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SPECTRAL GEOMETRY OF THE SECOND VARIATION OPERATOR OF HARMONIC MAPS

BY

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Dedicated to Professor Shingo Murakami on the occasion of his sixtieth birthday

Introduction

In this paper, we deal with the inverse spectral problem of the Hessian (the so called Jacobi operator) of the energy of a harmonic map.

The spectral geometry of the Laplace-Beltrami operator has developed greatly during the last twenty years [3]. It is well-known [2], [20], [24] that if the spectrum Spec(Δ) of the Laplace-Beltrami operator Δ of a compact Riemannian manifold (M, g) coincides with the one of the standard sphere (S^n, can), $n < 7$, then (M, g) is isometric to (S^n, can). Since the Laplace-Beltrami operator of (M, g) can be regarded as the Jacobi operator of a constant map of (M, g) into a circle, it is reasonable to investigate the spectral geometry for the Jacobi operator of a harmonic map.

In fact, since the Jacobi operator J_ϕ of a harmonic map ϕ is a second order elliptic differential operator acting on the space of sections of the induced bundle of the tangent bundle of the target manifold, the spectrum Spec(J_ϕ) of J_ϕ becomes a discrete set of the eigenvalues with finite multiplicities. Directly applying Gilkey's results [11], [12] about the asymptotic expansion of the trace of the heat kernel of a certain differential operator of a vector bundle to our case, we can determine some geometric spectral invariants of the Jacobi operator (§§2, 3). Using these results, we obtain a series of geometric results distinguishing typical harmonic maps, i.e., (0) constant maps, (1) geodesics, (2) isometric minimal immersions, (3) holomorphic maps between Kaehler manifolds, and (4) Riemannian submersions all of whose fibers are minimal.

The analogue of spectral geometry for minimal submanifolds has been studied by H. Donnelly [7], and T. Hasegawa [15].

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1. Preliminaries

In this section we summarize briefly some of Gilkey's results in [11], [12] concerning the asymptotic expansion of the trace of the heat kernel for a certain elliptic differential operator acting on the space of sections of a vector bundle (see also the paper [7] [15] by Donnelly and Hasegawa).

Let $E \to M$ be a real smooth vector bundle of fiber dimension r over a compact connected Riemannian manifold M of dimension m. Assume that E has a connection

$$\tilde{\nabla};\quad \Gamma(E) \to \Gamma(E \otimes T^*M)$$

which is compatible with the inner product (,) on each fiber of E, i.e.,

$$X(s, s') = \left(\tilde{\nabla}_X s, s'\right) + \left(s, \tilde{\nabla}_X s'\right)$$

for $s, s' \in \Gamma(E)$ and a vector field X on M. We consider the following second order elliptic differential operator D; $\Gamma(E) \to \Gamma(E)$ of the form

$$D = \tilde{\nabla}^*\tilde{\nabla} - L,$$

where L is an endomorphism of E and $\tilde{\nabla}^*\tilde{\nabla}$ is the rough Laplacian acting on $\Gamma(E)$ which is given by

$$\tilde{\nabla}^*\tilde{\nabla}s = -\sum_{j=1}^{m}\left(\tilde{\nabla}_{e_j}\tilde{\nabla}_{e_j} - \tilde{\nabla}_{\nabla_{e_j}e_j}\right)s,\quad s \in \Gamma(E).$$

Here $\{e_j;\ j = 1,\dots, m\}$ is a local orthonormal frame field on (M, g) and ∇ is the Levi-Civita connection of (M, g). Since D is self-adjoint and elliptic and M is compact, D has a discrete spectrum of eigenvalues with finite multiplicities, denoted by

$$\operatorname{Spec}(D) = \left\{\tilde{\lambda}_1 \leqq \tilde{\lambda}_2 \leqq \cdots \leqq \tilde{\lambda}_j \leqq \cdots \uparrow \infty\right\}.$$

Then the trace $Z(t) = \sum_{j=1}^{\infty}\exp(-t\tilde{\lambda}_j)$ of the heat kernel for the operator D has the asymptotic expansion

$$Z(t) \sim (4\pi t)^{-m/2}\left\{a_0(D) + a_1(D)t + a_2(D)t^2 + \cdots\right\} \tag{1.1}$$

as $t \to \infty$.

Moreover we have:

THEOREM 1.1 [11], [12], [7], [15]. *Under the above situation, we have*

$$a_0(D) = r\,\mathrm{Vol}(M, g),$$

$$a_1(D) = \frac{r}{6}\int_M \tau_g v_g + \int_M \mathrm{Tr}(L)\,v_g,$$

$$a_2(D) = \frac{r}{360}\int_M \left\{5\tau_g^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2\right\} v_g$$

$$+ \frac{1}{360}\int_M \left\{-30\|R^{\tilde\nabla}\|^2 + 60\tau_g\,\mathrm{Tr}(L) + 180\,\mathrm{Tr}(L^2)\right\} v_g,$$

where R_g, ρ_g, τ_g, *are the curvature tensor, the Ricci tensor and the scalar curvature of* (M, g), *respectively, and* v_g *is the volume element of* (M, g). *The operator* $R^{\tilde\nabla}$ *is the curvature tensor of the connection* $\tilde\nabla$ *on* E,

$$R^{\tilde\nabla}_{X,Y}(s) = -\left[\tilde\nabla_X, \tilde\nabla_Y\right]s + \tilde\nabla_{[X,Y]}s$$

for $s \in \Gamma(E)$ *and vector fields* X, Y *on* M. *The norm* $\|\ \|$ *is that induced from* $(\ ,\ )$ *and* g.

2. The spectral invariants of the Jacobi operator

In this section, we apply Gilkey's results to the Jacobi operator J_ϕ of a harmonic map ϕ. First let us recall the second variation formula of the energy for a harmonic map.

Let (M, g) be an m-dimensional compact Riemannian manifold without boundary and (N, h) an n-dimensional Riemannian manifold. A smooth map ϕ; $(M, g) \to (N, h)$ is said to be *harmonic* if it is a critical point of the energy $E(\cdot)$ defined by

$$E(\phi) = \int_M e(\phi)\,v_g, \tag{2.1}$$

$$e(\phi) = \frac{1}{2}\sum_{i=1}^m h(\phi_* e_i, \phi_* e_i), \tag{2.2}$$

where ϕ_* is the differential of ϕ. Namely, for every vector field V along ϕ,

$$\left.\frac{d}{dt}\right|_{t=0} E(\phi_t) = 0.$$

Here ϕ_t; $M \to N$ is a one parameter family of smooth maps with $\phi_0 = \phi$ and

$$\left.\frac{d}{dt}\right|_{t=0} \phi_t(x) = V_x \in T_{\phi(x)}N$$

for every point x in M.

The second variation formula of the energy E for a harmonic map ϕ is given by

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\phi_t) = \int_M h(V, J_\phi V) v_g. \tag{2.3}$$

Here J_ϕ is a differential operator (called the *Jacobi operator*) acting on the space $\Gamma(E)$ of sections of the induced bundle $E = \phi^{-1}TN$. The operator J_ϕ is of the form

$$J_\phi V = \tilde{\nabla}^* \tilde{\nabla} V - \sum_{i=1}^{m} R_h(\phi_* e_i, V)\phi_* e_i, \quad V \in \Gamma(E). \tag{2.4}$$

Here $\tilde{\nabla}$ is the connection of $E = \phi^{-1}TN$ (cf. [10, p. 4]) which is induced by

$$\tilde{\nabla}_X V = \nabla^h_{\phi_* X} V,$$

for $V \in \Gamma(E)$, X a tangent vector of M, and the Levi-Civita connection ∇^h of (N, h). R_h is the curvature tensor of (N, h) whose sign is the same as $R^{\tilde{\nabla}}$. Note that $\tilde{\nabla}$ is compatible with the metric h. Define the endomorphism L for our E by

$$L(V) = \sum_{i=1}^{m} R_h(\phi_* e_i, V)\phi_* e_i, \quad V \in \Gamma(E). \tag{2.5}$$

Then we have by definition,

$$\mathrm{Tr}(L) = \mathrm{Tr}_g(\phi^* \rho_h). \tag{2.6}$$

We denote also the spectrum of the Jacobi operator J_ϕ of the harmonic map ϕ by

$$\mathrm{Spec}(J_\phi) = \left\{ \tilde{\lambda}_1 \leqq \tilde{\lambda}_2 \leqq \cdots \leqq \tilde{\lambda}_j \leqq \cdots \uparrow \infty \right\}.$$

Then the trace $Z(t) = \exp(-t\tilde{\lambda}_j)$ of the heat kernel for the Jacobi operator J_ϕ

has the asymptotic expansion

$$(2.7)\quad Z(t) \sim (4\pi t)^{-m/2}\left\{a_0(J_\phi) + a_1(J_\phi)t + a_2(J_\phi)t^2 + \cdots\right\} \quad \text{as } t \to \infty.$$

Moreover we have by Theorem 1.1:

THEOREM 2.1. *For a harmonic map* ϕ; $(M^m, g) \to (N^n, h)$,

$$a_0(J_\phi) = n\,\mathrm{Vol}(M, g),$$

$$a_1(J_\phi) = \frac{n}{6}\int_M \tau_g v_g + \int_M \mathrm{Tr}_g(\phi^*\rho_h)\, v_g,$$

$$a_2(J_\phi) = \frac{n}{360}\int_M \left\{5\tau_g^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2\right\} v_g$$

$$+ \frac{1}{360}\int_M \left\{-30\|\phi^* R_h\|^2 + 60\tau_g\,\mathrm{Tr}_g(\phi^*\rho_h) + 180\|L\|^2\right\} v_g,$$

where, for tangent vectors $X, Y \in T_x M$, $(\phi^* R_h)_{X,Y}$ *is the endomorphism of* $T_{\phi(x)}N$ *given by* $(\phi^* R_h)_{X,Y} = R_{h\phi_* X, \phi_* Y}$.

Then we immediately have the following corollaries.

COROLLARY 2.2. *Let* (M, g) *be a compact Riemannian manifold and* (N, h), *an Einstein manifold with non-zero Einstein constant, i.e., whose Ricci curvature* ρ_h *satisfies* $\rho_h = kh$ *for some non-zero constant* k. *Let* ϕ, ϕ' *be two harmonic maps from* (M, g) *into* (N, h). *Assume that*

$$\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'}).$$

Then we have $E(\phi) = E(\phi')$. *In particular, if* ϕ *is a constant map, then so is* ϕ'.

COROLLARY 2.3. (i). *Let* ϕ, ϕ' *be two harmonic maps of compact Riemannian manifolds* $(M, g), (M', g')$ *into Riemannian manifolds* $(N, h), (N', h')$, *respectively. Assume that* $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$. *Then we have* $\dim(M) = \dim(M')$.

(ii) *Moreover let us consider two periodic geodesics*

$$\phi, \phi';\quad (S^1, \mathrm{can}) \to (N, h)$$

in an Einstein manifold (N, h) *with the non-zero Einstein constant. Assume that* $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$. *Then both geodesics* ϕ, ϕ' *have the same length, index, and nullity.*

Proof. The asymptotic expansion of the trace $Z(t)$ has of the form (2.7), so we get the first statement. The second follows from Corollary 2.2.

3. The precise spectral invariants

In this section, we assume that the target space (N, h) is either a space form $N^n(c)$, i.e., a Riemannian manifold of constant curvature c, or a complex n-dimensional Kaehler manifold of constant holomorphic sectional curvature c. In these cases, we will calculate the terms $a_1(J_\phi)$ and $a_2(J_\phi)$ of the asymptotic expansion (1.1) for the Jacobi operator J_ϕ.

3.1. First, let (N, h) be the space form $N^n(c)$. The curvature tensor R_h of $N^n(c)$ is given (cf. [17]) by $R_{h_{X,Y}}Z = -c\{h(Z,Y)X - h(Z,X)Y\}$, for tangent vectors X, Y, Z on $N^n(c)$. Therefore we obtain by a direct computation, that for a harmonic map ϕ; $(M, g) \to N^n(c)$,

$$\mathrm{Tr}(L) = 2(n-1)ce(\phi), \tag{3.1}$$

$$\|R^{\tilde{\nabla}}\|^2 = 2c^2\{4e(\phi)^2 - \|\phi^*h\|^2\}, \tag{3.2}$$

$$\mathrm{Tr}(L^2) = \|L\|^2 = c^2\{4(n-2)e(\phi)^2 + \|\phi^*h\|^2\}, \tag{3.3}$$

where $\|\phi^*h\|^2 = \sum_{i=1}^m \phi^*h(e_i, e_j)^2$. Therefore we have:

THEOREM 3.1. *Let ϕ; $(M^m, g) \to N^n(c)$ be a harmonic map of a compact Riemannian manifold (M, g) into a space form $N^n(c)$. Then the coefficients $a_0(J_\phi)$, $a_1(J_\phi)$, and $a_2(J_\phi)$ of the asymptotic expansion for the Jacobi operator J_ϕ are given as follows*:

$$a_0(J_\phi) = n\,\mathrm{Vol}(M, g), \tag{3.4}$$

$$a_1(J_\phi) = \frac{n}{6}\int_M \tau_g v_g - 2(n-1)cE(\phi), \tag{3.5}$$

$$\begin{aligned} a_2(J_\phi) = {} & \frac{n}{360}\int_M \{5\tau_g^2 + 2\|\rho_g\|^2 + 2\|R_g\|^2\} v_g \\ & + 2c^2 3^{-1}\int_M \{(3n-7)e(\phi)^2 + \|\phi^*h\|^2\} v_g \\ & + (n-1)c3^{-1}\int_M \tau_g e(\phi) v_g. \end{aligned} \tag{3.6}$$

As an application, we obtain:

COROLLARY 3.2. *Let ϕ, ϕ' be two harmonic maps from a compact Riemannian manifold (M, g) with constant scalar curvature into the n-dimensional* *space form $N^n(c)$ of non-zero constant sectional curvature c. Suppose that*

$$\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'}).$$

Then we have

$$E(\phi) = E(\phi') \tag{3.7}$$

and

$$\int_M \{(3n-7)e(\phi)^2 + \|\phi^* h\|^2\} v_g = \int_M \{(3n-7)e(\phi')^2 + \|\phi'^* h\|^2\} v_g. \tag{3.8}$$

3.2. In this subsection, we assume that (N^{2n}, h) is the complex n-dimensional Kaehler manifold $P^n(c)$ of constant holomorphic sectional curvature c. The curvature tensor R_h of $P^n(c)$ is given [17, p. 167] by

$$R_{h_{Z,W}} Y = -\frac{c}{4}\{h(Y,W)Z - h(Y,Z)W + h(Y,JW)JZ - h(Y,JZ)JW + 2h(Z,JW)JY\},$$

where J is the complex structure of $P^n(c)$. Then for a harmonic map

$$\phi;\ (M^m, g) \to P^n(c),$$

we obtain

$$\mathrm{Tr}(L) = (n+1)ce(\phi), \tag{3.9}$$

since $\rho_h = 2^{-1}(n+1)ch$. Moreover let $\{e_1', \dots, e_n', Je_1', \dots, Je_n'\}$ be a local orthonormal frame field on $P^n(c)$. Then since

$$\|R^{\tilde{\nabla}}\|^2 = \sum_{i,j=1}^{m}\sum_{k=1}^{n}\left\{\left\|R_{h\phi_* e_i, \phi_* e_j}(e_k')\right\|^2 + \left\|R_{h\phi_* e_i, \phi_* e_j}(Je_k')\right\|^2\right\},$$

$$\mathrm{Tr}(L^2) = \sum_{i,j=1}^{m}\sum_{k=1}^{n}\Big\{h\big(R_{h\phi_* e_i, e_k'}(\phi_* e_i), R_{h\phi_* e_j, e_k'}(\phi_* e_j)\big) + h\big(R_{h\phi_* e_i, Je_k'}(\phi_* e_i), R_{h\phi_* e_j, Je_k'}(\phi_* e_j)\big)\Big\},$$

by a straightforward computation we obtain

$$\|R^{\tilde{\nabla}}\|^2 = c^2 4^{-1}\{4e(\phi)^2 - \|\phi^* h\|^2 + (2n+5)\|\phi^*\Phi\|^2\}, \tag{3.10}$$

$$\mathrm{Tr}(L^2) = c^2 8^{-1}\{4(n+2)e(\phi)^2 + 5\|\phi^* h\|^2 - 3\|\phi^*\Phi\|^2\}, \tag{3.11}$$

where Φ is the Kaehler form of $P^n(c)$, i.e., $\Phi(X, Y) = h(X, JY)$, for vector fields X, Y on $P^n(c)$. Hence we have:

THEOREM 3.3. *Let ϕ be a harmonic map of a compact Riemannian manifold (M^m, g) into the complex n-dimensional Kaehler manifold of constant holomorphic sectional curvature c. Then the coefficients $a_0(J_\phi)$, $a_1(J_\phi)$, and $a_2(J_\phi)$ of the asymptotic expansion for the Jacobi operator J_ϕ are*

$$a_0(J_\phi) = 2n\,\mathrm{Vol}(M, g), \tag{3.12}$$

$$a_1(J_\phi) = 3^{-1}n\int_M \tau_g v_g + (n+1)cE(\phi), \tag{3.13}$$

$$\begin{aligned} a_2(J_\phi) = {} & \frac{n}{180}\int_M \{5\tau_g^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2\} v_g \\ & + 24^{-1}c^2\int_M \{(6n+10)e(\phi)^2 + 8\|\phi^*h\|^2 \\ & \qquad - (n+7)\|\phi^*\Phi\|^2\} v_g \\ & + 6^{-1}(n+1)c\int_M \tau_g e(\phi) v_g, \end{aligned} \tag{3.14}$$

where Φ is the Kaehler form of $P^n(c)$.

COROLLARY 3.4. *Let ϕ, ϕ' be two harmonic maps of a compact Riemannian manifold (M^m, g) with constant scalar curvature into the complex n-dimensional Kaehler manifold $P^n(c)$ of non-zero constant holomorphic sectional curvature c. Assume that*

$$\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'}).$$

Then we have

$$E(\phi) = E(\phi') \tag{3.16}$$

and

$$\begin{aligned} & \int_M \{(6n+10)e(\phi)^2 + 8\|\phi^*h\|^2 - (n+7)\|\phi^*\Phi\|^2\} v_g \\ & = \int_M \{(6n+10)e(\phi')^2 + 8\|\phi'^*h\|^2 - (n+7)\|\phi'^*\Phi\|^2\} v_g. \end{aligned} \tag{3.16}$$

4. Spectral geometry of a harmonic map

4.1 A counterexample. In this subsection, we construct harmonic maps with the same spectra for their Jacobi operator.

Let ϕ be a harmonic map of a compact Riemannian manifold (M, g) into a flat torus $(N, h) = (T^n, h)$. Take n parallel vector fields $\{X_1, \ldots, X_n\}$ on (T^n, h) which are linearly independent at each point of T^n. Defining elements $\tilde{X}_i$ in $\Gamma(\phi^{-1}TN)$ by $\tilde{X}_{ix} = X_{i\phi(x)}$, $x \in M$, we have

$$\Gamma(\phi^{-1}TN) = \left\{ \sum_{i=1}^{n} f_i \tilde{X}_i;\ f_i \in C^\infty(M),\ i = 1, \ldots, n \right\},$$

and the Jacobi operator J_ϕ is of the form

$$J_\phi V = \sum_{i=1}^{n} (\Delta f_i) \tilde{X}_i \quad \text{for } V = \sum_{i=1}^{n} f_i \tilde{X}_i \in \Gamma(\phi^{-1}TN),$$

where $\Delta = \delta d$ the (positive) Laplacian of the compact Riemannian manifold (M, g) acting on $C^\infty(M)$. Then we get immediately:

PROPOSITION 4.1. *For every harmonic map ϕ of a compact Riemannian manifold (M, g) into a flat torus (T^n, h), the spectrum* $\mathrm{Spec}(J_\phi)$ *of the Jacobi operator J_ϕ is given by* $\mathrm{Spec}(J_\phi) = n \times \mathrm{Spec}(\Delta)$, *namely, the eigenvalues of J_ϕ are the eigenvalues of the Laplacian Δ of (M, g) with multiplicity n.*

In particular, let ϕ, ϕ' be two harmonic maps of compact Riemannian manifolds $(M, g), (M', g')$ into flat tori $(T, h), (T', h')$, respectively. Then $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$ *if and only if*

$$\dim(T) = \dim(T') \quad \text{and} \quad \mathrm{Spec}(\Delta) = \mathrm{Spec}(\Delta'),$$

where Δ, Δ' are the Laplacians for $(M, g), (M', g')$, respectively.

Remark 4.2. By Proposition 4.1, we can never distinguish even a constant map by the spectrum $\mathrm{Spec}(J_\phi)$ of the Jacobi operator for a harmonic map into a flat torus (Compare with Corollary 2.2).

4.2 The homotopy determination by the spectrum.

PROPOSITION 4.3. *Let M^2, N^2 be compact Riemann surfaces with Kaehler metrics g, h, respectively. Let ϕ_1, ϕ_2; $M^2 \to N^2$ be holomorphic or anti-holomorphic maps. Assume that* $\mathrm{Spec}(J_{\phi_1}) = \mathrm{Spec}(J_{\phi_2})$. *Then* $|\deg(\phi_1)| = |\deg(\phi_2)|$. *In particular, when the genus of N^2 is zero, the maps ϕ_1, ϕ_2 are homotopic each other.*

Proof. We use the notation of [9]. We get $E(\phi) = \pm K(\phi) = \pm \mathrm{Vol}(N^2, h) \deg(\phi)$ when ϕ is holomorphic or anti-holomorphic, because

$K(\phi) = E'(\phi) - E''(\phi)$, $E(\phi) = E'(\phi) + E''(\phi)$, and by Corollary 8.12 in [9]. Therefore the assumption $\mathrm{Spec}(J_{\phi_1}) = \mathrm{Spec}(J_{\phi_2})$ implies $E(\phi_1) = E(\phi_2)$ and then $\deg(\phi_1) = \pm\deg(\phi_2)$. When the genus of N^2 is zero, ϕ_1 and ϕ_2 are homotopic due to the Hopf degree theorem [13], [14].

COROLLARY 4.4. *Let ϕ_1, ϕ_2 be two harmonic maps of the standard unit 2-sphere (S^2, g) into itself. Suppose that* $\mathrm{Spec}(J_{\phi_1}) = \mathrm{Spec}(J_{\phi_2})$. *Then ϕ_1, ϕ_2 are homotopic and $\phi_i^* g = \mu_i g$, $i = 1, 2$, with $\int_{S^2} \mu_1^2 v_g = \int_{S^2} \mu_2^2 v_g$.*

The proof follows from Corollary 3.2, using the facts every harmonic map of (S^2, g) into itself is holomorphic or anti-holomorphic [19, Corollary 2.9] and hence weakly conformal [19, Theorem 2.8].

5. Isometric minimal immersions into spheres

In this section, we give results distinguishing isometric minimal immersions into spheres:

THEOREM 5.1. *Let (M, g) be a compact Riemannian manifold whose scalar curvature is constant. Let (N, h) be a non-flat space form, i.e., whose sectional curvature is non-zero constant. Let ϕ, ϕ' be two harmonic maps of (M, g) into (N, h). Suppose that* $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$. *If ϕ is an isometric minimal immersion, then so is ϕ'. If (M, g) is isometric to (N, h), and ϕ is an isometry of (M, g) onto itself, then so is ϕ'.*

Remark 5.2. The set of all harmonic maps of constant energy into spheres is parametrized by Toth and G'Ambra [23]. Their results say the dimension of the parameter space of constant energy harmonic maps is larger than the one of the parameter space of isometric minimal immersions.

Proof. By the assumption, we have $\phi^* h = g$ and $n \geqq m \geqq 2$. We can also write

$$\phi'^* h = \mu g + t, \tag{5.1}$$

where μ is a smooth function on M and t is a trace-less symmetric 2-tensor field on M. Then we get

$$e(\phi) = m/2 \quad \text{and} \quad \|\phi^* h\|^2 = m, \tag{5.2}$$

and

$$e(\phi') = \tfrac{1}{2}\mathrm{Tr}_g(\phi'^* h) = \frac{m\mu}{2}, \tag{5.3}$$
$$\|\phi'^* h\|^2 = \mu^2 \|g\|^2 + \|t\|^2 = m\mu^2 + \|t\|^2.$$

Assume that $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$. Then, by (3.7), (3.8), (5.2) and (5.3), we obtain

$$\int_M \mu v_g = \mathrm{Vol}(M, g) \tag{5.4}$$

and

$$d\int_M \mu^2 v_g + \int_M \|t\|^2 v_g = d\,\mathrm{Vol}(M, g), \tag{5.5}$$

where $d := (3n - 7)(m/2)^2 + m$ (which is positive since $n \geqq m \geqq 2$). We claim that (5.4) and (5.5) imply that $\mu = 1$ and $t = 0$. In fact, let $\{P_k\}_{k=0}^{\infty}$ be a complete orthonormal basis of the L^2-space of real valued functions on M with respect to the inner product $(f_1, f_2) = \int_M f_1 f_2 v_g$, with $P_0 = \mathrm{Vol}(M, g)^{-1/2}$. Let

$$\mu = \sum_{k=0}^{\infty} a_k p_k$$

be the Fourier expansion of μ relative to $\{P_k\}_{k=0}^{\infty}$. Then (5.4), (5.5) imply

$$a_0 = P_0^{-1}, \quad d\sum_{k=0}^{\infty} a_k^2 + \int_M \|t\|^2 v_g = d\,\mathrm{Vol}(M, g),$$

respectively. These equalities yield $a_k = 0$ for $k = 1, 2, \ldots,$ and $\int_M \|t\|^2 v_g = 0$. Therefore we obtain $\phi'^* h = g$.

COROLLARY 5.3. *Assume that (M^2, g) is the 2-sphere of constant curvature. Then every full harmonic map of (M^2, g) into the standard sphere (S^{2n}, h), with the same spectrum of the Jacobi operator as a full isometric minimal immersion ϕ of (M^2, g) into (S^{2n}, h) coincides with ϕ up to an isometry of (S^{2n}, h).*

The proof follows immediately from Theorem 5.1 and Calabi's rigidity theorem of full isometric minimal immersions of 2-sphere into spheres (cf. [5], [6]).

6. Isometric minimal immersions into complex projective spaces

In this section we first state some results of characterizing holomorphic isometric immersions into complex projective spaces by the spectra of the Jacobi operator. Secondly, we characterize both holomorphic and totally real immersions.

THEOREM 6.1. *Let (M^{2m}, g) be a compact Kaehler manifold whose scalar curvature is constant, and (N, h) be a complex n-dimensional Kaehler manifold with non-zero constant holomorphic sectional curvature. Let ϕ, ϕ' be holomorphic or anti-holomorphic weakly conformal maps of (M^{2m}, g) into (N, h). Suppose that* $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$. *If ϕ is an isometric immersion or an isometry, then so is ϕ'.*

Here a map ϕ of (M, g) into (N, h) is said to be *weakly conformal* if $\phi^*h = \mu g$ where μ is a (not necessarily positive) smooth function on M.

COROLLARY 6.2. *Let ϕ, ϕ' be two harmonic maps of the standard 2-sphere (S^2, can) into the complex n-dimensional projective space $(P^n(c), h)$ with the Fubini-Study metric h. Suppose that* $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$. *If ϕ is a holomorphic isometric immersion, then $\phi' = \phi$ up to an isometry of $(P^n(c), h)$.*

Proof of Theorem 6.1. Since ϕ, ϕ' are holomorphic or anti-holomorphic, we have $\|\phi^*\Phi\| = \|\phi^*h\|$ and $\|\phi'^*\Phi\| = \|\phi'^*h\|$. Therefore by Corollary 3.4, the assumption $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$ yields

$$E(\phi) = E(\phi') \tag{6.1}$$

and

$$\int_M \left\{(6n + 10)e(\phi)^2 + (1 - n)\|\phi^*h\|^2\right\} v_g = \int_M \left\{(6n + 10)e(\phi')^2 + (1 - n)\|\phi'^*h\|^2\right\} v_g. \tag{6.2}$$

Moreover since $\phi'h = g$ and $\phi'^*h = \mu g$, where μ is a smooth function on M, we have

$$e(\phi) = m \quad \text{and} \quad \|\phi^*h\|^2 = 2m, \tag{6.3}$$

$$e(\phi') = m\mu \quad \text{and} \quad \|\phi'^*h\|^2 = 2m\mu^2. \tag{6.4}$$

Here m is the complex dimension of M. Therefore by (6.1)–(6.4), we obtain

$$\int_M \mu v_g = \mathrm{Vol}(M, g) \quad \text{and} \quad \int_M \mu^2 v_g = \mathrm{Vol}(M, g),$$

which yields $\mu = 1$ by the same method as in the proof of Theorem 5.1.

Proof of Corollary 6.2. Since ϕ is a holomorphic map of (S^2, can) into $P^n(c)$, $\mathrm{Index}(\phi) = 0$ [8], [9]. The assumption $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$ yields $\mathrm{Index}(\phi') = 0$. Hence ϕ' is also holomorphic or anti-holomorphic [25], [9, p.

60]. Moreover, since every harmonic map of (S^2, can) into $P^n(c)$ is weakly conformal [19, Theorem 2.8], due to Theorem 6.1, ϕ' is either a holomorphic (or anti-holomorphic) isometric immersion, or a holomorphic (or anti-holomorphic) isometry. Then there exists an isometry ρ of $(P^n(c), h)$ such that $\rho \circ \phi'$ is a holomorphic isometric immersion. Then ϕ and $\rho \circ \phi'$ can be expressed as $\phi = j_1 \circ \phi_1$ and $\rho \circ \phi' = j_2 \circ \phi_2$, where the ϕ_i $(i = 1, 2)$ are full holomorphic isometric immersions of (S^2, can) into $P^{n_i}(c)$, $n_i \leqq n$, and j_i, $i = 1, 2$, are the inclusions of $P^{n_i}(c)$. Due to Calabi's rigidity theorem [4, p. 18, Theorem 9], we obtain $n_1 = n_2$ and $\phi_1 = \phi_2$ up to an isometry of $P^{n_1}(c)$. We obtain the conclusion.

Next we show:

THEOREM 6.3. *Let ϕ, ϕ' be isometric minimal immersions of a compact Riemannian manifold (M, g) into the complex projective space $(P^n(c), h)$ with the Fubini-Study metric h. Assume that* $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$.

(i) *If ϕ is totally real, then so is ϕ'.*

(ii) *If ϕ is holomorphic with respect to a complex structure of M making g a Kaehler metric, then ϕ' is holomorphic with respect to a suitable complex structure of M.*

Here the immersion ϕ is *totally real* if $h(\phi_* X, J\phi_* Y) = 0$ for all vector fields X, Y on M.

To prove Theorem 6.3, we need the following lemma.

LEMMA 6.4. *Let (N, h, J) be a hermitian manifold with the Kaehler form Φ. Let ϕ be an isometric immersion of a compact Riemannian manifold (M, g) into (N, h). Then we have the inequality*

$$0 \leqq \int_M \|\phi^* \Phi\|^2 v_g \leqq \dim(M) \mathrm{Vol}(M, g).$$

Moreover:

(i) *The equality*

$$\int_M \|\phi^* \Phi\|^2 v_g = 0$$

holds if and only if the immersion ϕ; $(M, g) \to (N, h, J)$ is totally real.

(ii) *The equality*

$$\int_M \|\phi^* \Phi\|^2 v_g = \dim(M) \mathrm{Vol}(M, g)$$

holds if and only if there exists a complex structure on M for which ϕ is holomorphic.

Proof of Theorem 6.3. Since ϕ and ϕ' are isometric immersions, we obtain

$$e(\phi) = e(\phi') = \dim(M)/2 \quad \text{and} \quad \|\phi^* h\|^2 = \|\phi'^* h\|^2 = \dim(M).$$

Then, by Corollary 3.4, the condition $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$ yields the equality

$$\int_M \|\phi^*\Phi\|^2 v_g = \int_M \|\phi'^*\Phi\|^2 v_g.$$

Therefore by Lemma 6.4, we have Theorem 6.3.

Proof of Lemma 6.4. To prove the inequality in Lemma 6.4, we only have to prove

$$0 \leqq \|\phi^*\Phi\|^2 \leqq \dim(M),$$

at each point of M. We take an orthonormal basis $\{e_i;\ i = 1, \ldots, m\}$ of the tangent space $T_x M$, $x \in M$, $\dim(M) = m$. Then we get

$$\begin{aligned} \|\phi^*\Phi\|^2 &= \sum_{i,j=1}^{m} h(\phi_* e_i, J\phi_* e_j)^2 \\ &= \sum_{j=1}^{m} h(PJ\phi_* e_j, J\phi_* e_j) \\ &= \sum_{j=1}^{m} h(PJ\phi_* e_j, PJ\phi_* e_j), \end{aligned}$$

where P is the orthogonal projection of $T_{\phi(x)}N$ onto $\phi_* T_x M$ with respect to h. Since $\{\phi_* e_j;\ j = 1, \ldots, m\}$ is an orthonormal basis of $\phi_* T_x M$, we obtain the inequality

$$0 \leqq \|\phi^*\Phi\|^2 \leqq \sum_{j=1}^{m} h(J\phi_* e_j, J\phi_* e_j) = m.$$

If $\int_M \|\phi^*\Phi\|^2 v_g = 0$, we get

$$\begin{aligned} \|\phi^*\Phi\|^2 = 0 &\Leftrightarrow PJ\phi_* e_j = 0, \quad j = 1, \ldots, m \\ &\Leftrightarrow h(\phi_* X, J\phi_* Y) = 0 \quad \text{for all vector fields } X, Y \text{ on } M \\ &\Leftrightarrow \phi \text{ is totally real.} \end{aligned}$$

If $\int_M \|\phi^*\Phi\|^2 v_g = \dim(M)\mathrm{Vol}(M, g)$, we get

$$\begin{aligned} \|\phi^*\Phi\|^2 = \dim(M) &\Leftrightarrow PJ\phi_* e_j = J\phi_* e_j, \quad j = 1, \ldots, m, \\ &\Leftrightarrow J\phi_* T_x M \subset \phi_* T_x M, \quad \text{at each point } x \text{ in } M, \end{aligned}$$

which holds if and only if there is a complex structure on M with respect to which ϕ is holomorphic.

7. Harmonic morphisms and harmonic Riemannian submersions

In this section, we study spectral characterization of harmonic Riemannian submersions among the set of all harmonic morphisms.

A smooth map ϕ of M into N is a *harmonic morphism* (cf. [10], [16]) if for every harmonic function ν on an open subset U in N, $\nu \circ \phi$ is a harmonic function on $\phi^{-1}(U)$. Then Fuglede [10] and Ishihara [16] showed that:

THEOREM 7.1 [10], [16]. (i) *If* $\dim(M) < \dim(N)$, *every harmonic morphism is constant.*

(ii) *If* $\dim(M) \geqq \dim(N)$, *a smooth map* ϕ; $(M, g) \to (N, h)$ *is a harmonic morphism if and only if* ϕ *is semi-conformal and harmonic.*

Here a smooth map ϕ; $(M, g) \to (N, h)$ is *semi-conformal* if (i) the differential

$$\phi_{*x};\ T_xM \to T_{\phi(x)}N$$

is surjective at the point x with $e(\phi)(x) \neq 0$, and (ii) there exists a smooth function λ on M such that if $e(\phi)(x) \neq 0$, the pull back ϕ^*h satisfies

$$\phi^*h(X, Y) = \lambda^2(x)g(X, Y)$$

for all $X, Y \in H_x$, where H_x is the orthogonal complement of the kernel of the differential ϕ_{*x} with respect to g_x, $x \in M$. It is known (cf. [10]) that the set $\{x \in M;\ e(\phi)(x) \neq 0\}$ is open and dense in M and the function λ^2 is given by

$$\lambda^2 = 2e(\phi)\dim(N)^{-1},$$

and $\|\phi^*h\|^2 = \dim(N)\lambda^4$. A smooth map ϕ; $(M, g) \to (N, h)$ is a *Riemannian submersion* if it is semi-conformal with $\lambda = 1$, i.e., $e(\phi) = \dim(N)/2$, everywhere M.

Now we prove the following:

THEOREM 7.2. *Let* (M, g) *be a compact Riemannian manifold whose scalar curvature is constant, and let* (N, h) *be either the standard n-sphere* (S^n, can) *or the complex projective space* $(P^n(c), h)$ *with the Fubini-Study metric h. Let* ϕ, ϕ' *be harmonic morphisms of* (M, g) *into* (N, h) *with* $\operatorname{Spec}(J_\phi) = \operatorname{Spec}(J_{\phi'})$. *If* ϕ *is a Riemannian submersion, then so is* ϕ'.

Proof. We only have to show that the function λ^2 for ϕ' satisfies $\lambda^2 = 1$ everywhere M.

Case 1. $(N, h) = (S^n, \mathrm{can})$. In this case, $e(\phi') = 2^{-1}n\lambda^2$, $\|\phi'^*h\|^2 = n\lambda^4$, $e(\phi) = 2^{-1}n$, and $\|\phi^*h\|^2 = n$. By Corollary 3.2, the assumption $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$ yields

$$E(\phi') = E(\phi) \tag{1}$$

and

$$\int_M \{(3n-7)e(\phi')^2 + \|\phi'^*h\|^2\} v_g = \int_M \{(3n-7)e(\phi)^2 + \|\phi^*h\|^2\} v_g \tag{2}$$

Condition (1) is equivalent to $\int_M \lambda^2 v_g = \int_M v_g$ and condition (2) is equivalent to $\int_M \lambda^4 v_g = \int_M v_g$. Therefore we get $\lambda^2 = 1$ everywhere M by the Cauchy-Schwarz inequality.

Case 2. $(N, h) = (P^n(c), h)$. We first show that if ϕ is a harmonic morphism of (M, g) into $(P^n(c), h)$,

$$\|\phi^*\Phi\| = \|\phi^*h\| \quad \text{on } \{x \in M;\ e(\phi)(x) \neq 0\}$$

where Φ is the Kaehler form of $(P^n(c), h)$. In fact, at each point x in M with $e(\phi)(x) \neq 0$, we can define a linear transformation $\tilde{J}$ of H_x into itself such that $J \circ \phi_* = \phi_* \circ \tilde{J}$ and $\tilde{J}^2 = -I$, where I is the identity and J is the complex structure of $P^n(c)$. Then $g(\tilde{J}X, \tilde{J}Y) = g(X, Y)$, and $g(\tilde{J}X, Y) = 0$, $X, Y \in H_x$. We can choose $\{e_i, \tilde{J}e_i;\ i = 1, \ldots, n\}$ as an orthonormal basis of (H_x, g_x). Then we get

$$\begin{aligned}
\|\phi^*\Phi\|^2 &= \sum_{i,j=1}^n \phi^*\Phi(e_i, e_j)^2 + 2\sum_{i,j=1}^n \phi^*\Phi\left(e_i, \tilde{J}e_j\right)^2 + \sum_{i,j=1}^n \phi^*\Phi\left(\tilde{J}e_i, \tilde{J}e_j\right)^2 \\
&= \sum_{i,j=1}^n h(\phi_*e_i, J\phi_*e_j)^2 + \sum_{i,j=1}^n h\left(\phi_*\tilde{J}e_i, J\phi_*\tilde{J}e_j\right)^2 \\
&\qquad + 2\sum_{i,j=1}^n h\left(\phi_*e_i, J\phi_*\tilde{J}e_j\right)^2 \\
&= \|\phi^*h\|^2.
\end{aligned}$$

Now let ϕ, ϕ' be harmonic morphisms of (M, g) into $(P^n(c), h)$ with $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$, and assume that ϕ is a Riemannian submersion. Then, by Corollary 3.4, we have

$$E(\phi') = E(\phi) \tag{3}$$

and

$$\int_M \left\{ (6n+10) e(\phi')^2 + (1-n) \|\phi'^* h\|^2 \right\} v_g = \int_M \left\{ (6n+10) e(\phi)^2 + (1-n) \|\phi^* h\|^2 \right\} v_g, \tag{4}$$

together with the above. Then by a similar argument to that in Case 1, we obtain Theorem 7.2.

Remark 7.3. A non-trivial harmonic morphism is a holomorphic map from a compact Kaehler manifold into a compact Riemann surface (N, h) with Hermitian metric h [9, Corollary 8.17].

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