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# COMPACT HANKEL OPERATORS ON THE BERGMAN SPACE

BY

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## 1. Introduction

Let $\mathbf{D} = \{z \in \mathbf{C}: |z| < 1\}$ denote the open unit disk in the complex plane $\mathbf{C}$, and let $A$ denote the usual Lebesgue area measure on $\mathbf{C}$. For $1 \leq p < \infty$ and $f$: $\mathbf{D} \to \mathbf{C}$ Lebesgue measurable let $\|f\|_p = (\int_{\mathbf{D}} |f|^p \, dA/\pi)^{1/p}$. The Bergman space $L_a^p(\mathbf{D})$ is the Banach space of analytic functions $f$: $\mathbf{D} \to \mathbf{C}$ such that $\|f\|_p < \infty$. The Bergman space $L_a^2(\mathbf{D})$ is a Hilbert space; it is a closed subspace of the Hilbert space $L^2(\mathbf{D}, dA/\pi)$ with inner product given by

$$\langle f, g \rangle = \int_{\mathbf{D}} f(z)\overline{g(z)} \, dA(z)/\pi,$$

for $f, g \in L^2(\mathbf{D}, dA/\pi)$. Let $P$ denote the orthogonal projection of $L^2(\mathbf{D}, dA/\pi)$ onto $L_a^2(\mathbf{D})$. The map $I - P$ is the orthogonal projection of $L^2(\mathbf{D}, dA/\pi)$ onto $L_a^2(\mathbf{D})^\perp$ (the orthogonal complement of $L_a^2(\mathbf{D})$ in $L^2(\mathbf{D}, dA/\pi)$). For a function $f \in L^\infty(\mathbf{D}, dA/\pi)$, the Hankel operator $H_f$: $L_a^2(\mathbf{D}) \to L_a^2(\mathbf{D})^\perp$ is defined by

$$H_f g = (I - P)(fg), \quad g \in L_a^2(\mathbf{D}).$$

It is clear that $H_f$ is a bounded operator for every function $f \in L^\infty(\mathbf{D}, dA/\pi)$. In [2], Sheldon Axler raised the question of finding necessary and sufficient conditions on the function $f \in L^\infty(\mathbf{D}, dA/\pi)$ for the Hankel operator $H_f$ to be compact. Sheldon Axler answered a special case of this problem in [3] where he considered conjugate analytic symbols. The "little Bloch" space $\mathscr{B}_0$ is the set of all analytic functions $f$ on $\mathbf{D}$ for which

$$(1 - |z|^2)f'(z) \to 0 \quad \text{as } |z| \to 1^-.$$

Axler proved that for a function $f$ in $L_a^2(\mathbf{D})$ (perhaps unbounded) the (densely defined) Hankel operator $H_{\bar{f}}$ is compact if and only if $f \in \mathscr{B}_0$. In [8], Kehe Zhu characterized the functions $f \in L^\infty(\mathbf{D}, dA/\pi)$ such that both Hankel operators $H_f$ and $H_{\bar{f}}$ are compact. In this paper we will characterize the

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functions $f \in L^\infty(\mathbf{D}, dA/\pi)$ for which the Hankel operator $H_f$ is compact, thus obtaining a complete answer to Sheldon Axler's problem raised in [2].

In our characterization of the compact Hankel operators on the Bergman space the Möbius functions on the disk play a crucial role. For $\lambda \in \mathbf{D}$ let the Möbius function $\varphi_\lambda$: $\mathbf{D} \to \mathbf{D}$ be defined by

$$\varphi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}, \quad z \in \mathbf{D}.$$

The main result of this paper is Theorem 6, which gives several necessary and sufficient conditions on a function $f \in L^\infty(\mathbf{D}, dA/\pi)$ for the Hankel operator $H_f$ to be compact; one of these conditions states that the Hankel operator $H_f$ is compact if and only if

$$\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2 \to 0 \quad \text{as } |\lambda| \to 1^-.$$

In Section 2 we will give the preliminaries needed for the rest of this paper. In Section 3 we will discuss how the Hankel operators behave when their symbols are composed with Möbius functions. We will obtain an explicit formula for the image of the reproducing kernels under Hankel operators. This formula will be used in Section 4, where we prove the main result. We end with a discussion of some open problems in Section 5.

I am grateful to Sheldon Axler for many helpful conversations. The basis for the work in this paper (Section 3) was part of my Ph.D. dissertation that I wrote at Michigan State University under his excellent guidance.

Dechao Zheng has informed me that he has also solved Axler's problem and independently obtained results similar to the ones in this paper.

## 2. Preliminaries

Point evaluation is a bounded linear functional on the Hilbert space $L^2_a(\mathbf{D})$, thus for every $\lambda \in \mathbf{D}$ there exists a unique function $k_\lambda \in L^2_a(\mathbf{D})$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle \quad \text{for all } f \in l^2_a(\mathbf{D}).$$

These functions $k_\lambda (\lambda \in \mathbf{D})$ are called the reproducing kernels for $L^2_a(\mathbf{D})$. It is easy to verify that for every $\lambda \in \mathbf{D}$ the reproducing kernel $k_\lambda$ is given by the formula

$$k_\lambda(z) = \frac{1}{(1 - \bar{\lambda} z)^2} \quad \text{for } z \in \mathbf{D}.$$

Because of the reproducing property of $k_\lambda$ we have $\langle k_\lambda, k_\lambda \rangle = k_\lambda(\lambda)$. Using the above formula for $k_\lambda$ it follows at once that $\|k_\lambda\|_2 = 1/(1 - |\lambda|^2)$. For

$g \in L^2(\mathbf{D}, dA/\pi)$ and $z \in \mathbf{D}$ we have $(Pg)(z) = \langle Pg, k_z \rangle = \langle g, k_z \rangle$, so we get the following formula for the projection $Pg$:

$$(Pg)(z) = \int_{\mathbf{D}} \frac{g(w)}{(1 - \bar{w}z)^2}\, dA(w)/\pi \quad \text{for } z \in \mathbf{D}. \tag{1}$$

For $f \in L^\infty(\mathbf{D}, dA/\pi)$ and $g \in L^2_a(\mathbf{D})$, using (1) for the product $fg$ and for $g = Pg$ we get the following formula for the Hankel operator $H_f$:

$$(H_f g)(z) = \int_{\mathbf{D}} \frac{f(z) - f(w)}{(1 - \bar{w}z)^2} g(w)\, dA(w)/\pi \quad \text{for } z \in \mathbf{D}. \tag{2}$$

For a function $f \in L^\infty(\mathbf{D}, dA/\pi)$, and a point $\lambda \in \mathbf{D}$ we will call $f \circ \varphi_\lambda - f(\lambda)$ a Möbius transform of $f$. In the next section we will see how a Hankel operator transforms if its symbol is replaced by one of its Möbius transforms. First we will need some properties of the Möbius functions $\varphi_\lambda$. The function $\varphi_\lambda$ is easily seen to be its own inverse under composition: $(\varphi_\lambda \circ \varphi_\lambda)(z) = z$ for all $z \in \mathbf{D}$. The following identity can be obtained by straightforward computation:

$$\frac{1 - \bar{u}\varphi_\lambda(z)}{1 - \bar{u}\lambda} = \frac{1 - \overline{\varphi_\lambda(u)}\, z}{1 - \bar{\lambda}z} \quad (u, \lambda, z \in \mathbf{D}). \tag{3}$$

The special case that $u = \lambda$ yields

$$(1 - \bar{\lambda}\varphi_\lambda(z))(1 - \bar{\lambda}z) = 1 - |\lambda|^2 \quad (\lambda, z \in \mathbf{D}). \tag{4}$$

If we substitute $u = \varphi_\lambda(z)$ in (3) and make use (4) we obtain the well-known identity:

$$1 - |\varphi_\lambda(z)|^2 = \frac{(1 - |\lambda|^2)(1 - |z|^2)}{|1 - \bar{\lambda}z|^2} \quad (\lambda, z \in \mathbf{D}). \tag{5}$$

For a point $\lambda \in \mathbf{D}$ and $0 < r < 1$ the pseudo-hyperbolic disk $D(\lambda, r)$ with pseudo-hyperbolic center $\lambda$ and pseudo-hyperbolic radius $r$ is defined by $D(\lambda, r) = \varphi_\lambda(r\mathbf{D})$. The pseudo-hyperbolic disk $D(\lambda, r)$ is also a euclidean disk: its euclidean center and euclidean radius are

$$(1 - r^2)\lambda/(1 - r^2|\lambda|^2) \quad \text{and} \quad (1 - |\lambda|^2)r/(1 - r^2|\lambda|^2),$$

respectively (see, for example, page 4 in [6]).

For a Lebesgue measurable set $K \subset \mathbf{D}$, let $|K|$ denote the measure of $K$ with respect to the normalized Lebesgue measure $A/\pi$. It follows immediately that:

$$|D(\lambda, r)| = \frac{(1 - |\lambda|^2)^2}{(1 - r^2|\lambda|^2)^2} r^2. \tag{6}$$

For $\lambda \in \mathbf{D}$, the substitution $z = \varphi_\lambda(w)$ results in the Jacobian change in measure given by

$$dA(z)/\pi = |\varphi_\lambda'(w)|^2 \, dA(w)/\pi.$$

For a Lebesgue integrable or a non-negative Lebesgue measurable function $h$ on $\mathbf{D}$ we have the following change-of-variable formula:

$$\int_{D(\lambda, r)} h(z) \, dA(z)/\pi = (1 - |\lambda|^2)^2 \int_{D(0, r)} (h \circ \varphi_\lambda)(w) \frac{1}{|1 - \bar{\lambda} w|^4} \, dA(w)/\pi. \tag{7}$$

## 3. Möbius-transformations of the symbol

In this section we will prove that a Hankel operator transforms in a unitarily equivalent way if its symbol is replaced by one of its Möbius transforms. As a corollary of the proof we obtain an explicit formula for the image of the reproducing kernels under a Hankel operator. This formula will play a crucial role in the proof of our characterization of the compact Hankel operators.

THEOREM 1. *Let $f \in L^\infty(\mathbf{D}, dA/\pi)$. For each $\lambda \in \mathbf{D}$ the Hankel operators $H_f$ and $H_{f \circ \varphi_\lambda}$ are unitarily equivalent.*
*More precisely, there exist unitary operators*

$$U_1 \colon L^2_a(\mathbf{D}) \to L^2_a(\mathbf{D}) \quad \text{and} \quad U_2 \colon L^2_a(\mathbf{D})^\perp \to L^2_a(\mathbf{D})^\perp$$

*such that*

$$U_2 \circ H_{f \circ \varphi_\lambda} = H_f \circ U_1.$$

*Proof.* Take $g \in L^2_a(\mathbf{D})$ and let $\lambda \in \mathbf{D}$. By (2) we have, for $z \in \mathbf{D}$,

$$(H_{f \circ \varphi_\lambda} g)(z) = \int_{\mathbf{D}} \frac{f(\varphi_\lambda(z)) - f(\varphi_\lambda(w))}{(1 - \bar{w}z)^2} g(w) \, dA(w)/\pi. \tag{8}$$

In (8) make the substitution $u = \varphi_\lambda(w)$. Making use of identity (3) we have

$$\frac{1}{\left(1-\overline{\varphi_\lambda(u)}\,z\right)^2}\frac{1}{|1-\bar{\lambda}u|^4}$$

$$= \frac{(1-\bar{u}\lambda)^2}{(1-\bar{\lambda}z)^2(1-\bar{u}\varphi_\lambda(z))^2}\frac{1}{|1-\bar{\lambda}u|^4}$$

$$= \frac{1}{(1-\bar{\lambda}z)^2}\frac{1}{(1-\bar{u}\varphi_\lambda(z))^2(1-\bar{\lambda}u)^2},$$

so that change-of-variable formula (7) transforms (8) into

$$\begin{aligned}\left(H_{f\circ\varphi_\lambda}g\right)(z) &= \frac{(1-|\lambda|^2)^2}{(1-\bar{\lambda}z)^2}\int_{\mathbf{D}}\frac{f(\varphi_\lambda(z))-f(u)}{(1-\bar{u}\varphi_\lambda(z))^2}\frac{1}{(1-\bar{\lambda}u)^2}g(\varphi_\lambda(u))\,dA(u)/\pi \\ &= (1-|\lambda|^2)k_\lambda(z)\int_{\mathbf{D}}\frac{f(\varphi_\lambda(z))-f(u)}{(1-\bar{u}\varphi_\lambda(z))^2}(1-|\lambda|^2)k_\lambda(u) \\ &\quad\times(g\circ\varphi_\lambda)(u)\,dA(u)/\pi \\ &= (1-|\lambda|^2)k_\lambda(z)H_f\left((1-|\lambda|^2)k_\lambda(g\circ\varphi_\lambda)\right)(\varphi_\lambda(z)).\end{aligned}$$

Thus we have

$$H_{f\circ\varphi_\lambda}g = (1-|\lambda|^2)k_\lambda H_f\left((1-|\lambda|^2)k_\lambda(g\circ\varphi_\lambda)\right)\circ\varphi_\lambda. \tag{9}$$

Define the operator $U\colon L^2(\mathbf{D}, dA/\pi) \to L^2(\mathbf{D}, dA/\pi)$ by

$$Ug = (1-|\lambda|^2)k_\lambda(g\circ\varphi_\lambda) \quad \text{for } g \in L^2(\mathbf{D}, dA/\pi).$$

Since $(1-|\lambda|^2)k_\lambda = -\varphi'_\lambda$, we have for $g \in L^2(\mathbf{D}, dA/\pi)$,

$$\|Ug\|_2^2 = \int_{\mathbf{D}}|(g\circ\varphi_\lambda)(z)|^2|\varphi'_\lambda(z)|^2\,dA(z)/\pi = \|g\|_2^2,$$

so that $U$ is well-defined. For $g, h \in L^2(\mathbf{D}, dA/\pi)$ we have

$$\langle Ug, h\rangle = \int_{\mathbf{D}}(1-|\lambda|^2)k_\lambda(z)g(\varphi_\lambda(z))\overline{h(z)}\,dA(z)/\pi.$$

In the above integral make the substitution $u = \varphi_\lambda(z)$. We get

$$\langle Ug, h\rangle = \int_{\mathbf{D}} (1 - |\lambda|^2) k_\lambda(\varphi_\lambda(u)) g(u) \overline{h(\varphi_\lambda(u))}\, |\varphi'_\lambda(u)|^2 \, dA(u)/\pi.$$

Now using identity (4) it is easy to verify that $k_\lambda(\varphi_\lambda(u))|\varphi'_\lambda(u)|^2 = k_\lambda(u)$, so that

$$\langle Ug, h\rangle = \int_{\mathbf{D}} g(u) \overline{(1 - |\lambda|^2) k_\lambda(u) h(\varphi_\lambda(u))}\, dA(u)/\pi = \langle g, Uh\rangle.$$

Hence $U$ is a self-adjoint operator on $L^2(\mathbf{D}, dA/\pi)$.

Take $g \in L^2(\mathbf{D}, dA/\pi)$ and put $h = Ug$. Differentiating the identity $\varphi_\lambda(\varphi_\lambda(z)) = z$ we see that for each $z \in \mathbf{D}$, $(1 - |\lambda|^2)^2 k_\lambda(z) k_\lambda(\varphi_\lambda(z)) = 1$, so that

$$(Uh)(z) = (1 - |\lambda|^2)^2 k_\lambda(z) k_\lambda(\varphi_\lambda(z)) g(z) = g(z),$$

and thus $U \circ U = I$. Hence $U$ is a unitary operator on $L^2(\mathbf{D}, dA/\pi)$.

Observe that $U(L^2_a(\mathbf{D})) \subset L^2_a(\mathbf{D})$ and $U(L^2_a(\mathbf{D})^\perp) \subset L^2_a(\mathbf{D})^\perp$. The first of these inclusions is obvious from the definition of $U$. The second inclusion follows from the first since the operator $U$ is self-adjoint. Let

$$U_1 \colon L^2_a(\mathbf{D}) \to L^2_a(\mathbf{D}) \quad \text{and} \quad U_2 \colon L^2_a(\mathbf{D})^\perp \to L^2_a(\mathbf{D})^\perp$$

be the restrictions of $U$ to $L^2_a(\mathbf{D})$ and $L^2_a(\mathbf{D})^\perp$ respectively. Then both $U_1$ and $U_2$ are unitary operators. We claim that

$$U_2 \circ H_{f \circ \varphi_\lambda} = H_f \circ U_1.$$

Let $g \in L^2_a(\mathbf{D})$, then it follows from (9) that

$$H_{f \circ \varphi_\lambda} g = (1 - |\lambda|^2) k_\lambda (H_f \circ U_1)(g) \circ \varphi_\lambda,$$

so that

$$\begin{aligned}(U_2 \circ H_{f \circ \varphi_\lambda})(g) &= (1 - |\lambda|^2) k_\lambda \big((H_{f \circ \varphi_\lambda} g) \circ \varphi_\lambda\big) \\ &= (1 - |\lambda|^2)^2 k_\lambda (k_\lambda \circ \varphi_\lambda)(H_f \circ U_1)(g) \\ &= (H_f \circ U_1)(g),\end{aligned}$$

and our claim is verified. This completes the proof of Theorem 1. □

The following proposition, a corollary of the proof of Theorem 1, gives a formula for the image under $H_f$ of the reproducing kernels $k_\lambda$ for $\lambda \in \mathbf{D}$. This formula will play an important role in the proof of our characterization of the compact Hankel operators.

PROPOSITION 2. *Let $f \in L^\infty(\mathbf{D}, dA/\pi)$. For each $\lambda \in \mathbf{D}$ we have*

$$H_f(k_\lambda) = (f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda) k_\lambda. \tag{10}$$

*Proof.* Since $(1 - |\lambda|^2)^2 k_\lambda (k_\lambda \circ \varphi_\lambda) = 1$, it follows from (9) that

$$H_{f \circ \varphi_\lambda}(k_\lambda) = k_\lambda H_f(1) \circ \varphi_\lambda = k_\lambda (f \circ \varphi_\lambda - P(f) \circ \varphi_\lambda).$$

Replacing $f$ by $f \circ \varphi_\lambda$ we get formula (10). □

## 4. Compact Hankel operators

In this section we will state and prove our main result, Theorem 6. To show that the operator $H_f$ is compact we will actually consider the operator $H_f^* H_f$. The following proposition gives a convenient way to represent this operator.

PROPOSITION 3. *Let $f \in L^\infty(\mathbf{D}, dA/\pi)$. Then for $h \in H^\infty(\mathbf{D})$ and $\lambda \in \mathbf{D}$,*

$$(H_f^* H_f h)(\lambda) = \int_{\mathbf{D}} \frac{|f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^2}{(1 - \lambda \bar{z})^2} h(z)\, dA(z)/\pi.$$

*Proof.* Let $f \in L^\infty(\mathbf{D}, dA/\pi$, $h \in H^\infty(\mathbf{D})$ and fix a point $\lambda \in \mathbf{D}$. Then

$$\begin{aligned} (H_f^* H_f h)(\lambda) &= \langle H_f^* H_f h, k_\lambda \rangle \\ &= \langle fh - P(fh), H_f k_\lambda \rangle \\ &= \langle fh, H_f k_\lambda \rangle \quad \big(\text{since } P(fh) \perp H_f k_\lambda\big). \end{aligned}$$

Now, $P(f \circ \varphi_\lambda) \circ \varphi_\lambda \in L_a^2(\mathbf{D})$, thus $(P(f \circ \varphi_\lambda) \circ \varphi_\lambda) h \in L_a^2(\mathbf{D})$, so that we have

$$\langle (P(f \circ \varphi_\lambda) \circ \varphi_\lambda) h, H_f k_\lambda \rangle = 0.$$

Using this we get

$$\begin{aligned} (H_f^* H_f h)(\lambda) &= \langle fh, H_f k_\lambda \rangle \\ &= \langle (f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda) h, H_f k_\lambda \rangle \\ &= \langle (f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda) h, (f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda) k_\lambda \rangle \quad \text{(by (10))} \\ &= \int_{\mathbf{D}} \frac{|f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^2}{(1 - \lambda \bar{z})^2} h(z)\, dA(z)/\pi. \end{aligned}$$

□

*Remark.* The formula in Proposition 3 also holds for $h \in L_a^2(\mathbf{D})$: it can be shown that the operator given by the integral in Proposition 3 is bounded, and thus agrees with $H_f^* H_f$ on $L_a^2(\mathbf{D})$.

The following lemma will be used in the proof of Lemma 5. For an elementary proof we refer the reader to [3].

LEMMA 4.

$$\sup_{\lambda \in \mathbf{D}} \int_{\mathbf{D}} \frac{1}{|1 - \lambda \bar{w}|^{6/5} (1 - |w|^2)^{3/5}} \, dA(w)/\pi < \infty.$$

In the proof of Theorem 6 we will use the following estimate.

LEMMA 5. *Let $f \in L^\infty(\mathbf{D}, dA/\pi)$. Then there exists a finite positive constant $C$ (depending on $f$) such that for every $\lambda \in \mathbf{D}$,*

$$\int_{\mathbf{D}} \frac{|f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^4}{|1 - \lambda \bar{z}|^2 (1 - |z|^2)^{1/2}} \, dA(z)/\pi \le \frac{C}{(1 - |\lambda|^2)^{1/2}}.$$

*Proof.* Let $f \in L^\infty(\mathbf{D}, dA/\pi)$. In the integral at the left make the change-of-variable $w = \varphi_\lambda(z)$. Using (5) and (7) we get

$$\begin{aligned} &\int_{\mathbf{D}} \frac{|f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^4}{|1 - \lambda \bar{z}|^2 (1 - |z|^2)^{1/2}} \, dA(z)/\pi \\ &\quad = \frac{1}{(1 - |\lambda|^2)^{1/2}} \\ &\quad \times \int_{\mathbf{D}} |(f \circ \varphi_\lambda)(w) - P(f \circ \varphi_\lambda)(w)|^4 \frac{1}{|1 - \lambda \bar{w}|(1 - |w|^2)^{1/2}} \, dA(w)/\pi. \end{aligned}$$

Let $M$ denote the quantity of Lemma 4. Applying Hölder's inequality using conjugate indices 6 and 6/5 we see

$$\begin{aligned} &\int_{\mathbf{D}} \frac{|f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^4}{|1 - \lambda \bar{z}|^2 (1 - |z|^2)^{1/2}} \, dA(z)/\pi \\ &\quad \le \frac{M^{5/6}}{(1 - |\lambda|^2)^{1/2}} \left( \int_{\mathbf{D}} |f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)|^{24} \, dA/\pi \right)^{1/6} \\ &\quad \le \frac{C}{(1 - |\lambda|^2)^{1/2}}, \end{aligned}$$

since the Bergman projection $P$ maps $L^{24}(\mathbf{D}, dA/\pi)$ boundedly into $L_a^{24}(\mathbf{D})$ (for an elementary proof see [4]), and $\|f \circ \varphi_\lambda\|_{24} \le \|f\|_\infty$ for all $\lambda \in \mathbf{D}$. □

Now we are ready to prove the main result of this paper.

THEOREM 6. *Let* $f \in L^\infty(\mathbf{D}, dA/\pi)$ *and* $1 < p < \infty$. *The following statements are equivalent*:

(a) $H_f$ *is compact*;

(b) $\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_p \to 0$ *as* $|\lambda| \to 1^-$;

(c) $\frac{1}{|D(\lambda, r)|} \int_{D(\lambda, r)} |f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda|^p \, dA/\pi \to 0$ *as* $|\lambda| \to 1^-$ *for all* $r \in (0,1)$.

*Proof.* (a) ⇒ (b) Suppose that $H_f$ is compact. It is well known that $(1 - |\lambda|^2)k_\lambda \to 0$ weakly in $L_a^2(\mathbf{D})$ as $|\lambda| \to 1^-$ (for a proof see [3]). The compactness of $H_f$ implies that

$$(1 - |\lambda|^2)\|H_f k_\lambda\|_2 \to 0 \quad \text{as } |\lambda| \to 1^-.$$

Using Proposition 2 and change-of-variable formula (7) we get

$$\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2 = (1 - |\lambda|^2)\|H_f k_\lambda\|_2 \to 0 \quad \text{as } |\lambda| \to 1^-.$$

For $1 < p < \infty$, and application of Hölder's inequality yields the inequality

$$\begin{aligned}&\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_p^p \\ &\quad \le \|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2 \left(\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_{2p-2}\right)^{p-1}.\end{aligned}$$

Using the boundedness of $P$ we have

$$\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_{2p-2} \le C_{2p-2}\|f \circ \varphi_\lambda\|_{2p-2} \le C_{2p-2}\|f\|_\infty.$$

Thus

$$\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_p \to 0 \quad \text{as } |\lambda| \to 1^-.$$

(b) ⇒ (c) Suppose that (b) holds, and let $r \in (0,1)$. Using change-of-variable formula (7) and formula (6) it is easy to verify that

$$\begin{aligned}&\frac{1}{|D(\lambda, r)|} \int_{D(\lambda, r)} |f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda|^p \, dA/\pi \\ &\quad \le \frac{4}{r^2(1 - r)^2} \|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_p^p, \qquad (11)\end{aligned}$$

so that (c) follows.

(c) ⇒ (a) The proof of this implication is divided into several steps. To show that $H_f$ is compact it suffices to show that $H_f^*H_f$ is compact. We will do this by defining Hilbert-Schmidt operators $S_\rho$ $(0 < \rho < 1)$ for which we will then show that $S_\rho \to H_f^*H_f$ in operator norm as $\rho \to 1^-$.

*Step* 1. Suppose that (c) holds for some $p \in (1, \infty)$. To prove (a) we will need (c) for $p = 4$. Let $q = p/(p-1)$ be the conjugate index of $p$. Hölder's inequality, (11) and the fact that

$$\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_{3q} \le C_{3q}\|f\|_\infty$$

give the inequality

$$\frac{1}{|D(\lambda, r)|}\int_{D(\lambda, r)} |f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda|^4 \, dA/\pi$$

$$\le C_{p,r}\|f\|_\infty^3 \left( \frac{1}{|D(\lambda, r)|}\int_{D(\lambda, r)} |f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda|^p \, dA/\pi \right)^{1/p},$$

and our claim that (c) holds for $p = 4$ follows.

*Step* 2. Let $0 < \rho < 1$. Define the operator $S_\rho$: $L_a^2(\mathbf{D}) \to L^2(\mathbf{D}, dA/\pi)$ by

$$(S_\rho h)(\lambda) = \chi_{\rho\mathbf{D}}(\lambda)\int_{\mathbf{D}} \frac{|f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^2}{(1 - \lambda\bar{z})^2} h(z) \, dA(z)/\pi,$$

for $h \in L_a^2(\mathbf{D})$, $\lambda \in \mathbf{D}$. We claim that $S_\rho$ is a Hilbert-Schmidt operator. To prove this claim we need to show that the kernel of $S_\rho$ is square-integrable over $\mathbf{D} \times \mathbf{D}$. Using Fubini's Theorem we have

$$\int_{\mathbf{D}}\left( \int_{\mathbf{D}} \chi_{\rho\mathbf{D}}(\lambda) \frac{|f(z) - P(f \circ \varphi_\lambda)(\varphi_\lambda(z))|^4}{|1 - \lambda\bar{z}|^4} \, dA(z)/\pi \right) dA(\lambda)/\pi$$

$$= \int_{\rho\mathbf{D}} \frac{1}{(1 - |\lambda|^2)^2}\left( \int_{\mathbf{D}} |(f \circ \varphi_\lambda)(w) - P(f \circ \varphi_\lambda)(w)|^4 \, dA(w)/\pi \right) dA(\lambda)/\pi$$

(by (7))

$$\le \frac{\rho^2}{1 - \rho^2} \sup_{\lambda \in \rho\mathbf{D}} \|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_4^4 < \infty,$$

and our claim that $S_\rho$ is Hilbert-Schmidt is verified.

*Step* 3. Now let $0 < r < 1$. Using Proposition 3 and the definition of $S_\rho$ we see that for $h \in H^\infty(\mathbf{D})$ and $\lambda \in \mathbf{D}$,

$$\big((H_f^* H_f - S_\rho)h\big)(\lambda)$$
$$= \chi_{\mathbf{D}\setminus\rho\mathbf{D}}(\lambda)\int_{\mathbf{D}} \frac{|f(z) - P(f\circ\varphi_\lambda)(\varphi_\lambda(z))|^2}{(1-\lambda\bar{z})^2} h(z)\, dA(z)/\pi.$$

By Minkowski's inequality,

$$\|(H_f^* H_f - S_\rho)h\|_2$$
$$\leq \left[\int_{\mathbf{D}\setminus\rho\mathbf{D}}\left(\int_{D(\lambda,r)} \frac{|f(z) - P(f\circ\varphi_\lambda)(\varphi_\lambda(z))|^2}{|1-\lambda\bar{z}|^2}|h(z)|\, dA(z)/\pi\right)^2 dA(\lambda)/\pi\right]^{1/2}$$
$$+\left[\int_{\mathbf{D}\setminus\rho\mathbf{D}}\left(\int_{\mathbf{D}\setminus D(\lambda,r)} \frac{|f(z) - P(f\circ\varphi_\lambda)(\varphi_\lambda(z))|^2}{|1-\lambda\bar{z}|^2}|h(z)|\, dA(z)/\pi\right)^2 dA(\lambda)/\pi\right]^{1/2} \tag{12}$$

We will estimate the two expressions at the right hand side of (12) separately. This will be done in steps 4 and 5 respectively.

*Step* 4. To save some writing we introduce the notation

$$I(\lambda, r) = \frac{1}{|D(\lambda,r)|}\int_{D(\lambda,r)} |f - P(f\circ\varphi_\lambda)\circ\varphi_\lambda|^4\, dA/\pi.$$

By Cauchy-Schwarz

$$\left(\int_{D(\lambda,r)} \frac{|f(z) - P(f\circ\varphi_\lambda)(\varphi_\lambda(z))|^2}{|1-\lambda\bar{z}|^2}|h(z)|\, dA(z)/\pi\right)^2$$
$$\leq \int_{D(\lambda,r)} |f - P(f\circ\varphi_\lambda)\circ\varphi_\lambda|^4\, dA/\pi \times \int_{D(\lambda,r)} \frac{|h(z)|^2}{|1-\lambda\bar{z}|^4}\, dA(z)/\pi$$
$$\leq I(\lambda,r)\frac{r^2}{(1-r^2)^2}\int_{D(\lambda,r)} |h(z)|^2 \frac{(1-|\lambda|^2)^2}{|1-\lambda\bar{z}|^4}\, dA(z)/\pi \quad \text{(using (6))}.$$

Integrating the above inequality and applying Fubini's Theorem we get

$$\int_{\mathbf{D}\setminus\rho\mathbf{D}}\left(\int_{D(\lambda,r)}\frac{\left|f(z)-P(f\circ\varphi_\lambda)(\varphi_\lambda(z))\right|^2}{|1-\lambda\bar{z}|^2}|h(z)|\,dA(z)/\pi\right)^2 dA(\lambda)/\pi$$

$$\leq \frac{r^2}{\left(1-r^2\right)^2}\sup_{\lambda\in\mathbf{D}\setminus\rho\mathbf{D}} I(\lambda,r)$$

$$\times\int_{\mathbf{D}}|h(z)|^2\left(\int_{D(z,r)}\frac{\left(1-|\lambda|^2\right)^2}{|1-\lambda\bar{z}|^4}\,dA(\lambda)/\pi\right)dA(z)/\pi$$

A change-of-variable shows that the inner integral is less than

$$\int_{r\mathbf{D}}|1-\bar{u}z|^{-4}\,dA(u)/\pi$$

which is easily seen to be bounded by $r^2/(1-r^2)^2$. Hence we have the following estimate for the first integral at the right hand side of (12):

$$\left[\int_{\mathbf{D}\setminus\rho\mathbf{D}}\left(\int_{D(\lambda,r)}\frac{\left|f(z)-P(f\circ\varphi_\lambda)(\varphi_\lambda(z))\right|^2}{|1-\lambda\bar{z}|^2}|h(z)|\,dA(z)/\pi\right)^2 dA(\lambda)/\pi\right]^{1/2}$$

$$\leq \frac{r^2}{\left(1-r^2\right)^2}\sup_{\lambda\in\mathbf{D}\setminus\rho\mathbf{D}} I(\lambda,r)^{1/2}\,\|h\|_2. \qquad (13)$$

*Step* 5. Now we estimate the second integral at the right of (12):

$$\left(\int_{\mathbf{D}\setminus D(\lambda,r)}\frac{\left|f(z)-P(f\circ\varphi_\lambda)(\varphi_\lambda(z))\right|^2}{|1-\lambda\bar{z}|^2}|h(z)|\,dA(z)/\pi\right)^2$$

$$\leq \int_{\mathbf{D}\setminus D(\lambda,r)}\frac{\left|f(z)-P(f\circ\varphi_\lambda)(\varphi_\lambda(z))\right|^4}{|1-\lambda\bar{z}|^2\left(1-|z|^2\right)^{1/2}}\,dA(z)/\pi$$

$$\times\int_{\mathbf{D}\setminus D(\lambda,r)}|h(z)|^2\frac{\left(1-|z|^2\right)^{1/2}}{|1-\lambda\bar{z}|^2}\,dA(z)/\pi \text{ (by Cauchy-Schwarz)}$$

$$\leq \frac{C}{\left(1-|\lambda|^2\right)^{1/2}}\int_{\mathbf{D}\setminus D(\lambda,r)}|h(z)|^2\frac{\left(1-|z|^2\right)^{1/2}}{|1-\lambda\bar{z}|^2}\,dA(z)/\pi \text{ (by Lemma 5).}$$

Integrating the above inequality and applying Fubini's Theorem we get

$$\int_{\mathbf{D}\setminus\rho\mathbf{D}}\left(\int_{\mathbf{D}\setminus D(\lambda,r)}\frac{|f(z)-P(f\circ\varphi_\lambda)(\varphi_\lambda(z))|^2}{|1-\lambda\bar{z}|^2}|h(z)|\,dA(z)/\pi\right)^2 dA(\lambda)/\pi$$
$$\le C\int_{\mathbf{D}}|h(z)|^2(1-|z|^2)^{1/2}$$
$$\times\left(\int_{\mathbf{D}\setminus D(z,r)}\frac{1}{(1-|\lambda|^2)^{1/2}|1-\lambda\bar{z}|^2}\,dA(\lambda)/\pi\right)dA(z)/\pi. \tag{14}$$

In the inner integral in (14) make the change of variable $\lambda=\varphi_z(u)$. Using formula (7) and identity (5) we see

$$\int_{\mathbf{D}\setminus D(z,r)}\frac{1}{|1-\lambda\bar{z}|^2(1-|\lambda|^2)^{1/2}}\,dA(\lambda)/\pi$$
$$=\frac{1}{(1-|z|^2)^{1/2}}\int_{\mathbf{D}\setminus r\mathbf{D}}\frac{1}{|1-\bar{u}z|(1-|u|^2)^{1/2}}\,dA(u)/\pi$$
$$\le\frac{1}{(1-|z|^2)^{1/2}}\left(\int_{\mathbf{D}}\frac{1}{|1-u\bar{z}|^{6/5}(1-|u|^2)^{3/5}}\,dA(u)/\pi\right)^{5/6}|\mathbf{D}\setminus r\mathbf{D}|^{1/6}$$
(by Hölder's inequality)
$$\le\frac{1}{(1-|z|^2)^{1/2}}M^{5/6}(1-r^2)^{1/6} \qquad \text{(by Lemma 4).}$$

Combining this with (14) we get an estimate on the second integral in (12):

$$\left[\int_{\mathbf{D}\setminus\rho\mathbf{D}}\left(\int_{\mathbf{D}\setminus D(\lambda,r)}\frac{|f(z)-P(f\circ\varphi_\lambda)(\varphi_\lambda(z))|^2}{|1-\lambda\bar{z}|^2}|h(z)|\,dA(z)/\pi\right)^2 dA(\lambda)/\pi\right]^{1/2}$$
$$\le C(1-r^2)^{1/12}\|h\|_2. \tag{15}$$

*Step* 6. Combining our estimates (13) and (15) with inequality (12) we see that

$$\|(H_f^*H_f-S_\rho)h\|_2$$
$$\le\frac{r^2}{(1-r^2)^2}\sup_{\lambda\in\mathbf{D}\setminus\rho\mathbf{D}}I(\lambda,r)^{1/2}\|h\|_2+C(1-r^2)^{1/12}\|h\|_2.$$

Since $H^\infty(\mathbf{D})$ is dense in $L_a^2(\mathbf{D})$ we can conclude that

$$\|H_f^*H_f - S_\rho\| \le \frac{r^2}{(1-r^2)^2} \sup_{\lambda\in\mathbf{D}\setminus\rho\mathbf{D}} I(\lambda, r)^{1/2} + C(1-r^2)^{1/12}.$$

Since $I(\lambda, r) \to 0$ as $|\lambda| \to 1^-$ for every $0 < r < 1$ (by step 1) it follows easily that $S_\rho \to H_f^*H_f$ in operator norm as $\rho \to 1^-$. Because the $S_\rho$ are Hilbert-Schmidt, thus compact, it follows that $H_f^*H_f$ is compact, and therefore $H_f$ is compact. □

To state a corollary we need to introduce more notation. For $f \in L^\infty(\mathbf{D}, dA/\pi)$ define $\tilde{f}$, the Berezin symbol of $f$, by $\tilde{f}(\lambda) = \langle fk_\lambda/\|k_\lambda\|_2, k_\lambda/\|k_\lambda\|_2\rangle$ for $\lambda \in \mathbf{D}$, so that

$$\tilde{f}(\lambda) = (1-|\lambda|^2)^2 \int_{\mathbf{D}} f(z)\frac{1}{|1-\bar{\lambda}z|^4}\, dA(z)/\pi \quad \text{for } \lambda \in \mathbf{D}.$$

As a corollary we get some of Kehe Zhu's results [8].

COROLLARY 7. *Let $f \in L^\infty(\mathbf{D}, dA/\pi)$ and $1 < p < \infty$. The following statements are equivalent*:

(a) *$H_f$ and $H_{\bar{f}}$ are compact*;

(b) *$\|f\circ\varphi_\lambda - \tilde{f}(\lambda)\|_p \to 0$ as $|\lambda| \to 1^-$.*

*Proof.* First observe that $\tilde{f}(\lambda) = \int_{\mathbf{D}} f\circ\varphi_\lambda\, dA/\pi$ (by change-of-variable formula (7)), and thus

$$\tilde{f}(\lambda) = P(f\circ\varphi_\lambda)(0) = \overline{P(\bar{f}\circ\varphi_\lambda)\,(0)} \quad \text{for } \lambda \in \mathbf{D}.$$

We will also make use of the fact that for an analytic function $h$ on $\mathbf{D}$, $P(\bar{h}) = \bar{h}(0)$.

(a) ⇒ (b) Suppose that both $H_f$ and $H_{\bar{f}}$ are compact. Since $H_{\bar{f}}$ is compact we have

$$\left\|f\circ\varphi_\lambda - \overline{P(\bar{f}\circ\varphi_\lambda)}\right\|_p = \left\|\bar{f}\circ\varphi_\lambda - P(\bar{f}\circ\varphi_\lambda)\right\|_p \to 0 \quad \text{as } |\lambda| \to 1^-.$$

Using the boundedness of $P$ as an operator of $L^p(D, dA/\pi)$ into $L_a^p(\mathbf{D})$ we get

$$\left\|P(f\circ\varphi_\lambda) - \tilde{f}(\lambda)\right\|_p = \left\|P\left(f\circ\varphi_\lambda - \overline{P(\bar{f}\circ\varphi_\lambda)}\right)\right\|_p \to 0 \quad \text{as } |\lambda| \to 1^-.$$

The compactness of $H_f$ implies that $\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_p \to 0$ as $|\lambda| \to 1^-$, which combined with the above statement gives that (b) holds.

(b) $\Rightarrow$ (a) Suppose that $\|f \circ \varphi_\lambda - \tilde{f}(\lambda)\|_p \to 0$ as $|\lambda| \to 1^-$. Again using the boundedness of $P$ it follows that $\|P(f \circ \varphi_\lambda) - \tilde{f}(\lambda)\|_p \to 0$ as $|\lambda| \to 1^-$, thus

$$\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_p \to 0 \quad \text{as } |\lambda| \to 1^-.$$

By Theorem 6 $H_f$ is compact. Since also $\|\bar{f} \circ \varphi_\lambda - \tilde{\bar{f}}(\lambda)\|_p \to 0$ as $|\lambda| \to 1^-$, $H_{\bar{f}}$ is compact. □

## 5. Remarks and open questions

In this section we discuss some open questions and directions for further research.

(1) For $f \in L^1(\mathbf{D}, dA/\pi)$ (so $f$ is not necessarily bounded) we can consider $H_f$ as an operator $L^2_a(\mathbf{D}) \to L^2_a(\mathbf{D})^\perp$ densely defined by $H_f g = (I - P)(fg)$, $g \in H^\infty(\mathbf{D})$. It is possible that even for unbounded $f$ the operator $H_f$ is bounded. The question is to find necessary and sufficient conditions on $f$ for the operator $H_f$ to be bounded. For conjugate analytic functions on $\mathbf{D}$ the answer is known. The Bloch space $\mathscr{B}$ is the set of all analytic functions $f$ on $\mathbf{D}$ for which

$$\sup_{z \in \mathbf{D}} (1 - |z|^2)|f'(z)| < \infty.$$

In [3], Sheldon Axler proved that for a function $f$ in $L^2_a(\mathbf{D})$ the (densely defined) Hankel operator $H_{\bar{f}}$ is bounded if and only if $f \in \mathscr{B}$. The proof of Theorem 1 shows that for every $\lambda \in \mathbf{D}$ the (densely defined) Hankel operators $H_f$ and $H_{f \circ \varphi_\lambda}$ are unitarily equivalent (in the sense that there exist unitary operators $U_1$ and $U_2$ as in Theorem 1 such that also $U_1(H^\infty(\mathbf{D})) \subset H^\infty(\mathbf{D})$). Consequently, a condition on $f$ that is necessary and sufficient for the operator $H_f$ to be bounded has to be Möbius-invariant. I conjecture that an answer for the general case is that for any $1 < p < \infty$,

$$\sup_{\lambda \in \mathbf{D}} \|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_p < \infty.$$

(2) Find necessary and sufficient conditions on $f$ for the operator $H_f$ to be in the Schatten $p$-class $S^p$. For conjugate analytic functions on $\mathbf{D}$ the answer has been found by J. Arazy, S. Fisher, and J. Peetre [1]. Theorem 1 implies that the class of $f$ in $L^1(\mathbf{D}, dA/\pi)$ for which $H_f$ belongs to $S^p$ is again Möbius-invariant. Without proof we mention that for $f$ in $L^\infty(\mathbf{D}, dA/\pi)$ the Hankel operator $H_f$ is Hilbert-Schmidt if and only if

$$\int_{\mathbf{D}} \frac{\|f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)\|_2^2}{(1 - |\lambda|^2)^2} \, dA(\lambda)/\pi < \infty.$$

For a conjugate analytic function $f$ on $\mathbf{D}$ the above condition is easily seen to be equivalent to $\int_{\mathbf{D}} |f'(z)|^2 \, dA(z)/\pi < \infty$, i.e., $f$ belongs to the Dirichlet space (a special case of Arazy, Fisher and Peetre's results).

(3) For $f, g \in L^\infty(\mathbf{D}, dA/\pi)$ find necessary and sufficient conditions for the operator $H_f^* H_g$ to be compact. For conjugate analytic symbols $f$ and $g$ this question was raised in [3]. In this special case a necessary condition was found by Sheldon Axler and Pamela Gorkin [5] and, independently, Dechau Zheng [7] proved that this condition is necessary and sufficient. They found that for bounded analytic functions $f$ and $g$ the operator $H_{\bar f}^* H_{\bar g}$ is compact if and only if $(1 - |z|^2) \min\{|f'(z)|, |g'(z)|\} \to 0$ as $|z| \to 1^-$. It follows from the results in Zheng's paper that this condition is equivalent to

$$\int_{\mathbf{D}} |f \circ \varphi_\lambda - f(\lambda)| \, |g \circ \varphi_\lambda - g(\lambda)| \, dA/\pi \to 0 \text{ as } |\lambda| \to 1^-.$$

The proof of Theorem 6 can be adjusted to show that for $f, g \in L^\infty(\mathbf{D}, dA/\pi)$ the operator $H_f^* H_g$ is compact if

$$\int_{\mathbf{D}} |f \circ \varphi_\lambda - P(f \circ \varphi_\lambda)| \, |g \circ \varphi_\lambda - P(g \circ \varphi_\lambda)| \, dA/\pi \to 0 \quad \text{as } |\lambda| \to 1^-.$$

It is my guess that this condition is also necessary.

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