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THE q-PARTS OF DEGREES OF BRAUER CHARACTERS OF SOLVABLE GROUPS[1]

BY

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0. Introduction

All groups considered are finite and p and q denote primes. Assume $q \neq p$ and every $\varphi \in IBr_p(G)$ has q'-degree. In [11], we showed that if G is p-solvable, then G is in fact q-solvable with metabelian Sylow-q-subgroups. While, in general, G may not be q-solvable (e.g., $PSL(2, p)$ with $q = 2$), it remains open whether a Sylow-q-subgroup of G is necessarily metabelian. In Section 1 below, we assume that $q^{e+1} \nmid \varphi(1)$ for all $\varphi \in IBr_p(G)$ and give, for solvable G, a linear bound for both the derived length of a Sylow-q-subgroup of G and the q-length of G. In fact, if $N \trianglelefteq G$ and $\mu \in IBr_p(N)$, we bound the derived length of a Sylow-q-subgroup of G/N in terms of the largest power of q dividing $\varphi(1)/\mu(1)$ as φ varies over $IBr_p(G|\mu)$, the irreducible Brauer characters of G lying over μ.

Assume that $p^{e+1} \nmid \varphi(1)$ for all $\varphi \in IBr_p(G)$. If G is p-solvable, we give a linear bound for the p-rank of $G/O_p(G)$ and a logarithmic bound for the p-length of $G/O_p(G)$, but give no bound for the derived length of a Sylow-p-subgroup of $G/O_p(G)$. The methods here are different than for $q \neq p$ and we show that we cannot derive these bounds "locally," i.e., relative to a character of a normal subgroup. In closing, we do improve known bounds for the derived length of a Sylow-p-subgroup of p-solvable groups in terms of the degrees of ordinary characters.

All groups considered are finite. We let $l_p(H)$ and $r_p(H)$ denote the p-length and p-rank (respectively) of a p-solvable group H, i.e., $r_p(H)$ is the largest integer r such that p^r is the order of a p-chief factor of H. Also $dl_p(G0$ denotes the derived length of a Sylow-p-subgroup of G.

Section 1. $q \neq p$

In this section, for solvable G, we bound $dl_q(G)$ in terms of the largest power of q that divides the degree of some irreducible Brauer character of G.

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1.1 LEMMA. *Assume a solvable group G acts faithfully and completely reducibly on an elementary abelian q-group V. Suppose that $q \nmid |G : C_G(x)|$ for all $x \in V$.*

(i) *If $q \geq 5$, then $dl_q(G) \leq 1$.*
(ii) *If $q \leq 3$, then $dl_q(G) \leq 2$.*

Proof. We may assume that $G = O^{q'}(G) \neq 1$. We may also assume that V is an irreducible, faithful G-module. If V_N is homogeneous for all characteristic subgroups N of G, Theorem 1.8 of [11] implies that $q^2 \nmid |G|$ or $O^q(G)$ is cyclic. In this case, $dl_q(G) = 1$. Choose $C \trianglelefteq G$ maximal with respect to V_C not homogeneous and let $V_1, \dots, V_n$ be the homogeneous components of V_C. By Lemma 1.2 of [11], $q \leq 3$ and $q^2 \nmid |G/C|$. In particular conclusion (i) holds. It suffices to show that a Sylow-q-subgroup of C is abelian. Note that $q \nmid |\mathbf{F}(C)|$, since $q = \mathrm{char}(V)$ and V_C is completely reducible and faithful. We may assume that C is not metabelian. Then by Corollary 1.3 of [12], $q = 3$, $|V_i| = 3^2$ or 3^4, and $C/C_C(V_i)$ acts irreducibly on V_i. By Theorem 1.8 of [13], a Sylow-3-subgroup of $C/C_C(V_i)$ has order at most 3^2 and hence is abelian. Since $\bigcap C_C(V_i) = 1$, a Sylow-3-subgroup of C is abelian. ∎

1.2 THEOREM. *Assume that G is solvable, $N \trianglelefteq G$, $\alpha \in IBr_p(G)$, $q \neq p$, and*

$$q \nmid \chi(1)/\alpha(1) \quad \textit{for all } \chi \in IBr_p(G|\alpha).$$

Then:

(i) *$dl_q(G/N)$ is at most 3.*
(ii) *If $q \geq 5$, $dl_q(G/N)$ is at most 2.*

Proof. We argue by induction on $|G : N|$. If $N \leq K \trianglelefteq G$ and $\tau \in IBr_p(K|\alpha)$, then $q \nmid \tau(1)/\alpha(1)$ and $q \nmid \chi(1)/\tau(1)$ for all $\chi \in IBr_p(G|\tau)$. Without loss of generality $O_{q'}(G/N) = 1$ and $O^{q'}(G/N) = G/N$. The hypothesis on character degrees and Clifford's Theorem imply that $I_G(\alpha)$ contains a Sylow-q-subgroup of G. We thus assume that $I_G(\alpha) = G$.

Let $M/N = O_q(G/N) > 1$. Now each $\sigma \in IBr_p(M|\alpha)$ extends α. In particular, each $\delta \in IBr_p(M/N)$ is linear and, as $q \neq p$, M/N is abelian. By Glauberman's Lemma [13.8 of 8], there exists $\varphi \in IBr_p(M|\alpha)$ such that $I_G(\varphi)$ contains a Hall-q'-subgroup of G. The hypotheses imply that φ is G-invariant. Now $\lambda \to \lambda\varphi$ defines a bijection from $IBr_p(M/N)$ onto $IBr_p(M|\alpha)$. Then $I_G(\lambda\varphi) = I_G(\lambda)$ has q'-index. Since $\mathrm{Irr}(M/N) = IBr_p(M/N)$, we have

$$q \nmid |G : I_G(\lambda)| \quad \text{for all } \lambda \in \mathrm{Irr}(M/N).$$

Since $O_{q'}(G/N) = 1$, it follows that $M/N = \mathbf{F}(G/N)$. Let $N = N_0 < N_1 \cdots < N_m = M$ be such that N_i/N_{i-1} is a chief factor in G. Let $C_i = C_G(N_i/N_{i-1}) \geq M$. Since $M/N = \mathbf{F}(G/N)$ and G/N is solvable, $\bigcap C_i = M$.

For each i, N_i/N_{i-1} and $\mathrm{Irr}(N_1/N_{i-1})$ are faithful irreducible G/C_i-modules. For $\beta \in \mathrm{Irr}(N_i/N_{i-1})$, β is the restriction to N_i of some $\lambda \in \mathrm{Irr}(M/N)$ and hence $q \nmid |G : I_G(\beta)|$. By Lemma 1.1, $dl_q(G/C_i) \leq 2$ and if $q \leq 5$, then $dl_q(G/C_i) \leq 1$. Since $\bigcap C_i = M$, $dl_q(G/M)$ is at most 2, and if $q \geq 5$, at most 1. Since M/N is abelian, the result follows. ■

1.3 COROLLARY. *Suppose that G is solvable, $N \trianglelefteq G$, $\alpha \in IBr_p(N)$, $q \neq p$, and*

$$q^{e+1} \nmid \chi(1)/\alpha(1) \quad \textit{for all } \chi \in IBr_p(G|\alpha).$$

Then:

(a) $dl_q(G/N) \leq 4e + 3$.
(b) *If $q \geq 5$, then $dl_q(G/N) \leq 3e + 2$.*

Proof. We prove part (a) by induction on e and note that the proof for (b) is similar. By Theorem 1.2, we may assume that $e \geq 1$, $dl_q(G/N) \geq 4$, and choose $N \leq K \trianglelefteq G$ and $\tau \in IBr_p(K|\alpha)$ such that $dl_q(K/N) = 4$ and $q | \tau(1)/\alpha(1)$. Since

$$q^e \nmid \beta(1)/\tau(1) \quad \text{for all } \beta \in IBr_p(G|\tau),$$

the inductive hypothesis implies that $dl_q(G/K) \leq 4(e - 1) + 3$. Hence

$$dl_q(G/N) \leq dl_q(G/K) + dl_q(K/N) \leq 4e + 3. \qquad ■$$

For q-solvable H, a result of Hall and Higman shows that $l_q(H) \leq dl_q(H)$ provided $q \neq 2$ (see [6, Theorem IX.5.4(b)]). For $q = 2$, Bryukhanova [1] has obtained the same inequality. We combine this with Corollary 1.3 to obtain Corollary 1.4.

1.4 COROLLARY. *Assume the hypotheses of Corollary* 1.3. *Then*:

(i) $l_q(G/N) \leq 3e + 2$ *if* $q \geq 5$.
(ii) $l_q(G/N) \leq 4e + 3$.

If we let $N = 1$ in the above corollaries, we have linear bounds for $dl_q(G)$ and $l_q(G)$ for solvable G in terms of e, where q^e is the largest power of q dividing the degree of an irreducible Brauer character of G. If we choose p not to divide $|G|$, then we have a bound for $dl_q(G)$ in terms of f, where q^f is the largest power of q dividing the degree of an ordinary irreducible character. However, a better bound $dl_q(G) \leq 2f + 1$ was given by Isaacs [7] for solvable G and extended to q-solvable G by Gluck and the second author [3]. We shall see in the next section (Corollary 2.7) that this can be further improved.

Furthermore the first author [10] bounded $l_q(G)$ for q-solvable G as a logarithmic function of f by methods similar to those of the next section.

Section 2. $q = p$

In this section, we give an upper bound for $l_p(G)$ in terms of the largest power of p that divides the degree of some $\chi \in IBr_p(G)$. The techniques of the last section fail here and we start by showing there is no analogue of Theorem 1.2.

2.1 *Example.* Let p be a prime. For each non-negative integer i, there exists a solvable group G_i whose center Z_i is a cyclic p'-group and a faithful $\lambda_i \in \mathrm{Irr}(Z_i)$ such that

(i) $IBr_p(G_i|\lambda_i) = \{\chi_i\}$ and $p \nmid \chi_i(1)$,
(ii) $l_p(G_i/Z_i) = i$,
(iii) $O_{p'}(G_i/Z_i) = 1$.

Note. Observe that $dl_p(G_i/Z_i)$ tends to infinity.

Proof. By induction on i. For $i = 0$, let $G_0 = 1$. Assume that G_i has been chosen as above. We construct G_{i+1}. Let $q \neq p$ be a prime with $(q, |G_i|) = 1$ and q odd. For a sufficiently large n, G_i/Z_i can be embedded into $GL(n, q)$. Since

$$A \to \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$$

embeds $GL(n, q)$ into $Sp(2n, q)$, G_i/Z_i may be embedded into $Sp(2n, q)$. Let Q be an extra-special q-group of exponent q and order q^{2n+1}. Then G_i/Z_i acts faithfully on both Q and $Q/Z(Q)$, while centralizing $Z(Q)$. Since $(|G_i|, q) = 1$, Fitting's lemma implies that

$$Q/Z(Q) = C_Q(G_i)/Z(Q) \times D/Z(Q) \quad \text{where } D/Z(Q) = [Q/Z(Q), G_i].$$

Since $(|G_i|, q) = 1$ and Q/Z is abelian, it is an easy consequence of the three subgroups lemmas applied to $[G_i, Z(D), Q]$ that $Z(Q) = Z(D)$ and hence that D is extra-special. We may assume without loss of generality that $D = Q > Z(Q)$ and $Z(Q) = C_Q(G_i)$.

Now let H be the semi-direct product $Q \rtimes G_i$. Let $Z_{i+1} = Z(Q) \times Z_i = Z(H)$. Since $q \nmid |G_i|$, Z_{i+1} is a cyclic p'-group. Let $\lambda \in \mathrm{Irr}(Z(Q))$ be faithful and set

$$\lambda_{i+1} = \lambda \times \lambda_i \in \mathrm{Irr}(Z_{i+1}).$$

Let θ be the unique irreducible constituent of λ^Q and set

$$\tau = \theta \times 1_{Z_i} \in \mathrm{Irr}(Q \times Z_i).$$

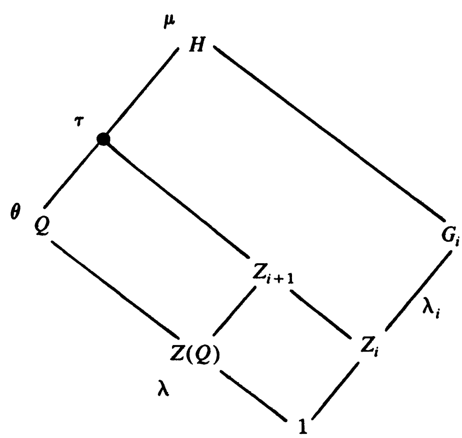
Since $(|H/QZ_i|, |QZ_i/\mathrm{ker}(\tau)|) = 1$ and τ is a H-invariant (ordinary and Brauer) character of $Q \times Z_i$, τ extends to $\mu \in IBr_p(H)$. Since μ extends $\theta \in IBr_p(Q)$, the mapping $\alpha \to \alpha\mu$ is an injection from $IBr_p(H/Q)$ into $IBr_p(H|\theta)$, (see [6, Theorem VII.9.12]), and this mapping is onto by Isaacs [9, Corollary 7.3]. If $\eta \in IBr_p(H|\theta \times \lambda_i)$, then $\eta \in IBr_p(H|\theta)$ and so $\eta = \alpha\mu$ for some $\alpha \in IBr(H/Q)$. Since $Z_i \leq \mathrm{ker}(\mu)$, we have

$$\alpha \in IBr(H|1_Q \times \lambda_i).$$

Since $H/Q \simeq G_i$, it follows from the inductive hypothesis that

$$IBr_p(H|1_Q \times \lambda_i) = \{\beta\} \quad \text{and} \quad p \nmid \beta(1).$$

Since $q \neq p$, we have $IBr_p(H|\theta \times \lambda_i) = \{\eta\}$ and $p \nmid \eta(1)$. Since $\theta \times \lambda_i$ is the unique irreducible constituent of λ_{i+1} induced to $Q \times Z_i$, we have $IBr_p(H|\lambda_{i+1}) = \{\eta\}$.



Note that $O_{q'}(H/Z_{i+1}) = 1$, Z_{i+1} is a p'-group and $p \neq q$. Also

$$O_q(H/Z_{i+1}) = QZ_{i+1}/Z_{i+1}.$$

We may choose an elementary abelian p-group E such that H/Z_{i+1} acts faithfully on E and $C_E(Q) = 1 = C_E(H)$. Then let $G_{i+1} = E \rtimes H$ and observe that $Z(G_{i+1}) = Z(H) = Z_{i+1}$ is a cyclic p'-group. Also, $O_{p'}(G_{i+1}/Z_{i+1}) = 1$ and

$$l_p(G_{i+1}/Z_{i+1}) = 1 + l_p(H/Z_{i+1}) = 1 + l_p(G_i/Z_i) = i + 1.$$

Since E is a p-group, $\sigma \to \sigma_H$ defines a bijection from $IBr_p(G_{i+1})$ onto $IBr_p(H)$. Consequently the last paragraph implies that

$$IBr_p(G_{i+1}|\lambda_{i+1}) = \{\chi_{i+1}\} \quad \text{and} \quad p \nmid \chi_{i+1}(1). \qquad \blacksquare$$

2.2 LEMMA. *Assume that a p-group P acts faithfully on a finite vector space V such that* $p \neq \mathrm{char}(V)$. *Then*:

(i) *There exists* $v \in V$ *such that* $|C_p(V)| \leq |P|^{1/p}$.

(ii) *If p is not two, Fermat, nor Mersenne, there exists* $v \in V$ *such that* $C_p(v) = 1$.

Proof. Passman [12] proves (ii) and in general shows the existence of a vector v such that $|C_p(v)| \leq |P|^{1/2}$. At the end of the paper, it is commented that the same techniques show that v can be chosen so that $|C_p(v)| \leq |P|^{1/p}$. ■

2.3 THEOREM. *Let G be solvable and let r be the p-rank of* $G/O_p(G)$. *If* $p^{e+1} \nmid \theta(1)$ *for all* $\theta \in IBr_p(G)$, *then*

$$r \leq (p/(p-1))e.$$

Proof. By induction on $|G|$. Without loss of generality, $O_p(G) = 1$. Let M be a minimal normal subgroup of G and $N/M = O_p(G/M)$. By the inductive hypothesis, we may assume that $N/M \neq 1$. Since $O_p(G) = 1$, M is an elementary abelian q-group with $q \neq p$ and N/M acts faithfully on M and $\mathrm{Irr}(M)$. If $p^t = |N:M|$, we apply Lemma 2.2 to conclude there exists $\theta \in \mathrm{Irr}(M) = IBr_p(M)$ such that

$$|I_N(\theta)/M| \leq p^{t/p}.$$

Since $N \trianglelefteq G$, $e \geq t - t/p$ or equivalently $t \leq pe/(p-1)$. By the inductive hypothesis, the p-rank s of G/N does not exceed $pe/(p-1)$. Since $r \leq \max\{s, t\}$, the theorem follows. ■

Huppert [4] bounded the p-length of a p-solvable group as a logarithmic function of the p-rank. The following improvement, due to the second author [13], gives best bounds whenever p is odd and not a Fermat prime.

2.4 LEMMA. *Let G be p-solvable of p-length l and p-rank r. Then*:

(i) $l \leq 1 + \log_p(r)$ *if p is not Fermat.*

(ii) $l \leq 2 + \log_s(r/(p-1))$ *where* $s = p - 1 + (1/p)$.

Combining Theorem 2.3 and Lemma 2.4, we get a corollary.

2.5 COROLLARY. *Let G be solvable and l be the p-length of $G/O_p(G)$. If $p^{e+1} \nmid \theta(1)$ for all $\theta \in IBr_p(G)$, then*:

(i) $l \leq 1 + \log_p(pe/(p-1))$ *if p is not Fermat.*

(ii) $l \leq 2 + \log_s(pe/(p-1)^2)$ *where* $s = p - 1 + (1/p)$.

Some comments are appropriate at this point.

1. Theorem 2.3 and Corollary 2.5 remain valid if we place the same restriction on the degrees of $\theta \in \mathrm{Irr}(G)$, instead of Brauer characters. It should be clear that the proof is identical (although one could be heavy handed and note this follows via the Fong-Swan Theorem and the above results).

2. If G is solvable and p is not two, Fermat, nor Mersenne; then we may conclude in Theorem 2.3 and Corollary 2.5 that $r \leq e$ and $l \leq 1 + \log_p(e)$. See Lemma 2.2 and use the same proof.

3. The bounds in Lemma 2.3 and Corollary 2.5 may not be exact bounds, but are reasonable. For each odd prime p and positive integer l, it is possible to construct a solvable group G with $O_p(G) = 1$, $l_p(G) = l$, and p-rank r such that

$$l = 1 + \log_p(r) \quad \text{and} \quad r = \left(\frac{p-1}{p}\right)e - \frac{1}{p}.$$

This can be done using wreath products.

4. If $p^{e+1} \nmid \theta(1)$ for all irreducible Brauer characters of a p-solvable group G, is it possible to bound $dl_p(G/O_p(G))$ in terms of e? We finish by giving an analogue for classical characters.

2.6 LEMMA. *Let $N \trianglelefteq G$ with G p-solvable and let $\theta \in \mathrm{Irr}(N)$. Assume that*

$$p^{e+1} \nmid \chi(1)/\theta(1) \quad \textit{for all } \chi \in \mathrm{Irr}(G|\theta).$$

Then

$$dl_p(G/N) \leq e + l_p(G/N).$$

Proof. By induction on $|G/N|$. We may assume that $O_{p'}(G/N) = 1$. Let

$$M/N = O_p(G/N) \neq 1$$

and choose $\tau \in \mathrm{Irr}(M|\theta)$ with $\tau(1)/\theta(1)$ maximal, say $\tau(1)/\theta(1) = p^f$. Since

M/N is a p-group, Lemma 1.1 of [2] implies that $dl(M/N) \leq f + 1$. Since

$$p^{e-f+1} \nmid \chi(1)/\tau(1) \quad \text{for all } \chi \in \operatorname{Irr}(G|\tau),$$

the inductive hypothesis implies that $dl_p(G/M) \leq e - f + l_p(G/M)$. Then

$$dl_p(G/N) \leq dl_p(G/M) + dl(M/N) \leq e + l_p(G/M) + 1 \leq e + l_p(G/N).$$

■

2.7 COROLLARY. *Assume G is p-solvable and $p^{e+1} \nmid \chi(1)$ for all $\chi \in$* $\operatorname{Irr}(G)$. *Then*:

(i) $dl_p(G) \leq e + 3 + \log_p(4e)$;

(ii) *If G is solvable, then*

$$dl_p(G) \leq e + 3 + \log_s\left(\frac{pe}{(p-1)^2}\right)$$

where $s = p - 1 + (1/p)$;

(iii) *If G is solvable and p is not two, Fermat, nor Mersenne, then*

$$dl_p(G) \leq e + 2 + \log_p(e).$$

Proof. To prove (ii) and (iii), see comments (1) and (2) after Corollary 2.5 and apply Lemma 2.6. For (i), apply Lemma 2.6 and the main theorem of [10]. ■

Added in Proof. You-Qiang Wang (Ph.D. thesis, Ohio University) has just recently given an affirmative answer to the question posed in note 4 after Corollary 2.5.

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