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HAUSDORFF DIMENSION AND PERRON-FROBENIUS THEORY

BY

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1. Introduction

In this paper we describe a method of calculating Hausdorff dimension of certain subsets of the unit interval, using the Perron-Frobenius theory of non-negative matrices. The sets in question are as follows. Let $b > 1$ be a fixed integer. Each $x \in (0,1)$ can then be expressed in base b as

$$x = \sum_{n=1}^{\infty} e_n(x) b^{-n} = 0.e_1(x)e_2(x)\ldots, \tag{1}$$

where $0 \leq e_n(x) \leq n - 1$. The functions $e_n(x)$ are called the digits of x in base b. If we stipulate that the e_n's have the property that for each x, $e_n(x) < b - 1$ for infinitely many n's, then the expansion in (1), i.e., all the functions $e_n(x)$, is uniquely determined. The lack of uniqueness is an issue only for countably many x's. Now, given two integers $0 \leq c \leq r$ we define the set $T_b(c, r)$ to be

$$T_b(c, r) = \left\{ x \in (0,1) \colon \sum_{j=1}^{r} e_{n+j}(x) \geq c,\ n = 0, 1, 2 \ldots \right\}.$$

In other words, $T_b(c, r)$ consists of those x's in $(0,1)$, for which any r consecutive base b digits sum up to at least c. We will show how to calculate the Hausdorff dimension of these sets. The interest in them arose from the paper [2] by one of the authors, in which a Fibonacci type of recurrence of sets was studied. The set arising in that paper was $T_2(1,2)$, the Hausdorff dimension of which turns out to be $\log_2(\frac{1}{2}(1 + \sqrt{5}))$. In order to keep the exposition and notation clear we restrict our attention to case $b = 2$, i.e., to the binary expansion. Extension of the method to arbitrary b's is completely routine.

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We sketch, very briefly, the definition and some basic properties of Hausdorff dimension. For a complete discussion see [1]. In what follows, I and J (with or without subscripts and/or superscripts) will always denote intervals in the real line, and $|I|$ will be the length of the interval I. If $E \subseteq [0,1]$ and $\varepsilon > 0$, define for each $\alpha \geq 0$,

$$\Lambda(E, \alpha, \varepsilon) = \inf\left\{ \sum_{n=1}^{\infty} |I_n|^\alpha : E \subset \bigcup_{n=1}^{\infty} I_n,\ |I_n| \leq \varepsilon,\ n = 1,2,\ldots \right\}.$$

It is clear that for fixed E and α, $\Lambda(E, \alpha, \varepsilon)$ increases as $\varepsilon \downarrow 0$. Now we define

$$\lambda(E, \alpha) = \lim_{\varepsilon \downarrow 0} \Lambda(E, \alpha, \varepsilon).$$

For each $\varepsilon \geq 0$ the set function $\lambda(\cdot, \alpha)$ is an outer measure in the sense of Caratheodory. If we keep E fixed and vary α the following happens: There exists α_0 such that

$$\lambda(E, \alpha) = \begin{cases} 0 & \text{if } \alpha > \alpha_0 \\ \infty & \text{if } \alpha < \alpha_0 \end{cases}.$$

The number α_0 is called the Hausdorff dimension of E and is denoted by $\dim(E)$. The following two facts are basic in calculating Hausdorff dimension of various sets.

LEMMA 1. *Let $E \subseteq \mathbf{R}$ and let $\alpha \geq 0$ be given. Suppose for each $\varepsilon > 0$ there is a sequence of intervals $\{I_n\}$ such that $E \subseteq \bigcup I_n$, $|I_n| \leq \varepsilon$ for all n, and $\sum_{n=1}^{\infty} |I_n|^\alpha \leq 1$. Then* $\dim(E) \leq \alpha$.

LEMMA 2. *Let $E \subseteq \mathbf{R}$ be a compact set and let $\alpha \geq 0$ be given. Suppose there is an $\varepsilon > 0$ with the following property: Given any finite collection of closed, non-overlapping intervals $I_1, I_2, \ldots, I_n$ such that $|I_n| \leq \varepsilon$ and $\sum |I_j|^\alpha \leq 1$, it follows that $\bigcup I_j$ does not cover E. Then* $\dim(E) \geq \alpha$.

Lemma 1 is obvious; for Lemma 2 see [1].

2. Notation and terminology

Recall that we are considering base 2 expansions only. Given a finite sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$ of 0's and 1's we define a cylinder $I(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ of rank N to be the closure of the set

$$\{x \in [0,1]: e_1(x) = \varepsilon_1, e_2(x) = \varepsilon_2, \ldots, e_N(x) = \varepsilon_N\}.$$

Each cylinder of rank N is of length 2^{-N} and two distinct cylinders of the same rank do not overlap. Let now $0 \le c < r$ be two fixed integers. If $N \ge r$ we say that a cylinder $I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N)$ of rank N is admissible if

$$\sum_{j=1}^{r} \varepsilon_{n+j} \ge c \quad \text{for } n = 0, 1, 2, \dots, N - r.$$

If $N < r$, any cylinder of rank N is admissible. Let $\mathscr{F}(N)$ be the collection of all admissible cylinders of rank N and let F_N be the union of all cylinders in $\mathscr{F}(N)$. Finally, let $F = \cap F_N$. A moment of reflection shows that F differs from $T_b(c, r)$ by at most countable number of points. Indeed, $T_b(c, r)$ is the intersection of sets G_N, where G_N is the union of admissible cylinders of rank N, whose right end point was removed. Thus

$$\dim(T_b(c, r)) = \dim(F).$$

From now on we will deal with the set F only. Let now $J = I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$ be a fixed cylinder of rank r. If N is an integer, $N \ge r$, and $I = I(\eta_1, \eta_2, \dots, \eta_N\}$ is any cylinder of rank N, we say that I is of type J if "the last r digits of I coincide with the digits of J", that is, if

$$\eta_{N-j} = \varepsilon_{r-j} \quad \text{for } j = 0, 1, \dots, r - 1.$$

Thus if $r = 3$, $I(01\underline{100})$ is of type $I(\underline{100})$.

Let s be the number of all admissible cylinders of rank r; s depends only on r and c (and, of course, on the base b in general case). We choose an arbitrary ordering of these cylinders, say $J_1, J_2, \dots, J_s$. This ordering will remain fixed for the rest of the paper. Any admissible cylinder of rank $N \ge r$ is of one of the types $J_1, J_2, \dots, J_s$.

Next, we introduce an $s \times s$ matrix $M = [m(i, j);\ i, j = 1, 2, \dots, s]$ as follows. Fix $1 \le i \le s$ and let $J_i = I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$ be the ith admissible cylinder of rank r as above. Consider two cylinders $I' = I(\varepsilon_2, \varepsilon_3, \dots, \varepsilon_r, 1)$ and $I'' = (\varepsilon_2, \varepsilon_3, \dots, \varepsilon_r, 0)$. The cylinder I' is admissible since $\varepsilon_2 + \varepsilon_3 + \cdots + \varepsilon_r + 1 \ge \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r$, so for some j_1 we have $I' = J_{j_1}$. We set $m(i, j_1) = 1$. The cylinder I'' may or may not be admissible, depending on whether $\varepsilon_2 + \varepsilon_3 + \cdots + \varepsilon_r + 0$ is $\ge c$ or $< c$. If I'' is admissible, I'' is one of the J's, say $I'' = J_{j_2}$ for some j_2. We put then $m(i, j_2) = 1$. There is no conflict with the definition of $m(i, j_1)$ since $j_1 \ne j_2$, because the last digits of I' and I'' are different. We then set $m(i, j) = 0$ for all the other entries not determined by the above procedure.

Finally, if X is any $s \times s$ matrix, the spectral radius of X, denoted by $\rho(X)$, is defined by

$$\rho(X) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } X\}.$$

3. The main result

The Hausdorff dimension of the set $T_2(c, r)$ can be calculated as follows.

THEOREM. *With matrix M defined above,*

$$\dim(T_2(c, r)) = \log_2(\rho(M)).$$

Moreover, M has one eigenvalue $\lambda_0 > 0$ such that $\lambda_0 = \rho(M)$, this eigenvalue is simple and the corresponding eigenvector $v = (v_1, v_2, \ldots, v_s)$ has each $v_i > 0$. For every other eigenvalue λ of M, $|\lambda| < \lambda_0$.

We remark that there are well developed numerical procedures to calculate eigenvalues of matrices such as M above. Also, constructing M is quite straightforward, once c and r are given, and the ordering of J's is agreed upon. Thus the values of $\dim(T_b(c, r))$ can be calculated explicitly. We now proceed with the proof.

LEMMA 3. *Let $N \geq r$ and let $\mathcal{F}$ be any collection of admissible cylinders of rank N. For each $1 \leq i \leq s$ let f_i be the number of cylinders in $\mathcal{F}$ which are of type J_i, and let f be the vector $(f_1, f_2, \ldots, f_s)$. Now let $\mathcal{G} = \mathcal{G}(\mathcal{F})$ be the collection of admissible cylinders of rank $N + 1$ which are contained in some cylinder of $\mathcal{F}$. Let g_i be the number of cylinders of $\mathcal{G}$ which are of type J_i and let g be the vector $g = (g_1, g_2, \ldots, g_s)$. Then*

$$g = fM, \tag{2}$$

where M is the matrix constructed in the previous section.

Proof. Each cylinder I of $\mathcal{F}$ contains exactly two cylinders of rank $N + 1$. At least one of them is admissible: the one whose last digit is 1. Given a fixed $I \in \mathcal{F}$ of type J_i, it contains I' of type J_j if and only if $m(i, j) = 1$. Thus for a fixed j we have $g_j = \Sigma_i f_i$, where the summation is taken over those i's for which $m(i, j) = 1$. But that is exactly the equation (2). (We multiply f by the jth column of M to get the jth entry of g.)

Denote by U the vector $(1, 1, \ldots, 1)$ (s 1's).

COROLLARY 1. *Let Λ_N denote the number of admissible cylinders of rank $N \geq r$. Then*

$$\Lambda_N = (UM^{N-r} \cdot U).$$

(*Here $\cdot$ denotes the ordinary dot product.*)

Proof. By definition, there are $s = U \cdot U$ admissible cylinders of rank r. Applying Lemma 3 ($N - r$ times) with $f = U$, the result follows.

COROLLARY 2. *Let I be a fixed admissible cylinder of rank $t \geq r$ and of type J_k, and let*

$$V = (0, \ldots, 0, 1, 0, \ldots, 0) \quad (1 \textit{ in the kth place}).$$

Let $\Lambda_N(I)$ be the number of admissible cylinders of rank $t + N$ which are contained in I. Then $\Lambda_N(I) = (VM^N \cdot U)$.

We now prove the second assertion of the theorem (the part after "Moreover"). This is, however, precisely the Perron-Frobenius theorem (see [3], p. 30), the only thing we must show is that M is irreducible. Given an $s \times s$ matrix $X = [x_{ij}]$ with non-negative entries, the definition of irreducibility is as follows. We construct a directed graph on s vertices $v_1, v_2, \ldots, v_s$ with an arrow going from v_i to v_j if and only if $x_{ij} > 0$. The matrix X is called irreducible if and only if the resulting graph is strongly connected, i.e., if there is a path from any vertex to any other vertex. For all this see [3], Chapter 2. In the case of our matrix M, $m(i, j) > 0$ if and only if the ith cylinder J_i has the form

$$I(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$$

and the jth cylinder has the form

$$I(\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_r, \eta) \quad \text{where } \eta = 0 \text{ or } 1.$$

Thus to show that M is irreducible we must show that given any two admissible cylinders

$$J_i = I(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r) \quad \text{and} \quad J_j = I(\eta_1, \eta_2, \ldots, \eta_r),$$

it is possible to get from J_i to J_j by an operation of "adding a digit at the end and shifting the remaining digits to the left", with each intermediate cylinder being admissible. Now, if $J_i = I(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$ is admissible, then $I(\varepsilon_2, \ldots, \varepsilon_r, 1)$ is also admissible since

$$\varepsilon_2 + \cdots + \varepsilon_r + 1 \geq \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r.$$

Thus it is possible to get from $J_i = I(\varepsilon_1, \ldots, \varepsilon_r)$ to $I(1, 1, \ldots, 1)$ going only through admissible cylinders. But now, $I(1, 1, \ldots, 1, \eta_1)$ is also admissible because

$$1 + 1 + \cdots + \eta_1 \geq \eta_1 + \eta_2 + \cdots + \eta_r.$$

For the same reason, $I(1,1,\dots,1,\eta_1,\eta_2)$ is admissible, etc. Continuing in this way we see that it is indeed possible to get from any J_i to any J_j, and thus M is irreducible so the Perron-Frobenius theorem applies. Finally, it remains to show that for any other eigenvalue λ of M we have $|\lambda| < \lambda_0$, or, in the terminology of the Perron-Frobenius theory, that the matrix M is irreducible. By [3], p. 43, Exercise 1, it is enough to show that one of the diagonal entries of M is positive. Now, one of the admissible cylinders of rank r is $I(1,1,\dots,1)$, say this cylinder is J_{k_1} on the list of all such cylinders. The cylinder $I(1,1,\dots,1,1)(r+1$ 1's) is again admissible, is a subset of J_{k_1} and is of type J_{k_1}, so $m(k_1,k_1)=1$. The assertion is thus proved.

Let now λ_0 be the simple eigenvalue of M of largest absolute value; we have just shown that $\lambda_0 > 0$.

LEMMA 4. *There exists a constant* $c > 0$ *such that as* $N \to \infty$, $\Lambda_N \sim c\lambda_0^N$. *In particular, for some* $0 < c_1 < c_2$, $c_1\lambda_0^N \le \Lambda_N \le c_2\lambda_0^N$. ($\Lambda_N$ *is the number of admissible cylinders of rank* N.)

Proof. Let $v=(v_1,\dots,v_s)$ be the eigenvector corresponding to the (simple) eigenvalue λ_0; all the v_j's are >0 (see [3], Theorem 2.1). By Corollary 1, $\Lambda_N = (U^{N-r}\cdot U)$, where $U=(1,\dots,1)$. The entire space $\mathbf{R}^s$ can be written as a direct sum

$$\mathbf{R}^s = \operatorname{span}(v) \oplus Y,$$

where Y is invariant under M, and the restriction of M to Y, denoted by M_Y, has all of its eigenvalues $<\lambda_0$ in absolute value. Choose a number a such that $v_i \le a$, $i=1,\dots,s$. If $U \in Y$ then, by the spectral radius theorem,

$$\begin{aligned}\lambda_0 &= \lim \|vM^N\|^{1/N} \le \overline{\lim}\, \|aUM^N\|^{1/N}\\ &\le \overline{\lim}\, \|M_Y^N\|^{1/N} = |\text{largest eigenvalue of } M_Y| < \lambda_0.\end{aligned}$$

Hence $U = bv + w$, for some $b \neq 0$ and $w \in Y$. Thus

$$(UM^{N-r}\cdot U) = b\lambda_0^{-r}(v\cdot U)\lambda_0^N + (wM^{N-r}\cdot U)$$

and the last term is $o(\lambda_0^N)$, since $wM = wM_Y$.

LEMMA 5. *There exists a constant* $c_3 > 0$ *with the following property. Let* I *be any admissible cylinder of rank* t *and let* $\Lambda_N(I)$ *be the number of admissible cylinders of rank* $t+N$ *which are contained in* I. *Then* $\Lambda_N(I) \le c_3\lambda_0^N$. *Moreover, this constant* c_3 *can be taken to be independent of* I (*and hence of* t).

Proof. It is enough to show that such a constant exists, depending only on the type of I (there are only finite number of types). By Corollary 2,

$\Lambda_N(I) = (VM^N \cdot U)$, where $V = (0, \ldots, 1, \ldots, 0)$ (1 in the kth place), and U is as above. Let $v = (v_1, \ldots, v_s)$ be the eigenvector corresponding to the (simple) eigenvalue λ_0, and let $\mu = \operatorname{Min}(v_1, \ldots, v_s) > 0$. Then it easily follows that

$$\mu^2 \Lambda_N(I) = \left(\mu VM^N \cdot \mu U\right) \leq \left(vM^N \cdot \mu U\right) = \lambda_0^N(v \cdot \mu U)$$

so c_3 may be taken to be $\mu^{-2}(v \cdot \mu U)$. QED.

We now show, using Lemma 1, that $\dim(F) \leq \log_2(\lambda_0)$. Let $\alpha > \log_2(\lambda_0)$. The family $\mathscr{F}(N)$ covers F, each interval of $\mathscr{F}(N)$ has length 2^{-N} and by Lemma 4 there are at most $c_2 \lambda_0^N$ intervals in $\mathscr{F}(N)$. Thus

$$\sum_{I \in \mathscr{F}(N)} |I|^\alpha \leq c_2 (\lambda_0 2^{-\alpha})^N. \tag{3}$$

Now, given $\varepsilon > 0$ choose N so large that the right side of (3) is < 1. This is possible because $\lambda_0 < 2^\alpha$. Thus $\dim(F) \leq \alpha$. Since $\alpha > \log_2(\lambda_0)$ was arbitrary, the result follows.

Finally, we show that $\dim(F) \geq \log_2(\lambda_0)$. The proof is an adaptation of techniques in [1]. Let $\alpha < \log_2(\lambda_0)$. We will construct $\varepsilon > 0$ such that if $\mathscr{U}$ is any finite collection of intervals I so that $\sum_{I \in \mathscr{U}} |I|^\alpha < 1$ and $|I| < \varepsilon$ for all $I \in \mathscr{U}$, then for N large enough $\mathscr{F}(N)$ will contain cylinders disjoined from any I in $\mathscr{U}$. Since each cylinder in $\mathscr{F}(N)$ intersects the set F, this will show that $\mathscr{U}$ does not cover F, and thus, by Lemma 2, $\dim(F) \geq \alpha$. Since $\alpha < \log_2(\lambda_0)$ was arbitrary this will give the result. Given α as above we will construct ε as follows. Since the series $\sum_n (2^{-\alpha}\lambda_0)^n$ converges, there is an integer t so that

$$\sum_{n \geq t} (2^{-\alpha}\lambda_0)^n \leq \frac{c_1}{100 c_3},$$

where c_1 and c_3 are as in Lemmas 4 and 5, respectively. Put then $\varepsilon = 2^{-t}$. Let $\mathscr{U}$ be now a family of intervals as described above. For each $p = 1, 2, 3, \ldots$, let $\mathscr{U}(p)$ contain those I's from $\mathscr{U}$ for which

$$\frac{1}{2^{t+p}} < |I| \leq \frac{1}{2^{t+p-1}}.$$

Each I in $\mathscr{U}(p)$ intersects at most 3 intervals from $\mathscr{F}(t + p - 1)$, and thus at most 6 intervals from $\mathscr{F}(t + p)$. Let γ_p denote the number of intervals in $\mathscr{U}(p)$. We have then

$$\gamma_p (2^{-t-p})^\alpha \leq \sum_{I \in \mathscr{U}(p)} |I|^\alpha \leq 1$$

or $\gamma_p \leq (2^{t+p})^\alpha$. Let $B(p)$ denote the number of intervals from $\mathscr{F}(t+p)$ which intersect some $I \in \mathscr{U}(p)$. By the above,

$$B(p) \leq 6\gamma_p \leq 6(2^{t+p})^\alpha.$$

Since $\mathscr{U}$ is a finite family, $\mathscr{U}(p) = \varnothing$ for $p > P_0$. Let now $N > P_0$. If $I \in \mathscr{F}(t+N)$ intersects $\mathscr{U}$, then it must intersect some $\mathscr{U}(p)$ for certain $1 \leq p \leq P_0$, and hence this I must be contained in some $J \in \mathscr{F}(t+p)$, where such J intersects $\mathscr{U}(p)$. Given a specific J in $\mathscr{F}(t+p)$, by Lemma 5, there are at most $c_3\lambda_0^{t+N-(t+p)}$ I's in $\mathscr{F}(N+t)$ which are contained in this J. Hence the number of I's in $\mathscr{F}(t+N)$ which intersect a specific $\mathscr{U}(p)$ is at most

$$6(2^{t+p})^\alpha c_3\lambda_0^{N-p} = 6(2^{-\alpha}\lambda_0)^{t+p} c_3\lambda_0^{t+N}.$$

Thus, the total number of I's in $\mathscr{F}(t+N)$ which intersect some $\mathscr{U}(p)$ is at most

$$6\lambda_0^{t+N}c_3 \sum_{p=1}^{P_0} (2^{-\alpha}\lambda_0)^{t+p} < 6c_3 \sum_{n \geq t} (2^{-\alpha}\lambda_0)^n \lambda_0^{t+N} < \tfrac{6}{100}c_1\lambda_0^{t+N} < \tfrac{1}{2}c_1\lambda_0^{t+N},$$

by the choice of t. The total number of intervals in $\mathscr{F}(t+N)$ is, however, larger than $c_1\lambda_0^{t+N}$ by Lemma 4. Thus the assertion follows and the proof of the theorem is complete.

As an illustration, we will calculate $\dim(T_2(,1\ 2))$, i.e., the dimension of the set of those x's for which $e_j(x) + e_{j+1}(x) \geq 1$ for all j's. There are 3 admissible cylinders of rank 2: $I(1,1) = J_1$, $I(1,0) = J_2$, and $I(0,1) = J_3$. Easy calculation shows that the matrix M is given by

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of M are 0, $\frac{1}{2}(1+\sqrt{5})$, and $\frac{1}{2}(1-\sqrt{5})$. Thus

$$\dim(T_2(1,2)) = \log_2\left(\tfrac{1}{2}(1+\sqrt{5})\right).$$

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