left|\frac{\psi(\zeta)}{B(\zeta)}\right| \leq c,\ \zeta \in \Gamma_0,$$

(II) $$0 < c_1 < \left|\frac{\alpha_m - \beta_m}{1 - \bar{\alpha}_m\beta_m}\right| < c_2 < 1,$$

and for $\omega_n = (\alpha_n + \beta_n)/2$,

(III) $\{\omega_n\}$ is an interpolation sequence and

(IV) $$0 < d_1 < \frac{1-|B(z,\{a_k\})|}{1-|B(z,\{\omega_m\})|} < d_2 < \infty, \text{ for } z \in \Delta.$$

Thus, for $z \in \Delta$,

$$|f'(z)| \leq c\sum\int_{\gamma_m}\frac{|d\zeta|}{|1-\bar{\zeta}z|^2}.$$

The use of property (II) above and an elementary estimate yields a constant $c > 0$ such that

$$|f'(z)| \leq c\sum\frac{1-|\omega_m|^2}{|1-\bar{\omega}_m z|^2}.$$

On the other hand, we claim that there exists $c > 0$ independent of $z \in \Delta$, such that

(2.3) $$\sum\frac{(1-|\omega_m|^2)(1-|z|^2)}{|1-\bar{\omega}_m z|^2} \leq c\left(1-|B(z,\{\omega_m\})|^2\right).$$

If $\varepsilon \in (0,1)$ and we consider only z in the set $\{|B(z,\{\omega_m\})| \geq \varepsilon\}$, then (2.3) follows (with c depending on ε) from the equality

$$1-|B(z,\{\omega_m\})|^2 = \sum_j \left|B\left(z,\{\omega_m\}_1^{j-1}\right)\right|^2 \frac{\left(1-|\omega_j|^2\right)\left(1-|z|^2\right)}{|1-\overline{\omega}_j z|^2}.$$

However, since $\{\omega_m\}$ is an interpolation sequence, the left-hand side of (2.3) is bounded above uniformly for all $z \in \Delta$. This shows that c may be chosen so that (2.3) holds for z in the set $\{|B(z,\{\omega_m\})| < \varepsilon\}$. The claim follows.

From (2.3), we conclude that

$$|f'(z)| \leq c\frac{1-|B(z,\{\omega_m\})|^2}{1-|z|^2}, \quad z \in \Delta,$$

and the proof is completed by applying property (IV).

Theorem 1.1 is an immediate consequence of Theorems 2.1 and 2.2. We note that condition (b′) of Theorem 2.1 makes sense if B is replaced by an arbitrary inner function φ, that is, a bounded analytic function having radial limits of modulus 1 a.e. $[d\theta]$. With $K_*(\varphi)$ defined analogously to $K_*(B)$, it is easy to show that a generalized form of Theorem C (see [6; Theorem 3.1]) and condition (b′) of Theorem 2.1 imply that $f(z)$ has an Ω-limit for every $f \in K_*(\varphi)$. The opposite direction is undoubtedly true, but in order to use the same method of proof as above, it would be necessary to generalize Theorem B.

The final theorem of this section provides a simple sufficient condition for the existence of a tangential limit. Cargo [4; Theorem 1] originally proved this theorem for $\psi_\Gamma(t) = ct^\alpha$, $\alpha \geq 1$, $c > 0$, generalizing a result of Frostman [8] for $\alpha = 1$.

THEOREM 2.3. *Suppose that $\psi = \psi_\Gamma$ is C^1-smooth. If $B(z) = B(z,\{a_k\})$ is a Blaschke product such that*

$$\sum \frac{1-|a_k|}{\psi(M|1-a_k|)} < \infty \tag{2.4}$$

for some $M \in (0,1)$, then every subproduct $f(z)$ of $B(z)$ has an Ω_Γ-limit of modulus 1.

Proof. If Γ is not tangential, then the theorem follows from Theorem A and the fact that the existence of a radial limit implies the existence of a nontangential limit for any function in H^∞.

For the remainder of the proof, we assume that Γ is tangential. We shall show (2.4) implies conditions (a) and (b) of Theorem B (with $|\operatorname{Arg} a_k|$

replaced by $\operatorname{Arg} a_k$). Condition (a) is an immediate consequence of (2.4). As before, this implies that

$$M|1 - a_k| \leq \operatorname{Arg}(a_k), \quad 0 < \operatorname{Arg} a_k < 2t,$$

for $t > 0$ sufficiently small. It now follows from (2.4) that

$$\sum \frac{1 - |a_k|}{\psi(\operatorname{Arg} a_k)} < \infty. \tag{2.5}$$

In addition, since ψ is C^1-smooth with $\lim_{t\to 0}\psi'(t) = 0$, the mean value theorem implies that

$$|\psi(\operatorname{Arg} a_k) - \psi(t)| < |t - \operatorname{Arg} a_k| \tag{2.6}$$

for $\frac{1}{2}t < \operatorname{Arg} a_k < 2t$ and $t > 0$ sufficiently small. Applying (2.5) and (2.6) yields condition (b). The required conclusion now follows from Theorem B, and the proof is complete.

3. Global conditions for tangential limits

In this section we prove Theorem 1.2. The proof is based on the local sufficient condition of Theorem 2.3 and a modification of an argument appearing in [5].

We shall use the following theorem concerning Hausdorff measure (see [11; Theorème III, Chapitre II, p. 27]). We assume throughout this section that $\omega \not\equiv 0$ is a continuous modulus of continuity that is C^1-smooth on $(0, 2\pi]$.

THEOREM D. *The following conditions on a Borel subset E of C are equivalent*:

(1) $H_\omega(E) > 0$.

(2) *E supports a finite positive Borel measure μ with $\omega_\mu(t) = O\,[\omega(t)]$.*

Recall that every finite positive Borel measure μ on C can be identified with the monotone nondecreasing function

$$\hat{\mu}(t) = \mu(\{e^{i\theta} : 0 \leq \theta \leq t\}),\ t \in [0, 2\pi].$$

Here and in what follows, ω_μ denotes the modulus of continuity of $\hat{\mu}$.

Proof of Theorem 1.2. Let $\Phi(t) = \psi(\mathrm{M}t)$. By Theorem 2.3, it suffices to show that

$$E = \left\{\eta \in C : \sum \frac{1 - |a_k|}{\Phi(|\eta - a_k|)} = \infty\right\}$$

satisfies $H_\omega(E) = 0$. For each positive integer m, let

$$O_m = \left\{\eta \in C\colon \sum_{k \geq m} \frac{1 - |a_k|}{\Phi(|\eta - a_k|)} \geq 1\right\}$$

and

$$G_m = \left\{\eta \in O_m\colon |\eta - a_k| \leq \Phi^{-1}(1 - |a_k|) \text{ for all } k \geq m\right\}.$$

By Theorem D, it is enough to show that for a finite positive measure μ on C such that $\omega_\mu(t) = O\,[\omega(t)]$, we have

$$\lim_{m \to \infty} \mu(O_m) = 0.$$

We shall consider G_m and $O_m \setminus G_m$ separately. First, we have

$$\begin{aligned}
\mu(G_m) &\leq \sum_{k \geq m} \mu\left(\left\{\eta \in C\colon |\eta - a_k| \leq \Phi^{-1}(1 - |a_k|)\right\}\right) \\
&\leq c \sum_{k \geq m} \omega \circ \Phi^{-1}(1 - |a_k|) \\
&= c \sum_{k \geq m} \omega\left[\frac{1}{M}\psi^{-1}(1 - |a_k|)\right] \\
&\leq c \sum_{k \geq m} \omega \circ \psi^{-1}(1 - |a_k|),
\end{aligned}$$

using the fact that ω is a modulus of continuity. Hence (1.2) implies that $\lim_{m \to \infty}\mu(G_m) = 0$.

Next let

$$\nu_k(t) = \mu(\{\zeta \in C\colon |\zeta - a_k| < t\}),\ t \geq 0,$$

for each positive integer k. Then

$$\begin{aligned}
\mu(O_m \setminus G_m) &\leq \int_{O_m \setminus G_m} \sum_{k \geq m} \frac{1 - |a_k|}{\Phi(|\zeta - a_k|)}\, d\mu(\zeta) \\
&= \sum_{k \geq m} \int_{\Phi^{-1}(1 - |a_k|)}^{2} \frac{1 - |a_k|}{\Phi(t)}\, d\nu_k(t).
\end{aligned}$$

Integrating by parts, we may express the kth term in the last sum as $T_k^{(1)} + T_k^{(2)}$, where

$$T_k^{(1)} = (1 - |a_k|) \int_{\Phi^{-1}(1 - |a_k|)}^{2} \frac{\Phi'(t)\nu_k(t)}{\Phi(t)^2}\, dt$$

and

$$T_k^{(2)} = (1 - |a_k|)\frac{\mu(C)}{\Phi(2)}.$$

It follows easily that

$$\lim_{m \to \infty} \sum_{k \geq m} T_k^{(2)} = 0$$

and it remains only to show that

$$\lim_{m \to \infty} \sum_{k \geq m} T_k^{(1)} = 0.$$

Since $\omega_\mu(t) = O\,[\omega(t)]$, we have

$$\begin{aligned} T_k^{(1)} &\leq c(1 - |a_k|)\int_{\Phi^{-1}(1-|a_k|)}^{2} \frac{\omega(t)\Phi'(t)}{\Phi(t)^2}\,dt \\ &\leq c(1 - |a_k|)\int_{1-|a_k|}^{\Phi(2)} \frac{\omega \circ \Phi^{-1}(s)}{s^2}\,ds, \end{aligned}$$

using the change of variables $\Phi(t) = s$.

We claim that there is a constant $r > 1$ such that for $s > 0$ sufficiently small,

$$\begin{aligned} \frac{\omega \circ \Phi^{-1}(s)}{s^2} &\leq -r\frac{d}{ds}\left[\frac{\omega \circ \Phi^{-1}(s)}{s}\right] \\ &= -r\left[\frac{s\omega' \circ \Phi^{-1}(s)\Phi^{-1}(s)' - \omega \circ \Phi^{-1}(s)}{s^2}\right]. \end{aligned} \tag{3.1}$$

After some algebra, it can be seen that (3.1) is equivalent to

$$\frac{\omega \circ \Phi^{-1}(s)}{s^2} \geq \frac{r}{r-1}\left[\frac{\omega' \circ \Phi^{-1}(s)\Phi^{-1}(s)'}{s}\right]. \tag{3.2}$$

Since $\Phi(t) = s$, inequality (3.2) is equivalent to

$$\frac{\omega(t)}{\Phi(t)} \geq \frac{r}{r-1}\frac{\omega'(t)}{\Phi'(t)}$$

for $t > 0$ sufficiently small. By (1.1) and the fact that

$$\lim_{r \to \infty} \frac{r}{r-1} = 1$$

we conclude that (3.1) holds for $s > 0$ sufficiently small and r sufficiently large.

To complete the proof, choose $s_0 > 0$ sufficiently small so that (3.1) holds for $0 < s < s_0$. Let m be sufficiently large so that $1 - |a_k| < s_0$ for $k \geq m$. Now by (3.1), we have

$$\int_{1-|a_k|}^{s_0} \frac{\omega \circ \Phi^{-1}(s)}{s^2}\, ds \leq c\left[\frac{\omega \circ \Phi^{-1}(1 - |a_k|)}{1 - |a_k|} - \frac{\omega \circ \Phi^{-1}(s_0)}{s_0}\right]$$

and

$$\int_{s_0}^{\Phi(2)} \frac{\omega \circ \Phi^{-1}(s)}{s^2}\, ds \leq c.$$

Thus

$$\sum_{k \geq m} T_k^{(1)} \leq c \sum_{k \geq m} \omega \circ \Phi^{-1}(1 - |a_k|) + c \sum_{k \geq m} (1 - |a_k|).$$

Once again using the fact that ω is a modulus of continuity and (1.2), we see that

$$\lim_{k \to \infty} \sum_{k \geq m} T_k^{(1)} = 0.$$

This completes the proof.

4. Construction of Blaschke products

In this section we prove Theorems 1.3 and 1.4 along with two other theorems in which conditions are placed on the behavior of the Blaschke products along all the rotates of Γ instead of $\tilde{\Omega}$.

We begin by stating a lemma (without proof) that is quite elementary but very useful. As before, it is assumed that $\psi = \psi_\Gamma$ is strictly increasing.

LEMMA 4.1. *Suppose that $a \in \Delta$ and let*

$$I_a = \{\eta \in C : a \in \eta\tilde{\Omega}\}. \tag{4.1}$$

If $1 - |a| < \psi_\Gamma(\pi)$, then I_a is an open arc centered at $a/|a|$ such that

$$|I_a| = 2\psi^{-1}(1 - |a|). \tag{4.2}$$

In the remainder of this section, we use the notation I_n or $I_{n,k}$ to denote the set defined in (4.1) when a is replaced by a_n or $a_{n,k}$.

We turn now to Theorem 1.3. The construction used to prove this theorem is a modification of one presented in [5; Theorem 2.5].

Proof of Theorem 1.3. Let E be a Borel subset of C such that $H_\omega(E) = 0$. Then for each positive integer n, we can find a cover of E by open arcs $\{A_{n,k}\}_{k=1}^{\infty}$ such that for $r_{n,k} = |A_{n,k}|/2$, we have

$$\sum_{k=1}^{\infty} \omega(r_{n,k}) < 2^{-n}. \tag{4.3}$$

Observe that if E is compact, we may assume that the cover is finite. Let $\eta_{n,k}$ be the midpoint of $A_{n,k}$ and define

$$a_{n,k} = |\Gamma(r_{n,k})|\eta_{n,k} = [1 - \psi(r_{n,k})]\eta_{n,k}, \quad k = 1, 2, \ldots .$$

It follows immediately from the definition of $a_{n,k}$ and (4.3) that

$$\sum_{n,k} \omega \circ \psi^{-1}(1 - |a_{n,k}|) = \sum_{n,k} \omega(r_{n,k}) < \infty.$$

Also, since ψ is increasing, we have $1 - |a_{n,k}| < \psi(\pi)$ and Lemma 4.1 implies that $A_{n,k} = I_{n,k}$ for each k, where $I_{n,k}$ is as in Lemma 4.1.

Let $B(z) = B(z, \{a_{n,k}\}_{n,k})$. Then for each $\eta \in E$, the set $\eta\tilde{\Omega}$ contains infinitely many zeros $a_{n,k}$. Thus

$$\liminf_{\substack{z \to \eta, \\ z \in \eta\tilde{\Omega}}} |B(z)| = 0, \quad \eta \in E. \tag{4.4}$$

In case E is compact, the observation that for each n, the sequence $\{a_{n,k}\}$ may be chosen to be finite makes it clear that there is a subsequence $\{n_j\}_1^{\infty}$ of the positive integers with large gaps such that if

$$f(z) = B\left(z, \{a_{n_j,k}\}_{j,k}\right),$$

then for each $\eta \in E$ we have

$$\limsup_{\substack{z \to \eta, \\ z \in \eta\tilde{\Omega}}} |f(z)| = 1$$

while (4.4) still holds with f replacing B. This completes the proof.

We proceed now to Theorem 1.4.

Proof of Theorem 1.4. Let $\{\eta_k\}_1^{\infty}$ be any sequence of points in C and let $a_k = t_k\eta_k$ for each k. It follows from Lemma 4.1 and (1.6) that $\sum|I_k| = \infty$ with $\{I_k\}_1^{\infty}$ as in the lemma. Thus we can clearly choose $\{\eta_k\}_1^{\infty}$ so that the I_k revolve around C infinitely many times and each point $\eta \in C$ is contained in infinitely many of them.

Let $B(z) = B(z, \{a_k\}_1^\infty)$ with $\{\eta_k\}_1^\infty$ chosen as above. Then for $\eta \in C$, the Blaschke product B has infinitely many zeros in $\eta\tilde{\Omega}$ so that (1.7) holds. The last assertion is easily proved by defining a subproduct f of B using a subsequence $\{a_{k_j}\}_1^\infty$ whose indices have large gaps for which each $\eta \in C$ is still contained in infinitely many I_{k_j}. The theorem is thereby established.

The remaining two theorems of this section deal with the limiting behavior of Blaschke products along the rotates of Γ. We shall need several lemmas.

Lemma 4.2. *For each $z \in \Delta$ and each positive integer j, the Blaschke product $B(z) = B(z, \{a_k\}_1^\infty)$ satisfies*

$$|B(z)| \leq \frac{|a_j - z|}{1 - |a_j|}.$$

Proof.

$$\begin{aligned} |B(z)| &= \left|B(z, \{a_k\}_{k \neq j})\right| \left| \left(\frac{\bar{a}_j}{|a_j|}\right)\left(\frac{a_j - z}{1 - \bar{a}_j z}\right)\right| \\ &\leq \frac{|a_j - z|}{1 - |a_j|}. \end{aligned}$$

For the next lemma, let

$$\Delta(a, r) = \{|z - a| < r\}$$

and

$$A_{\Delta(a,r)} = \{\eta \in C : \Delta(a, r) \cap \eta\Gamma \neq \varnothing\},$$

when $a \in \Delta$ and $r \in (0, 1 - |a|)$.

Lemma 4.3. *Suppose that $\psi = \psi_\Gamma$ is C^1-smooth and concave upward. If $t \in (0, \pi)$ and $0 < r < \psi(t)/2$, then*

$$\frac{r}{\psi'(t)} \leq |A_{\Delta[\eta\Gamma(t), r]}|, \quad \eta \in C.$$

Proof. Without loss of generality assume that $\eta = 1$. Let $x \in (0, 1)$. Then by the assumption on r and the concavity of ψ which implies $\psi(t) \leq t\psi'(t)$, we have $0 < r/\psi'(t) < t/2$. From the mean value theorem it follows that

$$\left|\psi(t) - \psi\left[t - \frac{xr}{\psi'(t)}\right]\right| \leq \frac{xr}{\psi'(t)}\psi'(t_0)$$

for some $t_0 \in (0, t)$. The concavity of ψ implies $\psi'(t_0)/\psi'(t) \leq 1$ so that

$$\left|\psi(t) - \psi\left[t - \frac{xr}{\psi'(t)}\right]\right| < r \tag{4.6}$$

Let

$$z = \left[1 - \psi\left(t - \frac{xr}{\psi'(t)}\right)\right] e^{it}.$$

Then since $\Gamma(t) = [1 - \psi(t)]e^{it}$, it follows from (4.6) that $z \in \Delta[\Gamma(t), r]$. Since

$$z = \exp\left(i\frac{xr}{\psi'(t)}\right)\Gamma\left[t - \frac{xr}{\psi'(t)}\right],$$

we conclude that

$$\exp\left(i\frac{xr}{\psi'(t)}\right) \in A_{\Delta[\Gamma(t), r]}$$

and hence

$$\frac{xr}{\psi'(t)} \leq |A_{\Delta[\Gamma(t), r]}|.$$

The desired inequality follows since $x \in (0, 1)$ was arbitrary and the proof is complete.

As a direct consequence of Lemma 4.3 we have the following estimate.

LEMMA 4.4. *Let $a \in \Delta$ with $1 - |a| < \psi(\pi)$ and suppose that $0 < r < (1 - |a|)/2$. Then*

$$\frac{r}{\psi'[\psi^{-1}(1 - |a|)]} \leq |A_{\Delta(a, r)}|.$$

Proof. Let $\theta = \operatorname{Arg} a$. If $t = \psi^{-1}(1 - |a|)$ and $\eta = e^{i(\theta - t)}$, then $a = \eta\Gamma(t)$, and we may apply Lemma 4.3.

We are now ready to prove an analogue of Theorem 1.4.

THEOREM 4.1. *Suppose that $\psi = \psi_\Gamma$ is C^1-smooth, concave upward, and satisfies*

$$\liminf_{t \to 0^+} \frac{\psi(t)}{t\psi'(t)} > 0. \tag{4.7}$$

If $\{t_k\}_1^\infty$ is a sequence in $[0, 1)$ such that $\Sigma(1 - t_k) < \infty$ and

$$\sum \psi^{-1}(1 - t_k) = \infty, \tag{4.8}$$

then there exists a Blaschke product $B(z) = B(z, \{a_k\})$ *such that* $|a_k| = t_k$, $k = 1, 2, \ldots,$ *and*

$$\liminf_{t \to 0^+} |B[\eta\Gamma(t)]| = 0, \quad \eta \in C. \tag{4.9}$$

Moreover, there is a subproduct f *of* B *for which* (4.9) *holds with* f *replacing* B *such that*

$$\limsup_{t \to 0^+} |f[\eta\Gamma(t)]| = 1, \quad \eta \in C. \tag{4.10}$$

In particular for every $\eta \in C$, *the function* f *fails to have an* $\eta\Gamma$*-limit.*

Proof of Theorem 4.1. By condition (4.8) it is possible to choose a sequence $\{r_k\}_1^\infty$ such that

$$\lim_{k \to \infty} \frac{r_k}{1 - t_k} = 0$$

and

$$\sum \frac{r_k}{1 - t_k} \psi^{-1}(1 - t_k) = \infty.$$

By the assumption (4.7) it follows that

$$\sum \frac{r_k}{\psi'[\psi^{-1}(1 - t_k)]} = \infty.$$

Thus if $\{\eta_k\}_1^\infty$ is an arbitrary sequence in C and $a_k = t_k \eta_k$ for each k, Lemma 4.4 implies that

$$\sum |A_{\Delta(a_k, r_k)}| = \infty.$$

As in the proof of Theorem 1.4, we can now select $\{\eta_k\}_1^\infty$ so that each point $\eta \in C$ lies in an infinite number of the $A_{\Delta(a_k, r_k)}$. Letting $B(z) = B(z, \{a_k\}_1^\infty)$, it follows from Lemma 4.2 and the choice of the r_k that

$$\lim_{k \to \infty} \max\left\{|B(z)| : z \in \overline{\Delta(a_k, r_k)}\right\} = 0.$$

Since for each $\eta \in C$, the curve $\eta\Gamma(t)$ intersects infinitely many of the disks $\Delta(a_k, r_k)$, we conclude that (4.9) holds. The last assertion is proved as in the proof of Theorem 1.4. Theorem 4.1 is thereby established.

In the final theorem, we drop the assumption of smoothness on ψ and assume only that it is tangential.

THEOREM 4.2. *If* Γ *is tangential, then there exists a Blaschke product* $B(z) = B(z, \{a_k\})$ *such that*

$$\liminf_{t \to 0^+} |B[\eta\Gamma(t)]| = 0, \quad \eta \in C. \tag{4.11}$$

To prove Theorem 4.2, we establish some notation and state without proof an elementary lemma. For each $\theta \in (0, \pi)$, let $l(\theta)$ be the line passing through 1 at an angle of θ with the vertical. Let t_θ be the smallest $t > 0$ such that $\Gamma(t) \in l(\theta)$. Such a t_θ must exist because Γ is tangential. Let $\alpha(\theta)$ be the smaller arc of C with endpoints 1 and e^{it_θ}.

LEMMA 4.5. *There exists a constant* $c > 0$ *such that*

$$\theta \geq c\frac{\psi(t_\theta)}{|\alpha(\theta)|}, \quad \theta \in (0, \pi). \tag{4.12}$$

Proof of Theorem 4.2. Select $(\theta_j)_1^\infty$ in $(0, \pi)$ such that

$$\sum 2^j\theta_j < \infty. \tag{4.13}$$

By Lemma 4.5, this implies

$$\sum 2^j\psi(t_{\theta_j}) < \infty. \tag{4.14}$$

Let m_j be the positive integer satisfying

$$\frac{2\pi}{|\alpha(\theta_j)|} < m_j \leq \frac{2\pi}{|\alpha(\theta_j)|} + 1. \tag{4.15}$$

For each positive integer j, let S_j be the set of points in the radial segments

$$R_{jk} = \left[|\Gamma(t_{\theta_j})|e^{2\pi ik/m_j}, e^{2\pi ik/m_j}\right], \quad k = 0, \ldots, m_j - 1,$$

at distances

$$\psi(t_{\theta_j})(1 - 1/2^j)^l, \quad l = 0, 1, \ldots \tag{4.16}$$

from C. Finally, let $\{a_m\}_1^\infty$ be an enumeration of the points in each of the sets $S_1, S_2, \ldots$.

By (4.13)–(4.16) we have

$$\begin{aligned}\sum(1-|a_m|) &= \sum_{j=1}^{\infty} m_j \sum_{l=0}^{\infty} \psi\left(t_{\theta_j}\right)\left(1-\frac{1}{2^j}\right)^l \\ &= \sum_{j=1}^{\infty} m_j 2^j \psi\left(t_{\theta_j}\right) \\ &\le \sum_{j=1}^{\infty} \frac{2\pi}{|\alpha(\theta_j)|} 2^j \psi\left(t_{\theta_j}\right) + \sum_{j=1}^{\infty} 2^j \psi\left(t_{\theta_j}\right) \\ &\le \sum_{j=1}^{\infty} \frac{2\pi}{c} 2^j \theta_j + \sum_{j=1}^{\infty} 2^j \psi\left(t_{\theta_j}\right) \\ &< \infty.\end{aligned}$$

Thus $B(z) = B(z, \{a_m\}_1^\infty)$ is a well-defined Blaschke product. From (4.15), we see that for each positive integer j and $\eta \in C$, there exists $k \in \{0, \dots, m_{j-1}\}$ such that $\eta\Gamma \cap R_{jk} \cap \Delta \neq \varnothing$. From Lemma 4.2, it follows that

$$|B(z)| \le 2^{1-j}, \quad z \in \Delta \cap \bigcup_{k=0}^{m_j} R_{jk},$$

for each j. We conclude that (4.11) holds and the proof is complete.

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