ILLINOIS JOURNAL OF MATHEMATICS
Volume 34, Number 2, Summer 1990

ON A GENERALIZATION OF CHEEGER-CHERN-SIMONS CLASSES[1]

BY

JOHAN L. DUPONT AND FRANZ W. KAMBER

In memory of Kuo-Tsai Chen

0. Introduction

For foliated bundles in the sense of Kamber-Tondeur [K-T2] there are well-known second order characteristic classes with real coefficients. For these the domain of definition is the cohomology of the truncated Weil algebra $W(G, K)_k$ where G is the structure group, $K \subseteqq G$ is maximal compact and k is the codimension of the foliation. On the other hand there are also associated Cheeger-Chern-Simons classes with $\mathbf{R}/L$ coefficients ($L \subseteqq \mathbf{R}$ is a subring, e.g., $L = \mathbf{Z}$ or $L = \mathbf{Q}$) defined for each invariant polynomial on the Lie algebra $\mathfrak{g}$ of degree greater than k together with a lift to $H^*(BG, L)$ of the corresponding primary characteristic class in $H^*(BG, \mathbf{R})$ (see [Cn-Si] or [Cr-Si]). In the present paper we combine these two ideas and define (in Section 2) a generalized Cheeger-Chern-Simons class corresponding to a class in the cohomology of the truncation ideal $FW(G, K) \subseteqq W(G, K)$ together with a lift to $H^*(BK, L)$ of the image in $H^*(W(G, K)) = H^*(BK, \mathbf{R})$. Thus, for instance, a real second order characteristic class corresponds to a Cheeger-Chern-Simons class with $L = 0$ via the coboundary map

$$H^*(W(G, K)_k) \xrightarrow{\delta} H^*(FW(G, K))$$

except that the latter has an indeterminacy consisting of primary classes.

One might hope that this approach for $L \neq 0$ gives other interesting invariants, but as we shall see in Section 3, at least for $L = \mathbf{Q}$ every generalized Cheeger-Chern-Simons class can be expressed in terms of real second order classes, "classical" Cheeger-Chern-Simons classes and primary classes. For $L = \mathbf{Z}$ the situation is not so clear.

In Section 4 we investigate the relation between generalized Cheeger-Chern-Simons classes and the rational homotopy invariants for foliations studied by Hurder [Hu1], [Hu2] and Hurder-Kamber [Hu-K]. Finally in

Received December 30, 1988.
[1]Work supported in part by the National Science Foundation.

Section 5 we give cochain formulas for Cheeger-Chern-Simons classes for flat bundles analogous to the formulas in Dupont [D1]; in particular we treat the integral Steifel-Whitney classes in this context.

It will be apparent that much of the material of this paper (in particular Sections 2 and 5) depends on ideas by J. Cheeger and J. Simons [Cr-Si]. We have tried to develop these further which seems to be justified by the recent growing interest in Cheeger-Chern-Simons classes both in Algebraic Geometry and Number Theory (see e.g., Soulé [So] or Karoubi [Ka] as well as in Gauge Theory (see, e.g., Floer [F]).

1. Preliminaries on simplicial manifolds

Let us recall the basic facts of Chern-Weil theory in the category of *simplicial manifolds* as developed in Dupont [D2].

A simplicial manifold is a simplicial set $X = \{X_p\}$, where the set of p-simplices X_p constitute a C^∞ manifold such that all face and degeneracy operators are C^∞ maps. Let $\Delta^p \subseteqq \mathbf{R}^{p+1}$ be the standard p-simplex; then the realization $\|X\|$ of X is the quotient of $\coprod_{p \geq 0} \Delta^p \times X_p$ with the identifications $(\varepsilon^i(t), x) \sim (t, \varepsilon_i x)$, $t \in \Delta^{p-1}$, $x \in X_p$, $i = 0, \dots p$, $p \geq 0$, where $\varepsilon^i\colon \Delta^{p-1} \to \Delta^p$ is the inclusion on the ith face and $\varepsilon_i\colon X_p \to X_{p-1}$ the corresponding face operator.

The most well-known example of a simplicial manifold is NG, G a Lie group, where $NG(p) = G \times \cdots \times G$ (p copies) and where ε_i is given by multiplying the ith and $(i+1)$st coordinates. Here $\|NG\| = BG$ is the classifying space for principal G-bundles.

For X a simplicial manifold and L any coefficient ring, the cohomology $H^*(\|X\|, L)$ can be computed as the cohomology of the total complex $C^*(X, L)$ of the double complex $C^{p,q}(X, L) = C^q(X_p, L)$ of cochains based on C^∞ singular simplices $\sigma\colon \Delta^q \to X_p$.

Also as in [D2] let $A^*(X)$ denote the DeRham complex of simplicial forms. Thus a simplicial k-form φ on X is a collection of k-forms $\varphi^{(p)}$ on $\Delta^p \times X_p$ such that $(\varepsilon^i \times \mathrm{id})^* \varphi^{(p)} = (\mathrm{id} \times \varepsilon_i)^* \varphi^{(p-1)}$ for all $i = 0, \dots, p$, and all $p = 0, \dots,$ i.e. φ is just a k-form on $\coprod \Delta^p \times X_p$ compatible with the identifications in the realization $\|X\|$. Recall from [D1, Corollary 2.8], that there is a natural chain map $\mathscr{I}\colon A^*(X) \to C^*(X, \mathbf{R})$ given by

$$\mathscr{I}(\varphi)(\sigma) = \int_{\Delta^p \times \Delta^q} (\mathrm{id} \times \sigma)^* \varphi^{(p)}, \sigma\colon \Delta^q \to X_p, \varphi \in A^{p+q}(X),$$

which induces an isomorphism of cohomology rings

$$\mathscr{I}\colon H(A^*(X)) \underset{\cong}{\to} H((C^*(X, \mathbf{R})) \cong H^*(\|X\|).$$

Now let G be a Lie group and $\pi\colon E \to B$ a simplicial principal bundle, i.e., a map of simplicial manifolds such that $\pi\colon E(p) \to B(p)$ is a principal

G-bundle for each p. A connection θ in E is now just a simplicial 1-form on E with coefficients in the Lie algebra $\mathfrak{g}$, such that the restriction of θ to $\Delta^p \times E(p)$ is a connection in the usual sense. Then as in [D1, §3] we get a Chern–Weil homomorphism $h = h(\theta)\colon I(G) \to A^*(B)$ and in turn on the cohomological level

$$w = \mathscr{I} \circ h\colon I(G) \to H(A^*(B)) \cong H^*(\|B\|).$$

Here, as usual, $I(G)$ is the ring of G-invariant polynomials on $\mathfrak{g}$.

Notice that the category of C^∞ manifolds is included in the simplicial category by associating to a manifold X the simplicial manifold (also denoted by X) given by $X(p) = X$ and all face and degeneracy operators are the identity. Similarly an ordinary C^∞-principal G-bundle $E \to M$ with connection θ gives rise in a trivial way to a simplicial principal G-bundle $E \to M$ with connection (given on $\Delta^p \times E$ by pulling θ back under the projection on E) and the Chern-Weil theories clearly correspond.

There is another simplicial manifold $N\bar{X}$ associated with a manifold X: Here $N\bar{X}(p) = X \times \cdots \times X$ (($p+1$) copies) and the ith face operator is just the projection leaving out the ith coordinate. The following is simple and well-known:

LEMMA. 1.1. *$\|N\bar{X}\|$ is contractible.*

Now for $\pi\colon E \to M$ an ordinary C^∞-principal G-bundle with connection θ we get a simplicial principal G-bundle $N\bar{E} \to N\bar{E}/G$ with the connection $\bar{\theta}$ defined by: the restriction of $\bar{\theta}$ to $\Delta^p \times N\bar{E}(p)$ is given by the convex combination

$$\bar{\theta} = \sum_{i=0}^{p} t_i\theta_i, \quad t = (t_0, \ldots, t_p) \in \Delta^p$$

where θ_i is the pullback of θ under the projection $\Delta^p \times N\bar{E}(p) \to E$ onto the ith coordinate. We shall denote the corresponding Chern-Weil map by $\bar{h} = h(\bar{\theta})$. In particular for the trivial G-bundle $G \to \mathrm{pt}$ with the obvious connection we get a canonical connection $\bar{\theta}$ in the simplicial bundle $\gamma\colon N\bar{G} \to N\bar{G}/G = NG$. It is well-known that the realization of this bundle is just the universal G-bundle $EG \to BG$. In the general case Lemma 1.1 implies:

PROPOSITION 1.2. *There is a commutative diagram of simplicial bundles with connections*

$$\begin{tikzcd} E \arrow[r, "\bar{\psi}"] \arrow[d] & N\bar{E} \arrow[d] & N\bar{G} \arrow[l, "\bar{\phi}_x"'] \arrow[d] \\ M \arrow[r, "\psi"] & N\bar{E}/G & NG \arrow[l, "\phi_x"'] \end{tikzcd}$$

where $\bar{\psi}$ is given by the diagonal and $\bar{\phi}_x$ is given by operating on the point $x \in E$. Here $\|\phi_x\|\colon BG \to \|N\bar{E}/G\|$ is a weak homotopy equivalence and the homotopy class is independent of the choice of $x \in E$.

It follows in particular that the cohomology of $\|N\bar{E}/G\|$ is canonically identified with the cohomology of BG.

Models for BG of the form $\|N\bar{E}/G\|$ are particularly useful in the category of algebraic varieties. This will be explored in Dupont-Hain-Zucker [D-Ha-Z].

2. Cheeger-Chern-Simons classes

In the following,

$$E \xrightarrow{\pi} M$$

denotes a smooth principal bundle over a paracompact smooth manifold M. Let $L \subset \mathbf{R}$ be some subring (e.g., $L = 0$, $\mathbf{Z}$ or $\mathbf{Q}$) and let $r\colon H^*(-, L) \to H^*(-, \mathbf{R})$ be the natural map in cohomology induced by the inclusion $L \subset \mathbf{R}$. Chern-Simons [Cn-Si, Theorem 3.16] showed the existence of some secondary characteristic classes

$$S_{P,u}(\theta) \in H^{2l-1}(M, \mathbf{R}/L)$$

associated to

$$E \underset{\pi}{\to} M$$

with connection θ for which the curvature form Ω satisfies $\Omega^l = 0$. Here the subscript (P, u) is a pair consisting of an invariant polynomial $P \in I^l(G)$ and a cohomology class $u \in H^{2l}(BG, L)$ such that

$$w(P) = \mathscr{I} \circ \bar{h}(P) = r(u) \in H^{2l}(BG, \mathbf{R}).$$

(We use the notation $S_{P,u}(\theta)$ following Cheeger-Simons [Cr-Si], see also Cheeger [Cr].)

We shall generalize the construction as follows: Let $K \subseteqq G$ be a maximal compact subgroup and let

$$W = W(G, K) = [\Lambda((\mathfrak{g}/\mathfrak{k})^*) \otimes S(\mathfrak{g}^*)]^K$$

be the relative Weil-algebra for (G, K) (see, e.g., [K-T2, Chapter 4]). Then there is a diagram of canonical isomorphisms:

$$\begin{array}{ccc} H^*(W(G,K)) & \xrightarrow[\cong]{w} & H^*(EG/K,\mathbf{R}) \\ \cong\downarrow & & \downarrow\cong \\ I(K) & \xrightarrow[\cong]{w} & H^*(BK,\mathbf{R}). \end{array}$$

Given a non-negative integer k consider the ideal

$$FW = F^{2(k+1)}W(G,K) \subseteqq W(G,K)$$

generated by $S^l(\mathfrak{g}^*)$, $l \geq k+1$ and let

$$j\colon FW \to W(G,K)$$

be the inclusion. Also as in [K-T2] let us consider the quotient

$$W_k = W(G,K)_k = W(G,K)/FW.$$

Notice that if $p\colon W \to W_k$ is the natural projection then the exact sequence

$$0 \to FW \xrightarrow{j} W \xrightarrow{p} W_k \to 0$$

gives rise to an exact triangle

$$\text{(2.1)} \qquad \begin{array}{ccc} H(FW) & \xrightarrow{j_*} & H(W) \cong I(K) \\ & \nwarrow^{\delta} \quad \swarrow_{p_*} & \\ & H(W_k) & \end{array}$$

where δ has degree $+1$.

Now for $L \subset \mathbf{R}$ a subring let us define

$$\mathscr{S}^*(G,K;L)_k = \{(v,u) \in H^*(FW) \times H^*(BK,L) | wj_*v = ru\},$$

that is, the pull-back of the diagram

$$\begin{array}{ccccc} & & & & H^*(BK,L) \\ & & & & \downarrow r \\ H^*(FW) & \xrightarrow{j_*} & H^*(W) & \xrightarrow[\cong]{w} & H^*(BK,\mathbf{R}). \end{array}$$

for $E \to M$ a G-bundle with connection θ satisfying $\Omega^{k+1} = 0$ we shall define secondary characteristic classes

$$S_{v,u}(\theta) \in H^*(M,\mathbf{R}/L)$$

associated to

$$(v, u) \in \mathscr{S}^*(G, K; L)_k.$$

However, in general they will have indeterminacy given by the primary characteristic classes

$$Ch^*(E) = \operatorname{Im}\left[H^*(BK, \mathbf{R}) \overset{\psi^*}{\to} H^*(M, \mathbf{R}) \overset{\rho}{\to} H^*(M, \mathbf{R}/L)\right],$$

where $\psi\colon M \to BK$ is the classifying map. Thus we have:

THEOREM 2.2. (1) *For*

$$E \overset{\pi}{\to} M$$

a G-bundle with connection θ satisfying $\Omega^{k+1} = 0$ there is a natural homomorphism

$$S(\theta)\colon \mathscr{S}^l(G, K; L)_k \to H^{l-1}(M, \mathbf{R}/L)/Ch^{l-1}(E);$$

that is, to a pair $([P], u) \in \mathscr{S}^l(G, K; L)_k$ (i.e., $P \in FW$, $u \in H^*(BK, L)$) there is associated a characteristic class

$$S_{P,u}(\theta) \in H^{l-1}(M, \mathbf{R}/L)$$

with indeterminacy $Ch^{l-1}(E)$.

(2) $\beta(S_{P,u}(\theta)) = -u(E)$, *where β: $H^{l-1}(M, \mathbf{R}/L) \to H^l(M, L)$ is the Bockstein homomorphism and $u(E) = \psi^*(u)$ is the characteristic class associated to E.*

(3) $Ch^{2l-1}(E) = 0$. *Hence, $S_{P,u}(\theta) \in H^{2l-1}(M, \mathbf{R}/L)$ has no indeterminacy.*

(4) *If $P \in I^l(G)$, $l > k$ then $S_{P,u}(\theta)$ is just the Chern-Simons class.*

(5) $\pi^* S_{P,u}(\theta) = \rho[\mathscr{I} TP(\theta, \Omega)] \in H^{l-1}(E, \mathbf{R}/L)$, *where $TP \in W(G)$ is a transgressive cochain of P, i.e., $dTP = P$.*

Proof. (1) Consider the diagram in Proposition 1.2 and again let $\bar{\theta}$ be the connection constructed there for the simplicial bundle $N\bar{E} \to N\bar{E}/G$, so that $\bar{\psi}^*\bar{\theta} = \theta$. Let $\bar{\Omega}$ be the curvature form for $\bar{\theta}$. Then as in [D1, §3] or [K-T2, Chapter 4] there is an induced map of differential graded algebras

$$k(\bar{\theta})\colon W(G, K) \to A^*(N\bar{E}/K),$$

giving rise to the isomorphism

$$w = \mathscr{I} \circ k(\bar{\theta})\colon H^*(W) \underset{\cong}{\to} H(A^*(N\bar{E}/K)) \cong H^*(\|N\bar{E}/K\|, \mathbf{R}).$$

Let us write $P(\bar{\theta}, \bar{\Omega})$ for the image by $k(\bar{\theta})$ of $P \in W(G, K)$. Also as in Proposition 1.2 the map

$$N\bar{K} \subseteqq N\bar{G} \overset{\phi_x}{\to} N\bar{E}$$

induces a weak homotopy equivalence $BK \to \|N\bar{E}/K\|$. Now let

$$([P], u) \in \mathscr{S}^l(G, K; L)_k,$$

i.e., $P \in FW \subseteqq W$ and $u \in H^l(BK, L)$ satisfying

$$wj_*[P] = ru. \tag{2.3}$$

Then we represent u by a cochain $\check{u} \in C^l(N\bar{E}/K, L)$, and by (2.3),

$$\mathscr{I}(P(\bar{\theta}, \bar{\Omega})) - r\check{u} = \delta\check{s} \tag{2.4}$$

for some cochain $\check{s} \in C^{l-1}(N\bar{E}/K, \mathbf{R})$. The reduction mod L of the cochain

$$\psi^*\check{s} \in C^{l-1}(E/K, \mathbf{R})$$

is a cocycle. In fact

$$\begin{aligned} \delta\psi^*\check{s} &= \psi^*(\mathscr{I}(P(\bar{\theta}, \bar{\Omega})) - r\check{u}) \\ &= \mathscr{I}(P(\theta, \Omega)) - r\psi^*\check{u} \\ &= -r\psi^*\check{u} \equiv 0 \mod L, \end{aligned} \tag{2.5}$$

since $P(\theta, \Omega) = 0$ when $P \in FW$ and $\Omega^{k+1} = 0$. Therefore $\psi^*\check{s} \bmod L$ defines a class in $H^{l-1}(E/K, \mathbf{R}/L)$. However the projection $E/K \to M$ induces an isomorphism in cohomology, so that we have obtained a class $S_{P,u}(\theta) \in H^{l-1}(M, \mathbf{R}/L)$.

Next let us find the indeterminacy: suppose

$$\mathscr{I}(P(\bar{\theta}, \bar{\Omega})) - r\check{u}_1 = \delta\check{s}_1 \tag{2.6}$$

is another choice as in (2.4). Then

$$\check{u} - \check{u}_1 = \delta t \quad \text{for some } t \in C^{l-1}(N\bar{E}/K, L)$$

and hence by (2.4) and (2.6)

$$\delta(\check{s} - \check{s}_1 + rt) = 0$$

so that $\check{s} - \check{s}_1 + rt$ represents an element $c \in H^{l-1}(N\bar{E}/K, \mathbf{R}) \cong$

$H^{l-1}(BK, \mathbf{R})$. Therefore

$$\psi^* \check{s} - \psi^* \check{s}_1 \equiv \psi^*(\check{s} - \check{s}_1 + rt) \mod L$$

represents

$$\psi^* c \in H^{l-1}(E/K, \mathbf{R}/L)$$

which clearly corresponds to an element in $Ch^{l-1}(E) \subseteqq H^{l-1}(M, \mathbf{R}/L)$.

Finally, let us show that $S_{P,u}(\theta)$ only depends on the class $[P] \in H^l(FW)$. Thus suppose $P = dQ$ in $FW \subseteqq W$; we want to show

$$S_{dQ,0}(\theta) = 0.$$

However, in this case we can take $\check{u} = 0$ and $\check{s} = \mathscr{I}(Q(\bar{\theta}, \bar{\Omega}))$ in (2.4), which yields

$$\psi^*(\check{s}) = \mathscr{I}(Q(\theta, \Omega)) = 0$$

since $Q \in FW$ and $\Omega^{k+1} = 0$. This ends the proof of (1).

(2) This is rather immediate from the above equation (2.5) and the definition of the Bockstein homomorphism. Notice by the way that the indeterminacy $Ch^*(E)$ by definition is contained in the image of the map $H^*(M, \mathbf{R}) \to H^*(M, \mathbf{R}/L)$ and hence goes to zero under β.

(3) Obvious from the fact that

$$H^{\text{odd}}(BK, \mathbf{R}) = 0.$$

(4) Since $I^l(G) \subseteqq FW$ for $l > k$, the classes $S_{P,u}(\theta) \in H^{2l-1}(M, \mathbf{R}/L)$ are defined in particular for $P \in I^l(G)$ and $u \in H^{2l}(GB, L)$ satisfying $w(P) = r(u)$. In this case the original construction of Chern-Simons [Cn-Si] is very similar to the above in (1). Only they used an "n-classifying bundle" instead of the more canonical simplicial construction in Proposition 1.2. To see that the resulting class is the same suppose that we have a map of principal G-bundles and connections

$$\begin{array}{ccc} E & \xrightarrow{\bar{\alpha}} & E' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha} & M' \end{array}$$

such that E' is highly connected. Then we obtain a commutative diagram of simplicial manifolds

(2.7)
$$\begin{array}{ccc} M & \xrightarrow{\alpha} & M' \\ \downarrow & & \downarrow \\ N\bar{E}/G & \longrightarrow & N\bar{E}'/G \end{array}$$

covered by a similar diagram of bundles and connections as in Proposition 1.2. Now if we make similar choices as in (2.4) for the cochains $\check{u}$ and $\check{s}$ in $N\bar{E}'/G$ it follows easily from (2.7) that our construction agrees with the original one.

(5) For the projection $\bar{\pi}$: $N\bar{E} \to N\bar{E}/K$, we have $\bar{\pi}^*(\check{u}) = \delta\check{c}$, $\check{c} \in C^{l-1}(N\bar{E}, L)$ by (1.1) and hence from (2.4), $\delta\bar{\pi}^*(\check{s}) = \delta(\mathscr{I}TP(\bar{\theta}, \bar{\Omega}) - r\check{c})$. Using (1.1) again, we have

$$\bar{\pi}^*(\check{s}) = \left(\mathscr{I}TP(\bar{\theta}, \bar{\Omega}) - r\check{c}\right) + \delta\check{c}'$$

in $C^{l-1}(N\bar{E}, \mathbf{R})$. Therefore by definition of $S_{P,u}(\theta)$,

$$\begin{aligned}\pi^*S_{P,u}(\theta) &= \rho\bar{\psi}^*(\bar{\pi}^*(\check{s}))\\ &= \rho\left(\mathscr{I}TP(\theta, \Omega) + \delta\bar{\psi}^*\check{c}'\right)\\ &= \rho\left(\mathscr{I}TP(\theta, \Omega)\right) + \delta\left(\rho\bar{\psi}^*\check{c}'\right)\end{aligned}$$

in $C^{l-1}(E, \mathbf{R}/L)$. ∎

Remarks. (1) If $1/p \in L$ whenever BK has p-torsion, then

$$r\colon H^*(BK, L) \rightarrowtail H^*(BK, \mathbf{R})$$

is injective by the Universal Coefficient theorem. Therefore

$$\mathscr{S}^*(G, K; L)_k \subseteqq H^*(FW)$$

is the lattice of classes $[P]$ with $wj_*[P] \in H^*(BK, L)$ and u is uniquely determined by P, so we shall sometimes write $S_{P,u}(\theta) = S_P(\theta)$.

(2) Now if $L = \mathbf{Z}$ and $[P] \in H^*(FW)$ has $w(P)$ represented by an integral class u then $S_{P,u}(\theta)$ depend on the integral lift u in the following way: If u' is another integral lift of $w(P)$ then $u - u'$ is in the image of

$$\beta\colon H^{l-1}(BK, \mathbf{Q}/\mathbf{Z}) \to H^l(BK, \mathbf{Z}),$$

and

$$S_{P,u}(\theta) - S_{P,u'}(\theta) \equiv S_{0,u-u'}(\theta) \equiv -\beta^{-1}(u - u')(E) \pmod{Ch^*(E)}$$

is a primary characteristic class.

We have now the following properties of $S_{P,u}(\theta)$:

THEOREM 2.8. (1) (*Rigidity*) *Let θ_t, $t \in [0,1]$, be a family of connections in E with curvature forms Ω_t satisfying $\Omega_t^{k+1} = 0$. Then if*

$$P \in \operatorname{Im}\left[H(F^{2(k+2)}W \to H(F^{2(k+1)}W)\right],$$

we have

$$S_{P,u}(\theta_0) = S_{P,u}(\theta_1).$$

(2) (*Multiplicativity*) *If*

$$([P], u) \in \mathscr{S}^*(G, K; L)_{-1} \cong H^*(BK, L)$$

and

$$([Q], v) \in \mathscr{S}^*(G, K; L)_k,$$

then for a bundle E with connection θ satisfying $\Omega^{k+1} = 0$ we have

$$S_{P\cdot Q, u \cup v}(\theta) = (-1)^{\deg(u)} u(E) \cup S_{Q,v}(\theta).$$

Although the proof is similar to the proof of the corresponding statements in Cheeger-Simons [Cr-Si], we include it for completeness.

Proof of Theorem 2.8(1) Let θ_t, $t \in [0,1]$, be the given family of connections. Then as in Section 1 this gives a family of connections $\bar{\theta}_t$ in the simplicial bundle $N\bar{E} \to N\bar{E}/G$. In other words, we have a connection $\tilde{\theta}$ in $E \times [0,1]$ which is the pull-back of a connection $\bar{\tilde{\theta}}$ in $N\bar{E} \times [0,1]$. Let $(P, u) \in \mathscr{S}^l(G, K; L)_k$. Then in $A^*(N\bar{E}/K)$ we have the equation

$$P(\bar{\theta}_1, \bar{\Omega}_1) - P(\bar{\theta}_0, \bar{\Omega}_0) = \int_{t=0}^{1} i_{d/dt}\left(dP(\bar{\tilde{\theta}}, \bar{\tilde{\Omega}})\right) + d\int_{t=0}^{1} i_{d/dt}\left(P(\bar{\tilde{\theta}}, \bar{\tilde{\Omega}})\right), \tag{2.9}$$

where $i_{d/dt}$ is the usual interior product in the t-variable and where $\bar{\Omega}_0$, $\bar{\Omega}_1$ and $\bar{\Omega}$ denote the curvature forms of $\bar{\theta}_0$, $\bar{\theta}_1$ and $\bar{\theta}$ respectively. Since $dP = 0$, the first term on the right hand side of (2.9) vanishes. Therefore if

$$\mathscr{I}\left(P(\bar{\theta}_0, \bar{\Omega}_0)\right) - r\check{u} = \delta\check{s}_0,$$

then we can choose

$$\check{s}_1 = \check{s}_0 + \mathscr{I}\left(\int_{t=0}^{1} i_{d/dt}\left(P(\bar{\tilde{\theta}}, \bar{\tilde{\Omega}})\right)\right)$$

in order to obtain

$$\mathscr{I}\left(P(\bar{\theta}_1, \bar{\Omega}_1)\right) - r\check{u} = \delta\check{s}_1.$$

In that case $S_{P,u}(\theta_1) - S_{P,u}(\theta_0)$ is represented in $C^*(E/K, \mathbf{R}/L)$ by

$$\psi^* s_1 - \psi^* s_0 = \mathscr{I}\left(\int_{t=0}^{1} i_{d/dt}\left(P(\tilde{\theta}, \tilde{\Omega})\right)\right). \tag{2.10}$$

Now let Ω_t be the curvature form for θ_t in E. Then the curvature form $\tilde{\Omega}$ for $\tilde{\theta}$ in $E \times [0,1]$ is given by

$$\tilde{\Omega}_t = \Omega_t + \left(\frac{d}{dt}\theta_t\right) \wedge dt$$

so that lth powers of $\tilde{\Omega}$ involve $(l-1)$th powers of Ω_t. From this it clearly follows that the right hand side of (2.10) vanishes if $P \in F^{2(k+2)}W$, which proves (1).

For the proof of (2) it is necessary first to compare the exterior product of differential forms with the Alexander-Whitney cup-product of singular cochains in a precise way. Thus for M any manifold let $\mathscr{A}^*(M)$ be the usual DeRham complex ad $C^*(M)$ the singular cochain complex with coefficients in $\mathbf{R}$ (and with only smooth singular simplices). Let $\cup$: $C^*(M) \otimes C^*(M) \to C^*(M)$ be the usual Alexander-Whitney cup-product, and $\mathscr{I}$: $\mathscr{A}^*(M) \to C^*(M)$ the usual integration map. Then we have:

LEMMA 2.11. *There is a functorial map* τ: $\mathscr{A}^* \otimes \mathscr{A}^* \to C^*$ *of degree* -1 *such that for all closed forms* $\omega_1, \omega_2 \in \mathscr{A}^*(M)$,

$$\mathscr{I}(\omega_1 \wedge \omega_2) - \mathscr{I}(\omega_1) \cup \mathscr{I}(\omega_2) = \delta\tau(\omega_1, \omega_2).$$

Proof. More generally for S any simplicial set let $A^*(S)$ be the simplicial DeRham complex as in Dupont [D1] or [D2] and we shall find a functorial map τ: $A^*(S) \otimes A^*(S) \to C^*(S)$ which is a chain homotopy between $\cup \circ (\mathscr{I} \otimes \mathscr{I})$ and $\mathscr{I} \circ \wedge$. Then the lemma follows by taking S to be the singular simplicial set for M. Now as in [D2, Theorem 2.16], $\mathscr{I}$: $A^* \to C^*$ is a natural chain equivalence with natural inverse $\mathscr{E}$: $C^* \to A^*$, and by the proof of [D2, Theorem 2.33] the Alexander-Whitney map is naturally chain homotopic with the map $\mathscr{I} \circ (\mathscr{E} \wedge \mathscr{E})$: $C^* \otimes C^* \to C^*$. Therefore by composing with $\mathscr{I}$ and using the natural chain homotopy between $\mathscr{E} \circ \mathscr{I}$ and the identity we obtain the natural chain homotopy τ. This proves the lemma. ■

Proof of Theorem 2.8 (2) Now let $E \to M$ be a principal G-bundle with connection θ satisfying $\Omega^{k+1} = 0$ and let $([P], u)$ and $([Q], v)$ be given as in the theorem. Let θ be induced from θ' a connection in some other bundle $E' \to M'$ with E' highly connected; i.e., we have a map of bundles and connections

$$\begin{array}{ccc} E & \xrightarrow{\bar{\alpha}} & E' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha} & M'. \end{array}$$

Since E'/K approximates BK we can represent u and v by cochains $\check{u}$ and $\check{v}$ and choose $\check{s}$ and $\check{t}$ in $C^*(E'/K, \mathbf{R})$ such that

$$\mathscr{I}(P(\theta', \Omega')) - r\check{u} = \delta\check{s},$$
$$\mathscr{I}(Q(\theta', \Omega')) - r\check{v} = \delta\check{t}.$$

Then using lemma 2.11 we obtain

$$\begin{aligned}\mathscr{I}&(P(\theta', \Omega') \wedge Q(\theta', \Omega')) - r(\check{u} \cup \check{v}) \\ &= \mathscr{I}(P(\theta', \Omega')) \cup \mathscr{I}(Q(\theta', \Omega')) - r(\check{u} \cup \check{v}) + \delta\tau(P(\theta', \Omega'), Q(\theta', \Omega')) \\ &= \delta\tilde{s}\end{aligned}$$

where

$$\tilde{s} = (-1)^{\deg u} r\check{u} \cup \check{t} + \check{s} \cup \mathscr{I}(Q(\theta', \Omega')) + \tau(P(\theta', \Omega'), Q(\theta', \Omega')).$$

Therefore $S_{P \cdot Q, u \cup v}(\theta)$ is represented in $C^*(E/K, \mathbf{R})$ by

$$\begin{aligned}\alpha^*\tilde{s} &= (-1)^{\deg u} r(\alpha^*\check{u}) \cup \alpha^*\check{t} + \alpha^*\check{s} \cup \mathscr{I}(Q(\theta, \Omega)) + \tau(P(\theta, \Omega), Q(\theta, \Omega)) \\ &= (-1)^{\deg u} r(\alpha^*\check{u}) \cup \alpha^*\check{t}\end{aligned}$$

which clearly represents $(-1)^{\deg u} u(E) \cup S_{Q, v}(\theta)$, and thus we have finished the proof. ∎

Remark. In view of the multiplicative property it would be interesting to know the generators of $H^*(FW)$ as a module over $H^*(W(G, K)) = H^*(BK, \mathbf{R})$. This question is dealt with in the next section.

We end this section by investigating the relation between the classes $S_{P,u}$ and the usual secondary characteristic classes defined for foliated bundles as in Kamber-Tondeur [K-T2, Chapter 4].

Thus let π: $E \to M$ be a *foliated G-bundle*, that is, there are given foliations $\bar{\mathscr{F}}$ on E and $\mathscr{F}$ on M, such that

(i) the action by G on E permutes the leaves of $\bar{\mathscr{F}}$, and
(ii) for each $x \in E$ the differential π_*: $T_xE \to T_xM$ maps the tangent space for the leaf of $\bar{\mathscr{F}}$ isomorphically onto the tangent space for the leaf of $\mathscr{F}$.

The codimension k of the foliation $\mathscr{F}$ is called the codimension of the foliated bundle. In this situation we can choose an *adapted* connection in the bundle $E \to M$, that is a connection θ which annihilates the vector fields tangent to the foliation $\bar{\mathscr{F}}$ (see [K-T2, Chapter 1]). The curvature form Ω satisfies $\Omega^{k+1} = 0$, so that the classes $S_{P,u}(\theta)$ are defined for $(P, u) \in \mathscr{S}^*(G, K; L)_k$. On the other hand, as explained in [K-T2, Chapter 4], the adapted connection θ gives rise to a map $k(\theta)$ of the truncated Weil algebra $W(G, K)_k$ into the DeRham complex of E/K, thus defining

$$\Delta_*: H(W(G, K)_k) \to H^*_{DR}(E/K, \mathbf{R}) \cong H^*_{DR}(M, \mathbf{R})$$

where H_{DR} denotes the DeRham cohomology.

We then have:

THEOREM 2.12. *Let $E \to M$ be a foliated principal G-bundle of codimension k, and let θ be an adapted connection. Then*:

(1) *The classes $S_{P,u}(\theta) \in H^*(M, \mathbf{R}/L)$ are independent of the choice of adapted connection θ.*

(2) *The diagram*

$$\begin{array}{ccc} \mathscr{S}^l(G, K; L)_k & \xrightarrow{S(\theta)} & H^{l-1}(M, \mathbf{R}/L)/Ch^{l-1}(E) \\ {\scriptstyle\delta'}\uparrow & & \uparrow{\scriptstyle\rho} \\ H^{l-1}(W(G, K)_k) & \xrightarrow{\Delta_*} & H^{l-1}(M, \mathbf{R}) \end{array} \tag{2.13}$$

commutes, where δ' associates the pair $([dP], 0) \in \mathscr{S}^*(G, K; L)_k$ to any $P \in W(G, K)$ with $dP \in F^{2(k+1)}W$.

The proof of (1) is very similar to the proof of the "rigidity property" in (1) of Theorem 2.8 above. Part (2) follows directly by checking through the definitions (compare Theorem 2.2(5)).

Remark. If we take $L = 0$ in Theorem 2.12 then

$$\mathscr{S}^*(G, K; 0)_k = \operatorname{Im}[\delta: H(W_k) \to H(FW)] \cong H(W_k)/p_*H(W)$$

by the exact sequence (2.1), so that in this case $S(\theta)$ is the same map as Δ_* modulo primary characteristic classes.

3. The structure of $\mathscr{S}^*(G, K; L)_k$

In this section we study the structure of the graded algebra (without unit) $\mathscr{S}^*(G, K; L)_k$ introduced in Section 2. In view of the module property of the Cheeger-Chern-Simons classes $S_{P,u}$ with respect to $\mathscr{S}^*(G, K; L)_{-1} \cong H^*(BK, L)$ in part (2) of Theorem 2.8, it is of interest to find a minimal system of generators of $H^*(FW(G, K)_k)$ over the algebra $I(K)$. Such a system $\mathscr{V}$ of generators is described in Theorem 3.14 under some mild conditions on G. In order to simplify exposition we formulate Theorem 3.14 for $L = \mathbf{Q}$, but we note that it remains valid if L satisfies the condition in

Remark (1) following (2.7). In particular, the mapping S is completely determined by its values on a L-lattice $\mathscr{V}_L$ of $\mathscr{V}$.

Throughout this section we assume that $G_F \subseteqq GL(n, F)$ is a reductive linear algebraic group defined over $\mathbf{Q}$ with maximal compact subgroup

$$K \overset{i}{\hookrightarrow} G = G^0_{\mathbf{R}}.$$

In order to carry out our program it is necessary to recall some earlier results on the cohomology-structure of the relative truncated Weil algebra $W(G, K)_k$ [K-T, 1-3].

Let $P \subset (\Lambda \mathfrak{g}^*)^G$ be the space of primitive elements of $\mathfrak{g}$ and consider a Samelson decomposition $P = \hat{P} \oplus \tilde{P}$, i.e. $\hat{P} = P \cap (\Lambda(\mathfrak{g}/\mathfrak{k})^*)^K$. The symmetric pair (G, K) satisfies the condition (CS) in [K-T2, 5.101], [K-T3, 2.4]; i.e., there exists a transgression τ: $P \to I(G)$ such that the corresponding space of indecomposable elements $V = \tau P = \hat{V} \oplus \tilde{V} \subset I(G)$ satisfies the condition

$$\ker(i^*\colon I(G) \to I(K)) = \operatorname{Ideal}(\hat{V}). \tag{3.1}$$

Thus we have a commutative diagram

$$\begin{tikzcd} I(G) \arrow[rr] & & I(K) \\ S(V) = S(\hat{V}) \otimes S(\tilde{V}) \arrow[u, "\cong"] \arrow[r] & S(\tilde{V}) \arrow[ur, phantom, "\cup"] \end{tikzcd} \tag{3.2}$$

We observe that (3.2) has a rational form given by the commutative diagram

$$\begin{tikzcd} I(G) \arrow[r, "i^*"] \arrow[d, "\cong"] & I(K) \arrow[d, "\cong"] \\ H^*(BU, \mathbf{R}) \arrow[r, "(Bi')^*"] & H^*(BK, \mathbf{R}), \end{tikzcd} \tag{3.3}$$

where $U \subset G_{\mathbf{C}}$ is the (unique) compact rational form of the complex group $G_{\mathbf{C}}$ and

$$K = G \cap U \overset{i'}{\hookrightarrow} U$$

is he canonical inclusion. We denote by $I(G)_{\mathbf{Q}}$ etc. the corresponding algebra of invariant polynomials over $\mathbf{Q}$ and by $V_{\mathbf{Q}}$ etc. the corresponding vector spaces over $\mathbf{Q}$ spanned by rational (resp. integral) generators of V. Thus we have, for example,

$$I(G)_{\mathbf{Q}} \cong S(V)_{\mathbf{Q}} \cong S_{\mathbf{Q}}(V_{\mathbf{Q}}).$$

We now use the results in [K-T1], [K-T2, Chapter 5], [K-T3] to show that diagram (2.1) is obtained by the tensor product $\otimes_{S(\tilde{V})} I(K)$ from the exact cohomology triangle of the exact sequence

$$0 \to F\hat{A} \xrightarrow{\hat{j}} \hat{A} \xrightarrow{\hat{p}} \Lambda \hat{P}^{(2k)} \otimes \hat{A}_k \to 0. \tag{3.4}$$

Here $P^{(2k)}$ and $P_{(2k)}$ denote the vector spaces spanned by elements of degree $> 2k$ and $\leq 2k$ respectively in a graded vector space P. The algebras $\hat{A}$, $\hat{A}_k$, $F\hat{A}$ are Koszul complexes given by

$$\hat{A} = \Lambda \hat{P} \otimes I(G), \; F\hat{A} = \Lambda \hat{P} \otimes FI(G),$$
$$\hat{A}_k = \Lambda \hat{P}_{(2k)} \otimes I(G)_k,$$

with the differentials determined as usual by the transgression

$$\tau \colon P \xrightarrow{\cong} V.$$

We have isomorphisms

$$\begin{array}{ccc} H^*(W_k) \cong & (\Lambda \hat{P}^{(2k)} \otimes H^*(\hat{A}_k)) \otimes_{S(\tilde{V})} I(K) \\ \big\downarrow \delta & \big\downarrow \hat{\delta} \otimes \mathrm{id} \\ H^*(FW) \cong & H^*(F\hat{A}) \otimes_{S(\tilde{V})} I(K), \end{array} \tag{3.5}$$

while the cohomology of $F\hat{A}$ is determined by the exact cohomology-triangle of (3.4):

$$0 \to \Lambda \hat{P}^{(2k)} \otimes H(\hat{A}_k)/S(\tilde{V})_k \xrightarrow{\hat{\delta}} H(F\hat{A}) \overset{\hat{j}_*}{\twoheadrightarrow} FS(\tilde{V}) \to 0 \tag{3.6}$$

(observe that $S(\tilde{V}) \cong H^*(\hat{A})$ and that (3.6) is split over $S(\tilde{V})$ in a natural way). From (3.2) and [K-T3, §3] we further obtain

$$FS(\tilde{V}) \otimes_{S(\tilde{V})} I(K) \cong i^* FI(G) \cdot I(K).$$

It follows now that (2.1) is determined by the exact tensor product $\otimes_{S(\tilde{V})} I(K)$ applied to (3.6) and that $\mathscr{S}^*(G, K; \mathbf{Q})_k$ is determined by a suitable pull-back operation. This is summarized in the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \delta H(W_k) \arrow[r] \arrow[d, equal] & \mathscr{S}^*(G, K; \mathbf{Q})_k \arrow[r] \arrow[d, tail] & i^* FI(G)_{\mathbf{Q}} \cdot I(K)_{\mathbf{Q}} \arrow[r] \arrow[d, tail] & 0 \\ 0 \arrow[r] & \delta H(W_k) \arrow[r] & H^*(FW) \arrow[r, "j_*"] & i^* FI(G) \cdot I(K) \arrow[r] & 0. \end{tikzcd} \tag{3.7}$$

Using the multiplicativity of S over $I(K)$ (Theorem 2.8(2)), it follows that S is completely determined by the restriction $\hat{S}$ to the algebra $\hat{\mathscr{S}}^*(G, K; \mathbf{Q})_k$ defined by the pull-back diagram

(3.8)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda\hat{P}^{(2k)} \otimes H(\hat{A}_k)/S(\tilde{V})_k & \longrightarrow & \hat{\mathscr{S}}^*(G, K; \mathbf{Q})_k & \longrightarrow & FS(\tilde{V})_{\mathbf{Q}} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda\hat{P}^{(2k)} \otimes H(\hat{A}_k)/S(\tilde{V})_k & \xrightarrow{\hat{\delta}} & H^*(F\hat{A}) & \xrightarrow{\hat{j}_*} & FS(\tilde{V}) & \longrightarrow & 0
\end{array}$$

or in fact by its values on a *minimal system of generators* over $S(\tilde{V})$ of the algebra $H^*(F\hat{A})$. It remains therefore to describe such a minimal system of generators in $H(F\hat{A})$, resp. in $H(\hat{A}_k)/S(\tilde{V})_k$ and $FS(\tilde{V})$.

The structure of these algebras as $S(\tilde{V})$-modules has been described in [K-T1], [K-T2, 5.110], [K-T3] as follows. Let $\hat{Z}_k \subset \hat{A}_k$ be the subalgebra spanned over $\mathbf{R}$ by the *admissible* cocycles

$$\hat{z}_{(I|J)} = \hat{y}_I \otimes c_J \in \hat{A}_k, \quad \hat{y}_I = \hat{y}_{i_1} \wedge \cdots \wedge \hat{y}_{i_s}, \quad c_J = c_1^{j_1} \dots c_r^{j_r},$$

where $V_{\mathbf{Q}} = \mathbf{Q}(c_1, \dots, c_r)$, $\hat{V}_{\mathbf{Q}} = \mathbf{Q}(\hat{c}_i)$, $\tilde{V}_{\mathbf{Q}} = \mathbf{Q}(\tilde{c}_l)$ denote rational bases (the c_j's consist of the $\hat{c}_i$'s and the $\tilde{c}_l$'s) and $y_j = \sigma(c_j)$ is the suspension. Explicitly, the $\hat{z}_{(I|J)}$'s satisfy

$$\text{(3.9)} \qquad \deg c_J \leq 2k, \quad \text{i.e., } c_J \in S\big(V_{(2k)}\big),$$

$$\text{(3.10)} \qquad \deg \hat{c}_{i_1} c_J > 2k \quad \text{(cocycle condition)},$$

$$\text{(3.11)} \qquad j_\alpha = 0 \text{ for } \alpha < i_1 \text{ (for all } \alpha \text{ if } s = 0) \quad \text{with } c_\alpha = \hat{c}_\alpha \in \hat{V}_{\mathbf{Q}}.$$

Then the inclusion $\hat{Z}_k \hookrightarrow \hat{A}_k$ induces an isomorphism

$$\text{(3.12)} \qquad \hat{Z}_k \underset{\cong}{\to} H\big(\hat{A}_k\big)$$

and the subalgebra $\hat{Z}_k^+$ consisting of the $\hat{z}_{(I|J)}$'s with $s > 0$ describes an $\mathbf{R}$-basis of $H(\hat{A}_k)/S(\tilde{V})_k$, while $\hat{Z}_k^0 = \{z_{(\varnothing, J)}\} \cong S(\tilde{V})_k$. Furthermore, the cocycles $\hat{z}_{(I|J)}$ satisfying the additional property

$$\text{(3.13)} \qquad \deg \hat{c}_{i_1} c_J - \deg \tilde{c}_\alpha \leq 2k \quad \text{for } \tilde{c}_\alpha | c_J,\ s > 0,$$

form a minimal system of generators of $\hat{Z}_k^+$ (*indecomposable classes*). In the absolute case, the corresponding algebra generated by *admissible* cocycles is denoted by

$$Z_k \subset A_k = \Lambda P_{(2k)} \otimes I(G)_k.$$

It is then clear that a minimal system of generators of $FS(\tilde{V})_{\mathbf{Q}}$ over $S(\tilde{V})_{\mathbf{Q}} = S(\tilde{V}_{\mathbf{Q}})$ is given by the generators $\tilde{c}_j \in \tilde{V}_{\mathbf{Q}}^{(2k)}$ and the class $\tilde{c}_j\tilde{c}_J \in S(\tilde{V})_{\mathbf{Q}}$ such that

$$\tilde{z}_{(j|J)} = \tilde{y}_j \otimes \tilde{c}_J \in Z_k^+;$$

i.e., $\tilde{z}_{(j|J)}$ is an admissible cocycle in A_k.

Summarizing, we have shown:

THEOREM 3.14. (1) *A minimal system of generators of $H(F\hat{A})$ over $S(\tilde{V})$ (resp. of $H(FW(G, K))$ over $I(K)$) is given by the classes of the following types:*

(I) $\tilde{c}_j\tilde{c}_J \in FS(\tilde{V}_{(2k)})_{\mathbf{Q}}$, *with* $\tilde{z}_{(j|J)} = \tilde{y}_j \otimes \tilde{c}_J \in Z_k^+$; *i.e.*, $\tilde{c}_j\tilde{c}_J$ *is indecomposable in* $FS(\tilde{V})$;

(II) $\tilde{c}_j \in \tilde{V}_{\mathbf{Q}}^{(2k)}$;

(III) $\delta(z_{(I|J)})$, $\hat{z}_{(I|J)} \in \hat{Z}_k^+$, *with* $\hat{z}_{(I|J)}$ *indecomposable*;

(IV) $\delta(\hat{y}_I \otimes \hat{z})$ *with* $\hat{y}_I \in \Lambda^+\hat{P}^{(2k)}$, $\hat{z} = \hat{z}_{(I|J)} \in \hat{Z}_k^+$ *as in* (III) *or* $\hat{z} = 1$.

(2) *The Cheeger-Chern-Simons map $S(\theta)$ in Section 2 is completely determined by its restriction to $\mathscr{V}_{\mathbf{Q}} \subset \hat{\mathscr{S}}(G, K; \mathbf{Q})_k$, where $\mathscr{V}_{\mathbf{Q}}$ is the pullback to $\hat{\mathscr{S}}(G, K; \mathbf{Q})_k$ in* (3.8) *of the real vector space $\mathscr{V} \subset H(F\hat{A})$ spanned by the classes of type* I *and* II.

Remarks. (1) From (3.8) or Theorem 3.14 we see that $\mathscr{S}^*(G, K; \mathbf{Q})_k$, resp. $\hat{\mathscr{S}}^*(G, K; \mathbf{Q})_k$ consists of the usual Chern-Simons classes (type I and II) and the secondary characteristic classes (type III and IV) of the foliated G-bundle $E \to M$. They are all of degree $l > 2k + 1$.

(2) The generators of type I and II in $\mathscr{V}_{\mathbf{Q}} \subset \hat{\mathscr{S}}^*(G, K; \mathbf{Q})_k$ are of even degree $l > 2k + 1$. Thus S has no indeterminacy of these classes (Theorem 2.2 (3)). For the generators of type III and IV the indeterminacy of S is explained by the diagrams (2.1), (2.13) and (3.8).

Example 3.15. *The normal bundle Q of a foliation*. In this case, we have

$$(G, K) = (GL(k, \mathbf{R}), O(k))$$

and $V = \mathbf{R}(c_1, \ldots, c_k)$ is spanned by the Chern-polynomials c_i given by

$$\det\left(\mathrm{Id} + \frac{t}{\pi}A\right) = \sum_{i=1}^{k} c_i(A)t^i, \quad A \in \mathfrak{gl}(k, \mathbf{R}).$$

Thus $V^{(2k)} = 0$. Further $i^*(c_{2j+1}) = 0$, whereas $i^*(c_{2j}) = p_j$ are the Pontrjagin-polynomials which correspond to integral classes in $H^*(BO(k), \mathbf{R})$. Thus

$$\hat{V} = \mathbf{R}(c_{2j+1}), \quad \tilde{V} = \mathbf{R}(c_{2j}) \quad \text{and} \quad H^*(FW) \cong H^*(\hat{A}_k) \oplus FS(\tilde{V}) \quad (\text{see } (3.8)),$$

where $\hat{A}_k = \Lambda(y_1, y_3, \ldots) \otimes \mathbf{R}[c_1, \ldots, c_k]_k$ is the usual Koszul-algebra, also denoted by WO_q. It follows from Theorem 3.14 that there are only classes of type I and III. In the oriented case for $k = 2m$ there is in addition the Euler-class $e_m \in I(SO(2m))$ which is not in the image of i^* but satisfies $e_m^2 = p_m = i^* c_{2m}$, and the formula for $H^*(FW)$ has to be modified according to (3.8). As $O(k)$ has only 2-torsion, we might in this case replace $\mathbf{Q}$ by $L = \mathbf{Z}[\frac{1}{2}]$. Finally S has no indeterminacy since all primary classes are of degree $\leq 2k$.

Example 3.16. *Flat bundles.* For a flat G-bundle

$$E \xrightarrow{\pi} M,$$

we have $k = 0$. Therefore $P = P^{(2k)}$, $P_{(2k)} = 0$ and $\hat{A}_k \cong \mathbf{R}$. From (3.8) it follows that

$$\hat{\mathscr{S}}^*(G, K; \mathbf{Q})_0 \cong \hat{\delta}\Lambda\hat{P} \oplus F^2 S(\tilde{V})_{\mathbf{Q}}.$$

From Theorem 3.14 we see that there are only classes of type II and IV, namely $\tilde{c}_j \in \tilde{V}_Q$ and $\hat{\delta}(\hat{y}_I)$, $\hat{y}_I \in \Lambda^+\hat{P}$, the latter corresponding via (2.13) to the usual characteristic classes Δ_*: $H^*(\mathfrak{g}, K) \to H^*(M, \mathbf{R})$ of E.

4. Cheeger-Chern-Simons classes and dual homotopy invariants of foliated bundles

In the spirit of Hurder's work on dual homotopy invariants of foliations [Hu1], [Hu2], [Hu-K], we proceed now to study the homotopy version of the map

$$S\colon \mathscr{S}^l(G, K; \mathbf{Q})_k \to H^{l-1}(M, \mathbf{R}/\mathbf{Q}).$$

As $\mathscr{S}^*(G, K; \mathbf{Q})_k$ is an algebra without unit and S is not an algebra homomorphism, it is not feasible to try to construct dual homotopy groups associated to $\mathscr{S}_k^*$ directly. It turns out however, that a suitable carrier for such invariants already exists, namely the dual homotopy groups of the truncated Weil algebra $W(G)_k$.

Let us recall the basic notions of the theory of minimal models and rational homotopy following Sullivan [Su] (see also Griffith-Morgan [G-M] or Halperin [Hal]). We shall always use real coefficients unless otherwise specified. Thus to any graded commutative differential graded algebra (DGA) A over $\mathbf{R}$ there is associated a unique *minimal model*, i.e., a minimal DGA $\mathscr{M}(A)$ with a DGA-map

(4.1) $$\mathscr{M}(A) \to A$$

which is a quasi-isomorphism (that is, induces an isomorphism in cohomology); compare e.g. Griffith-Morgan [G-M, Theorems 9.5 and 12.1], Halperin [Hal, Ch. XV] or Sullivan [Su]. The algebraic (dual) homotopy groups of A are defined for $i > 1$ by

$$\tilde{\pi}^i(A) = (\overline{\mathscr{M}}/\overline{\mathscr{M}}\cdot\overline{\mathscr{M}})^i, \tag{4.2}$$

where $\overline{\mathscr{M}} = \overline{\mathscr{M}}(A)$ consists of the elements of positive degree in $\mathscr{M} = \mathscr{M}(A)$. In this context the algebraic (dual) Hurewicz map $\tilde{\mathscr{H}}^*\colon H^*(A) \to \tilde{\pi}^*(A)$ is defined as the composite

$$\tilde{\mathscr{H}}^*\colon H^i(A) \cong H^i(\mathscr{M}) \to H^i(\overline{\mathscr{M}}/\overline{\mathscr{M}}\cdot\overline{\mathscr{M}}) = \tilde{\pi}^i(A), \tag{4.3}$$

where the last equality follows from the minimality of $\mathscr{M}$.

For X any topological space we now put

$$\mathscr{M}(X) = \mathscr{M}(A^*(X)) \quad\text{and}\quad \tilde{\pi}^i(X) = \tilde{\pi}^i(A^*(X)), \quad i > 1, \tag{4.4}$$

where $A^*(X) = A^*(\mathscr{S}(X))$ is the simplicial DeRham complex as in Section 1 for the singular complex $\mathscr{S}(X)$ of X. If X is assumed to have the weak homotopy type of a C.W.-complex of finite type and it X is simply connected (or more if generally of X is *nilpotent* in the sense that $\pi_1(X)$ is nilpotent and acts nilpotently on $\pi_i(X)$, $i > 1$), then there is a natural isomorphism

$$\tilde{\pi}^i(X) \cong \pi^i(X) \equiv \mathrm{Hom}(\pi_i(X), \mathbf{R}), \quad i > 1. \tag{4.5}$$

When X is not of finite type at least there is a natural homomorphism generalizing (4.5):

$$\chi\colon \tilde{\pi}^i(X) \to \pi^i(X), \quad i > 1 \tag{4.6}$$

given as follows: For $g\colon S^i \to X$ representing $[g] \in \pi_i(X)$ we consider the induced map

$$g^{\#}\colon \tilde{\pi}^i(X) \to \tilde{\pi}^i(S^i),$$

and identifying $\tilde{\pi}^i(S^i) \cong \mathrm{Hom}(\pi_i(S^i), \mathbf{R})$ via (4.5) we put

$$\chi(a)([g]) = \langle g^{\#}(a), \iota_i\rangle, \quad a \in \tilde{\pi}^i(X),$$

where $\iota_i \in \pi_i(S^i)$ is the canonical generator. With these definitions it easily follows that the diagram

$$
\text{(4.7)} \qquad \begin{tikzcd} H^*(X) \arrow[r, "\tilde{\mathscr{H}}^*"] \arrow[dr, "\mathscr{H}^*"'] & \tilde{\pi}^*(X) \arrow[d, "\chi"] \\ & \pi^*(X) \end{tikzcd}
$$

commutes, where $\mathscr{H}^*$ is the dual of the usual Hurewicz homomorphism.

Let us remark that this theory combines nicely with the Quillen +-construction. Thus (see Berrick [Be1, Ch. 5]) to any C.W.-complex X there is associated a cofibration $f\colon X \to X^+$ precisely annihilating the maximal perfect subgroup

$$P(\pi_1(X)) \subset \pi_1(X)$$

and inducing an isomorphism in homology (with any local coefficients induced from X^+). In particular the corresponding map of simplicial DeRham complexes

$$f^*\colon A^*(X^+) \to A^*(X)$$

is a quasi-isomorphism and hence

$$\mathscr{M}(X^+) = \mathscr{M}(X). \tag{4.8}$$

Combining with (4.6) above we obtain a natural homomorphism

$$\chi^+\colon \tilde{\pi}^i(X) \to \pi^i(X^+), \quad i > 1, \tag{4.9}$$

which is an isomorphism if X^+ is a simply connected (or more generally a nilpotent) C.W.-complex of finite type. Using (4.7) for X^+ we obtain a commutative diagram

$$
\begin{tikzcd} H^i(X) \arrow[r, "\tilde{\mathscr{H}}^*"] & \tilde{\pi}^i(X) \arrow[d, "\chi^+"] \\ H^i(X^+) \arrow[u, "f^*", "\cong"'] \arrow[r, "\mathscr{H}^*"] & \pi^i(X^+). \end{tikzcd}
$$

Let us note in passing that for X a differentiable manifold $A^*(X)$ in (4.4) can be replaced by the usual DeRham complex $\mathscr{A}^*(X)$, since these two complexes are quasi-isomorphic. Similarly, if X is a simplicial manifold, then $\mathscr{M}(\|X\|)$ is the minimal model for the simplicial DeRham complex $A^*(X)$ as in Section 1.

For the study of the homotopy invariants for foliated bundles we need the following result.

LEMMA 4.10. *Let G be a connected Lie group and π: $E \to X$ a principal G-bundle over a topological space X with ψ: $X \to BG$ the classifying map. Then there is an exact sequence for $i > 1$:*

$$\cdots \to \pi^i(BG) \xrightarrow{\psi^{\#}\circ\chi^{-1}} \tilde{\pi}^i(X) \to \tilde{\pi}^i(E) \to \cdots.$$

Proof. Since G is a simple space (i.e., $\pi_1(G)$ is abelian and acts trivially on $\pi_i(G)$, $i > 1$) we clearly have $\pi^*(G) \cong \tilde{\pi}^*(G) = P$, the finite dimensional vector space of primitive elements in $H^*(G, \mathbf{R}) \cong \Lambda(P)$. (In the notation of Section 3, $P \subset (\Lambda \mathfrak{k}^*)^K$ is the space of primitive elements for the maximal compact subgroup K.) Now elements in P are transgressive in the fibration π: $E \to X$ since this is the case in the universal fibration $EG \to BG$. It follows that the extension of DGA's

$$A^*(X) \to A^*(E)$$

is equivalent to a Koszul-Sullivan extension in the sense of Halperin [Hal, Ch. I]

$$A^*(X) \to A^*(X) \otimes \Lambda(P)$$

and furthermore this can be chosen such that for each $x \in P$, $dx = 1 \otimes \psi^* y$ for some $y \in A^*(BG)$ with $dy = 0$. It follows from Halperin [Hal, Ch. X] that there is an exact sequence

$$\cdots \to P^{i-1} \xrightarrow{d^{\#}} \tilde{\pi}^i(X) \to \tilde{\pi}^i(E) \to \cdots. \tag{4.11}$$

Furthermore by naturality we have a commutative diagram

$$\begin{array}{ccccccc}
\longrightarrow & P^{i-1} & \xrightarrow{d^{\#}} & \tilde{\pi}^i(X) & \longrightarrow & & \\
 & \| & & \uparrow{\scriptstyle\psi^{\#}} & & & \\
0 \longrightarrow & P^{i-1} & \underset{\cong}{\xrightarrow{d^{\#}}} & \tilde{\pi}^i(BG) & \longrightarrow & 0 &
\end{array}$$

which together with (4.11) clearly proves the lemma. ∎

Remarks. (1) Note that P only has odd-dimensional elements, so that $\pi^i(BG) = P^{i-1} = 0$ for i odd, and $\tilde{\pi}^i(X) \to \tilde{\pi}^i(E)$ is surjective for i even.

(2) If X^+ is a nilpotent space of finite type then by Berrick [Be1, Ch. 4] or [Be2] also E^+ is nilpotent and $X^+ \to BG$ is a fibration with fibre E^+. In this case the exact sequence in Lemma 4.10 reduces to the dual of the usual homotopy sequence for this fibration.

(3) All we have said so far obviously works just as well with $\mathbf{Q}$ coefficients instead of $\mathbf{R}$. Thus if the algebra A is quasi-isomorphic to $A_{\mathbf{Q}} \otimes \mathbf{R}$ for a

rational DGA $A_{\mathbf{Q}}$, then $\mathscr{M}(A_{\mathbf{Q}})$ is a rational DGA-lattice in $\mathscr{M}(A)$; i.e., $\mathscr{M}(A) = \mathscr{M}(A_{\mathbf{Q}}) \otimes_{\mathbf{Q}} \mathbf{R}$ and

$$\tilde{\pi}^*(A_{\mathbf{Q}}) \subset \tilde{\pi}^*(A)$$

is clearly a rational lattice. Hence we can define

$$\tilde{\pi}^*(A, \mathbf{R}/\mathbf{Q}) = \tilde{\pi}^*(A)/\tilde{\pi}^*(A_{\mathbf{Q}}).$$

In particular for a topological space X we also have the rational simplicial DeRham complex $A^*(X, \mathbf{Q})$ (see e.g. Dupont [D2, p. 37]) and we can define

$$\tilde{\pi}^*(X, \mathbf{R}/\mathbf{Q}) = \tilde{\pi}^*(A(X))/\tilde{\pi}^*(A(X, \mathbf{Q})).$$

With this notation Lemma 4.10 is valid also with $\mathbf{Q}$- or $\mathbf{R}/\mathbf{Q}$-coefficients.

We now return to the situation of a foliated bundle

$$E \xrightarrow{\pi} M$$

with adapted connection θ. The *algebraic homotopy class* of the characteristic homomorphisms

$$\begin{array}{ccc} W(G)_k & \xrightarrow{k(\theta)} & \mathscr{A}^*(E) \\ \cup & & \uparrow{\scriptstyle \pi^*} \\ I(G)_k & \xrightarrow{h(\theta)} & \mathscr{A}^*(M) \end{array} \tag{4.12}$$

is independent of θ. This follows from the homotopy formulas in [K-T4, §3]. Thus the maps in (4.12) determine lifts to the corresponding minimal models (unique up to algebraic homotopy) and in particular unique homomorphisms

$$\begin{array}{ccc} \tilde{\pi}^*(W(G)_k) & \xrightarrow{k^\#} & \tilde{\pi}^*(E) \\ \uparrow & & \uparrow{\scriptstyle \pi^\#} \\ \tilde{\pi}^*(I(G)_k) & \xrightarrow{h^\#} & \tilde{\pi}^*(M). \end{array} \tag{4.13}$$

If fact, for the normal bundle Q of $\mathscr{F}$, Hurder [Hu1, §2] shows that $h^\#$ depends only on the concordance class of $\mathscr{F}$. We are going to exploit the remarkable fact that $k^\#$ is substantially determined by $h^\#$ (cf. (4.24), (4.26)).

In order to define a mapping

$$\lambda \colon \mathscr{S}^l(G, K; Q)_k \to \tilde{\pi}^{l-1}(W(G)_k),$$

we consider the commutative diagram with exact rows

$$(4.14)\qquad \begin{tikzcd} 0 \arrow[r] & FW(G,K) \arrow[r, "j"] \arrow[d, hook, "\iota"] & W(G,K) \arrow[r, "p"] \arrow[d, hook, "\iota"] & W(G,K)_k \arrow[r] \arrow[d, hook, "\iota_k"] & 0 \\ 0 \arrow[r] & FW(G) \arrow[r] & W(G) \arrow[r] & W(G)_k \arrow[r] & 0 \end{tikzcd}$$

where the vertical maps are inclusions. As $\overline{H}(W(G)) = 0$, we obtain a diagram of connecting homorphisms

$$(4.15)\qquad \begin{tikzcd} H^{l-1}(W(G)_k) \arrow[r, "\delta_0", "\cong"'] & H^l(FW(G)) \\ H^{l-1}(W(G,K)_k) \arrow[r, "\delta"] \arrow[u, "\iota_{k*}"] & H^l(FW(G,K)), \arrow[u, "\iota_*"] \end{tikzcd}$$

with δ_0 an isomorphism for $l > 1$. Thus we may define λ as the composition

$$(4.16)\qquad \lambda\colon H^l(FW(G,K)) \xrightarrow{\delta_0^{-1}\circ\iota_*} H^{l-1}(W(G)_k) \xrightarrow{\tilde{\mathscr{H}}^*} \tilde{\pi}^{l-1}(W(G)_k).$$

Similarly, we define the mapping $\hat{S}^{\#}$ as the composition

$$(4.17)\qquad \hat{S}^{\#}\colon \mathscr{S}^l(G,K;\mathbf{Q})_k \xrightarrow{S} H^{l-1}(M,\mathbf{R}/\mathbf{Q}) \xrightarrow{\tilde{\mathscr{H}}^*} \tilde{\pi}^{l-1}(M,\mathbf{R}/\mathbf{Q}).$$

The relation between λ and $\hat{S}^{\#}$ is as follows.

PROPOSITION 4.18. *The diagram below is commutative*:

$$(4.19)\qquad \begin{tikzcd} H^l(FW(G,K)) \arrow[d, "\lambda"] & \supset & \mathscr{S}^l(G,K;\mathbf{Q})_k \arrow[d, "\hat{S}^{\#}"] \\ \tilde{\pi}^{l-1}(W(G)_k) \arrow[d, "k^{\#}"] & & \tilde{\pi}^{l-1}(M,\mathbf{R}/\mathbf{Q})/\mathrm{im}(\psi^{\#}) \arrow[d, hook, "\pi^{\#}"] \\ \tilde{\pi}^{l-1}(E,\mathbf{R}) \arrow[rr, "\rho"] & & \tilde{\pi}^{l-1}(E,\mathbf{R}/\mathbf{Q}); \end{tikzcd}$$

i.e.,

$$\pi^{\#}\circ\hat{S}^{\#} = \rho\circ k^{\#}\circ\lambda.$$

In addition, *we have*

$$(4.20)\qquad \hat{S}^{\#}\circ\delta' = \rho\circ\Delta^{\#}\circ\tilde{\mathscr{H}}^*$$

on $H(W(G,K)_k)$.

Proof. The commutativity of (4.19) is essentially a consequence of Theorem 2.2 (5). In fact, let $([P], u) \in \mathcal{S}^l$ and let $TP \in W(G)$ be a transgressive cochain of P; i.e., $dTP = P$. Then

$$\begin{aligned}\pi^{\#}\big(\hat{S}^{\#}_{P,u}(\theta)\big) &= \tilde{\mathscr{H}}^*\big(\pi^*(S_{P,u}(\theta))\big)\\ &= \tilde{\mathscr{H}}^*\big(\rho[\mathscr{I}TP(\theta,\Omega)]\big)\\ &= \rho\big(\tilde{\mathscr{H}}^*(k_*[TP])\big)\\ &= \rho\big(k^{\#}\big(\tilde{\mathscr{H}}^*\delta_0^{-1}\iota_*[P]\big)\big)\\ &= \rho\big(k^{\#}(\lambda[P])\big),\end{aligned}$$

where we used $\delta_0[TP] = \iota_*[P]$. The second formula follows from (2.13) and

$$\text{(4.21)} \qquad \lambda \circ \delta = \tilde{\mathscr{H}}^* \circ \iota_{k*}\colon H^*(W(G,K)_k) \to \tilde{\pi}^*(W(G)_k).$$

Note that the induced map $\pi^{\#}$ in (4.19) is injective. This follows from Lemma 4.10 (and Remark (3) following it) applied to the fibration $E \to E/K \simeq M$. ∎

The maps $\hat{S}^{\#}$ and λ are far from being injective. In fact, the multiplicative property of S in part (2) of Theorem 2.8 and the vanishing of $\tilde{\mathscr{H}}^*$ on products imply that $\hat{S}^{\#}$ is zero on $I(K)_{\mathbf{Q}}$-decomposable elements. Similarly λ is zero on $I(K)$-decomposable elements and thus factors as follows (cf. Theorem 3.14.(1)):

$$\text{(4.22)} \qquad \begin{array}{ccc} H^*(FW(G,K)) & \xrightarrow{\lambda} & \tilde{\pi}^{*-1}(W(G)_k) \\ \big\downarrow & & \big\uparrow \hat{\lambda} \\ H^*(FW(G,K)) \otimes_{I(K)} \mathbf{R} \cong & H^*(F\hat{A}) \otimes_{S(\tilde{V})} \mathbf{R} & \cong \mathscr{V}. \end{array}$$

Even on $\mathscr{V}$, $\hat{\lambda}$ is not injective if $\hat{P}^{(2k)} \neq 0$. In fact, the generators of type IV in Theorem 3.14 are products unless $\mathrm{I} = (i)$ and $\hat{z} = 1$. Let $\mathscr{V}' \subset \mathscr{V}$ be the subspace spanned by the classes of type I–III and IV′:

$$\text{(IV}'\text{)} \qquad \hat{\delta}(\hat{y}_i) = \hat{c}_i, \quad \hat{y}_i \in \hat{P}^{(2k)}.$$

The following theorem shows that there are no further identifications under $\hat{\lambda}$. In the course of the proof, we will also explicitly identify the range of $\hat{\lambda}$.

THEOREM 4.23. *$\hat{\lambda}\colon \mathscr{V}' \to \tilde{\pi}^{*-1}(W(G)_k)$ is injective.*

Proof. Recall from Section 3 (and references given there) that there is a quasi-isomorphism $\Lambda P^{(2k)} \otimes A_k \to W(G)_k$, inducing an isomorphism

$$\Lambda P^{(2k)} \otimes H^*(A_k) \underset{\cong}{\to} H^*(W(G)_k).$$

Thus there is a canonical isomorphism

$$P^{(2k)} \oplus \tilde{\pi}^*(A_k) \underset{\cong}{\to} \tilde{\pi}^*(W(G)_k). \tag{4.24}$$

Further $\tilde{\pi}^*(A_k)$ and $\tilde{\pi}^*(\hat{A}_k)$ have been explicitly determined [Hae], [Hu1] [Hu-K]. Recall then that the subalgebra $Z_k \subset A_k$ of admissible cocycles is quasi-isomorphic to A_k and has trivial products ($Z_k^{+2} = 0$). Hence A_k is biformal and $\tilde{\pi}^*(A_k)$ is essentially the dual of the free graded Lie algebra generated by the dual of Z_k^+:

$$\tilde{\pi}^*(A_k) \cong \operatorname{Hom}\big(s\mathbf{L}(s^{-1}Z_k^{+*}), \mathbf{R}\big),$$

where s denotes homotopy suspension. In particular, the dual Hurewicz homomorphism

$$\tilde{\mathscr{H}}^*\colon H^+(A_k) \to \tilde{\pi}^*(A_k) \tag{4.25}$$

is injective in positive degrees and we denote the image under $\tilde{\mathscr{H}}^*$ of an admissible cocycle $z_{(I|J)}$ by $\hat{u}_{(I|J)} \in \tilde{\pi}^*(A_k)$. Further, the dual homotopy groups of $I(G)_k$, A_k and $\hat{A}_k$ are related by split-exact sequences in the diagram

(4.26)

$$\begin{tikzcd}
 & 0 & & & \\
 & \tilde{V}_{(2k)} \arrow[u] & & 0 & \\
0 \arrow[r] & V_{(2k)} \arrow[r] \arrow[u] & \tilde{\pi}^*(I(G)_k) \arrow[r, shift left] & \tilde{\pi}^*(A_k) \arrow[r] \arrow[u] \arrow[l, shift left] & 0 \\
0 \arrow[r] & \hat{V}_{(2k)} \arrow[r] \arrow[u] & \tilde{\pi}^*(I(G)_k) \arrow[r, shift left] \arrow[u, equal] & \tilde{\pi}^*(\hat{A}_k) \arrow[r] \arrow[u, "\iota_k^\#"'] \arrow[l, shift left] & 0 \\
 & 0 \arrow[u] & & \tilde{V}_{(2k)} \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

(observe that all generators in $\tilde{\pi}^*(A_k)$ are of degree $\geq 2k+1$). By checking through the definitions, we find now that

$$\hat{\lambda}(\tilde{c}_j\tilde{c}_J) = \tilde{\mathscr{H}}^*(\tilde{z}_{(j|J)}) = \tilde{u}_{(j|J)} \in \tilde{\pi}^*(A_k)$$

for classes of type I,

$$\hat{\lambda}(\tilde{c}_j) = \tilde{y}_j \in \tilde{P}^{(2k)}_{\mathbf{Q}}$$

for classes of type II and

$$\hat{\lambda}(\hat{\delta}\hat{y}_i) = \hat{y}_i \in \hat{P}^{(2k)}$$

for classes of type IV′. These classes are clearly linearly independent in

$$\tilde{\pi}^*(W(G)_k) \cong P^{(2k)} \oplus \tilde{\pi}^*(A_k)$$

and so it remains to determine the images of the classes of type III, i.e., the proper foliation classes. Now by (4.21) we have $\hat{\lambda}(\hat{\delta}\hat{z}_{(I|J)}) = \tilde{\mathscr{H}}^*(\iota_{k*}\hat{z}_{(I|J)})$ and so we have to understand the mappings in the commutative diagram

$$\begin{array}{ccc} \hat{Z}_k^+ \cong H^+(\hat{A}_k) & \xrightarrow{\mathscr{H}^*} & \tilde{\pi}^*(\hat{A}_k) \\ \downarrow{\scriptstyle \iota_{k*}} & & \downarrow{\scriptstyle \iota_k^\#} \\ Z_k^+ \cong H^+(A_k) & \xrightarrow{\tilde{\mathscr{H}}^*} & \tilde{\pi}^*(A_k). \end{array} \tag{4.27}$$

Note that the indicated maps are injective (in degrees $> 2k$) by (4.25) and (4.26). The preceding arguments together with the following lemma complete then the proof of Theorem 4.23. ∎

LEMMA 4.28. (1) $\ker(\iota_{k*}) = \ker(\tilde{\mathscr{H}}^*) \subset \hat{Z}_k^+$ *is given by the linear span of decomposable cocycles.*

(2) *The images under* $\hat{\lambda}$ *of the classes of type* I *and* III *are linearly independent.*

Proof. Clearly the decomposable cocycles $\hat{Z}_k^+ \cdot F^2S(\tilde{V})$ are in $\ker(\iota_{k*}) = \ker(\tilde{\mathscr{H}}^*)$. Suppose then that $\hat{z}_{(I|J)} \in \hat{Z}_k^+$ is an indecomposable cocycle (3.13) of the form $\hat{z}_{(I|J)} = \hat{y}_{i_1} \wedge \cdots \wedge \hat{y}_{i_s} \otimes c_J$ and let j denote the smallest number so that $c_j | c_J$, i.e., $c_J = c_j \cdot c_K$ (the c_j's are supposed to be order by increasing degree). If $i_1 \leq j$, then $z = \iota_k(\hat{z}_{(I|J)}) \in Z_k^+$, i.e., z is an admissible cocycle in A_k, and we set $z_1 = z$, $z_\alpha = 0$, $\alpha > 1$. If $j < i_1$, then $c_j = \tilde{c}_j \in \tilde{V}_{(2k)}$ by (3.11) and we set

$$z_\alpha = \tilde{y}_j \wedge \hat{y}_{i_1} \wedge \cdots \wedge \hat{y}_{i_{\alpha-1}} \wedge \hat{y}_{i_{\alpha+1}} \wedge \cdots \wedge \hat{y}_{i_s} \otimes \hat{c}_{i_\alpha} c_K, \quad \alpha = 1, \ldots, s.$$

Then $z_\alpha \in Z_k^+$ and $z_1 \neq 0$ by (3.13). We observe that the mapping $\hat{Z}_k^+ \longrightarrow Z_k^+$ defined by $\hat{z}_{(I|J)} \mapsto z_1$ is 1-1 on indecomposable generators and therefore injective modulo decomposables. Setting $w = \tilde{y}_j \wedge \hat{y}_I \otimes c_K$, one verifies that

$$dw = z - \sum_{\alpha=1}^{S} (-1)^{\alpha-1} z_\alpha$$

and therefore

$$\iota_{k*}(\hat{z}_{(I|J)}) = z_1 + \sum_{\alpha=2}^{S} (-1)^{\alpha-1} z_\alpha \in Z_k^+ \cong H^+(A_k).$$

With respect to lexicographic ordering of the admissible cocycles in Z_k^+, the mapping

$$z_1 \mapsto z_1 + \sum_{\alpha=2}^{s} (-1)^{\alpha-1} z_\alpha$$

is in triangular form with eigenvalues $+1$, therefore injective and (1) follows.
For (2) we simply observe that

$$\hat{\lambda}\left(\hat{\delta}\hat{z}_{(I|J)}\right) = \tilde{\mathscr{H}}^*(\hat{y}_I \otimes c_J)$$

or

$$\hat{\lambda}\left(\hat{\delta}\hat{z}_{(I|J)}\right) = \sum_{\alpha=1}^{s} (-1)^{\alpha-1} \tilde{\mathscr{H}}^*(z_\alpha),$$

where the z_α are of the form $y_I \otimes \hat{c}_i c_K$. Thus these classes are linearly independent from the images of the classes of type I determined earlier. ■

Remark. All generators in $\mathscr{V}_{\mathbf{Q}}'$ are of degree $l > 2k + 1$; they have even degree, except those of type III. Thus $\hat{S}^{\#}$ has an indeterminacy at most for generators of type III. If $\pi^l(BK, \mathbf{R}) \cong \pi^{l-1}(K, \mathbf{R}) = 0$ for $l > 2k$, then $\hat{S}^{\#}$ has no indeterminacy at all.

Using (4.24) we define now a $\mathbf{Q}$-subspace in $\tilde{\pi}^*(W(G)_k)$ by

$$(4.29) \quad \tilde{\pi}^*(W(G)_k)_{\mathbf{Q}} \equiv \hat{P}^{(2k)} \oplus \tilde{P}_{\mathbf{Q}}^{(2k)} \oplus \tilde{\pi}^*(A_k) \subseteq \tilde{\pi}^*(W(G)_k).$$

THEOREM 4.30. *There is a unique mapping* $S^{\#}$: $\tilde{\pi}^*(W(G)_k)_{\mathbf{Q}} \to \tilde{\pi}^*(M, \mathbf{R}/\mathbf{Q})$ *such that* $\pi^{\#} \circ S^{\#} = \rho \circ k^{\#}$ *in diagram* (4.19) *and such that* *diagram* (4.31) *below is commutative.*

$$\begin{tikzcd} & \mathcal{V}'_{\mathbf{Q}_J} \arrow[dl, "\hat{\lambda}"'] \arrow[dr, "\hat{S}^{\#}"] & \\ \tilde{\pi}^*(W(G)_k)_{\mathbf{Q}} \arrow[rr, "S^{\#}"] & & \tilde{\pi}^*(M, \mathbf{R}/\mathbf{Q}) \\ \tilde{\pi}^*(I(G)_k) \arrow[u] \arrow[rr, "h^{\#}"] & & \tilde{\pi}^*(M, \mathbf{R}). \arrow[u, "\rho"'] \end{tikzcd} \tag{4.31}$$

If M^+ is nilpotent of finite type, the homomorphism χ^+ in (4.9) *is an isomorphism and $S^{\#}$ takes values in $\pi^*(M^+, \mathbf{R}/\mathbf{Q})$.*

Proof. Using the unique splitting in (4.26), we define $S^{\#}$ on $\tilde{\pi}^*(A_k)$ by $\rho \circ h^{\#}$ so that the lower part of (4.31) is commutative and $\pi^{\#} \circ S^{\#} = \rho \circ k^{\#}$ on $\tilde{\pi}^*(A_k)$ by (4.13). We further define

$$S^{\#}(\hat{y}_i) = \rho\big(\Delta^{\#}(\hat{y}_i)\big) \quad \text{for } \hat{y}_i \in \hat{P}^{(2k)}$$

and

$$S^{\#}(\tilde{y}_j) = S^{\#}\big(\hat{\lambda}\tilde{c}_j\big) = \hat{S}^{\#}(\tilde{c}_j) \quad \text{for } \tilde{y}_j \in \tilde{P}_{\mathbf{Q}}^{(2k)}.$$

The equation $\pi^{\#} \circ S^{\#} = \rho \circ k^{\#}$ on these latter classes, as well as the commutativity of the upper part of (4.31) follow now by direct calculation from (4.19), (4.20) and the previous explicit description of $\hat{\lambda}$. Finally, the uniqueness of $S^{\#}$ is a direct consequence of Theorem 4.23, (4.26) and the commutativity of (4.31). ■

Note that $S^{\#}$ on $\tilde{P}_{\mathbf{Q}}^{(2k)}$ is only defined modulo $\mathbf{Q}$. In fact, the restriction to the fibre of $k^{\#}\colon \tilde{P}_{\mathbf{Q}}^{(2k)} \to \tilde{\pi}^*(E, \mathbf{R})$ is given by the canonical map

$$\tilde{P}_{\mathbf{Q}}^{(2k)} \to \tilde{\pi}^*(K, \mathbf{Q})$$

and it vanishes only after reduction modulo $\mathbf{Q}$. This gives an alternative definition of

$$S^{\#}\colon \tilde{P}_{\mathbf{Q}}^{(2k)} \to \tilde{\pi}^*(M, \mathbf{R}/\mathbf{Q})$$

via the exact homotopy sequence

$$0 \to \tilde{\pi}^l(M, \mathbf{R}/\mathbf{Q}) \xrightarrow{\pi^{\#}} \tilde{\pi}^l(E, \mathbf{R}/\mathbf{Q}) \to \tilde{\pi}^l(K, \mathbf{R}/\mathbf{Q}) \to \cdots$$

of $E \to E/K \simeq M$, for $l > 1$ odd.

Example 4.32. *The normal bundle Q* (compare 3.15). $\mathscr{V}'_{\mathbf{Q}}$ is in this case spanned by the indecomposable elements $\tilde{c}_j\tilde{c}_J \in S(\tilde{V})_{\mathbf{Q}}$ (type I) and the boundaries of the indecomposable classes

$$z_{(I|J)} \in \hat{Z}_k^+ \cong H^+(A_k) \quad (\text{type III}),$$

which are mapped isomorphically under $\hat{\lambda}$ onto a subspace of

$$\tilde{\pi}^*(W(GL(k))_k)_{\mathbf{Q}} \cong \tilde{\pi}^*(A_k)$$

(observe that $P^{(2k)} = 0$ and use (4.24)). Thus from (4.13), (4.26) and (4.31) we conclude that $S^\# = \rho \circ h^\#$; i.e., the homotopy C-S-S invariants are completely determined by the Chern-homomorphism h: $I(G)_k \to A^*(M)$. In view of Hurder's results on independent variation [Hu1], [Hu2], one sees that $h^\#$ is not rational in many cases and thus $S^\# \not\equiv 0$. Observe that in this case there is no indeterminacy at all, since

$$\tilde{\pi}^*(O(k)) \cong \mathbf{R}(p_1, \ldots, p_m) \quad \text{for } k = 2m + 1$$

and

$$\tilde{\pi}^*(O(k)) \cong \mathbf{R}(p_1, \ldots, p_{m-1}, e_m) \quad \text{for } k = 2m,$$

and so all the primary invariants are of degree $\leq 2k$.

Example 4.33. *Flat bundles* (compare 3.16). The subspace $\mathscr{V}'_{\mathbf{Q}}$ is given by

$$\hat{\delta}\hat{P} \oplus \tilde{V}_{\mathbf{Q}} = \hat{V} \oplus \tilde{V}_{\mathbf{Q}}$$

and $\hat{\lambda}$: $\mathscr{V}'_{\mathbf{Q}} \to \tilde{\pi}^*(W(G)_0)_{\mathbf{Q}} = \hat{P} \oplus \tilde{P}_{\mathbf{Q}}$ is an isomorphism (suspension). Hence the homotopy invariants given by $S^\#$ in diagram (4.31) are expressed by the homologically defined map $\hat{S}^\#$. For the universal flat G-bundle $\tilde{E} = EG_d \times_{G_d} G \to BG_d$ (where G_d is the underlying discrete group of G) and any flat G-bundle $E \xrightarrow{\pi} X$ with classifying map φ: $X \to BG_d$, we obtain therefore a commutative diagram, using (4.9):

$$\text{(4.34)} \qquad \begin{array}{ccc} \tilde{\pi}^*(W(G)_0)_{\mathbf{Q}} & \xrightarrow{S_G^\#} & \pi^*((BG_d)^+, \mathbf{R}/\mathbf{Q}) \\ {\scriptstyle S^\#}\big\downarrow & & \big\downarrow{\scriptstyle (\varphi^+)^\#} \\ \tilde{\pi}^*(X, \mathbf{R}/\mathbf{Q}) & \xrightarrow{\chi^+} & \pi^*(X^+, \mathbf{R}/\mathbf{Q}). \end{array}$$

Here the homorphism $S_G^\# = \chi^+ \circ S^\#$ is associated to the group G. Notice that $(BG_d)^+$ is quite a different space than GB_d. Thus if G is a semi-simple connected Lie group for example, then by Chevalley [Cv, Ch. IV, §XII], G_d is a perfect group and hence $(BG_d)^+$ is simply connected and so must have non-zero higher homotopy groups (since the homology is non-zero). In particular for $G = SL(N, F)$, $F = \mathbf{R}$, $\mathbf{Q}$ or $\mathbf{H}$,

$$\pi_i\left(\left(BSL(N, F)_d\right)^+\right) \cong K_i(F) \quad (i \ll N)$$

is Quillen's algebraic K-theory of F so that (4.34) gives invariants for these groups. In this context, the invariants (of type IV′) corresponding to elements in

$$\hat{P} \subseteqq \tilde{\pi}^*(W(G)_0)_{\mathbf{Q}}$$

were introduced in Borel [Bor], and are widely known as the "Borel regulators". For the invariants (of type II) corresponding to $\tilde{P}_{\mathbf{Q}} \subset \tilde{\pi}^*(W(G)_0)_{\mathbf{Q}}$, it is not known whether they have non-trivial realizations (mod $\mathbf{Q}$).

The results of this section may be summarized as follows: The Cheeger-Chern-Simons classes determine dual homotopy-invariants of type I—III and IV′ which are *independently* realized by $\hat{\lambda}(\mathscr{V}'_{\mathbf{Q}}) \subset \tilde{\pi}^*(W(G)_k)_{\mathbf{Q}}$. For any foliated bundle the characteristic homomorphism $k^{\#}$ determines a natural homomorphism

$$S^{\#}: \tilde{\pi}^*(W(G)_k)_{\mathbf{Q}} \to \tilde{\pi}^*(M, \mathbf{R}/\mathbf{Q}),$$

which may properly be viewed as the homotopy version of the C-S-S classes.

5. Flat bundles and Eilenberg-MacLane cochains

As mentioned in Examples 3.16 and 4.33, Theorem 2.2 gives characteristic classes for *flat G-bundles* ($k = 0$). In the universal case we obtain classes in $H^*(BG_d, \mathbf{R}/L)$ (modulo the image of the map $H^*(BG, \mathbf{R}) \to H^*(BG_d, \mathbf{R}/L)$, where G_d is the discrete group underlying G. These classes generalize the classes considered by Cheeger [Cr]:

$$S_{P,u} \in H^{2l-1}(BG_d, \mathbf{R}/L)$$

defined for $P \in I^l(G)$ and $u \in H^{2l}(BG, L)$ with $w(P) = ru \in H^{2l}(BG, \mathbf{R})$. Since $H^*(BG_d, \mathbf{R}/L)$ is naturally isomorphic with the Eilenberg-MacLane group cohomology of G_d it is natural to ask for a more concrete description of representing cochains for these classes, analogous to the description of the primary real classes given in [D1] (see also [D2, Chapter 9] and Shulman-Tischler [Sh-Ti]).

There is a more direct way of constructing such cochains: Suppose G acts on a manifold V (on the left); for $q \geq 0$ define a *q-filling* of V to be a family of C^∞-singular simplices,

$$\sigma(g_1,\dots,g_p)\colon \Delta^p \to V,\, g_1,\dots,g_p \in G,\, p = 0,1,\dots,q,$$

such that

$$\sigma(g_1,\dots,g_p)\circ\varepsilon^i = \begin{cases} g_1\cdot\sigma(g_2,\dots,g_p), & i = 0,\\ \sigma(g_1,\dots,g_ig_{i+1},\dots,g_p), & 0 < i < p,\\ \sigma(g_1,\dots,g_{p-1}), & i = p.\end{cases}$$

This is simply the formal definition of a section of the associated bundle with fibre V corresponding to the universal G-bundle pulled back to $BG_d = \|NG_d\|$. An example for $V = G/K$ and $q = \infty$ is provided by the set of *geodesic simplices* $\Delta(g_1,\dots,g_p)$ defined inductively as the geodesic cone on $g_1\Delta(g_2,\dots,g_p)$ with top-point $\mathcal{O} = \{K\}$ (considering Δ^p as the cone on $\varepsilon^0(\Delta^{p-1})$). (See e.g. [D1] for details).

LEMMA 5.1. *q-fillings exist if V is $(q-1)$-connected and two q-fillings are homotopic if V is q-connected.*

Now suppose V is $(q-1)$-connected and let ω be a closed G-invariant q-form on V. Further let $L \subseteqq \mathbf{R}$ be the lattice generated by

$$\left\{\int_z \omega \,|\, z \text{ a singular cycle in } V\right\}.$$

Then for any choice of q-fillings the cochain $\mathscr{I}(\omega) \in C^q(NG_d, \mathbf{R}/L)$ given by

$$\mathscr{I}(\omega)(g_1,\dots,g_q) = \int_{\Delta^q}\sigma(g_1,\dots,g_q)^*\omega$$

gives a well-defined cohomology class in the Eilenberg-MacLane group cohomology $H^q(BG_d, \mathbf{R}/L)$, independent of the choice of q-filling.

Let us recall the results of [D1] and relate them to the framework of Cheeger-Chern-Simons classes. Thus let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, and let $\theta\colon \mathfrak{p} \to \mathfrak{g}$ be the inclusion considered as a $\mathfrak{g}$-valued linear form on $\mathfrak{p}$. Then $\Omega = -\frac{1}{2}[\theta,\theta]$ is a $\mathfrak{k}$-valued linear 2-form on $\mathfrak{p}$. We identify the G-invariant differential forms on G/K with the corresponding K-invariant forms $\Lambda((\mathfrak{g}/\mathfrak{k})^*)^K$ on $\mathfrak{p}$. We have then, using the ∞-filling of G/K by geodesic simplices.

THEOREM 5.2. (1) *For $P \in I^l(K)$ the corresponding primary characteristic class $w(P) \in H^{2l}(BG_d, \mathbf{R})$ is given by*

$$w(P) = \mathscr{I}\big(P(\Omega^l)\big)$$

(2) *For $P \in \ker(I^l(G) \to I^l(K))$ the corresponding secondary characteristic class $S_{P,0} \in H^{2l-1}(BG_d, \mathbf{R})$ is given by*

$$S_{P,0} = \mathscr{I}(\sigma P(\theta)),$$

where

$$\sigma P(\theta) = \frac{2^{2(l-1)}}{\binom{2l-1}{l}} P(\theta \wedge \Omega^{l-1}) \in \Lambda^{2l-1}((\mathfrak{g}/\mathfrak{k})^*)^K$$

is the suspension of P.

(3) *Any class of the form $\mathscr{I}(\omega)$ is a sum of products of the classes in* (1) *and* (2).

Proof. Part (1) is just [D1, Corollary 1.3] and (2) is a reformulation of [D1, Theorem 1.1] in view of the remark following Theorem 2.12 above. Part (3) follows from Theorems 2.8(2), 2.12(2), and 3.14. ■

From now on assume for simplicity G compact and take $L = \mathbf{Z}$. In that case, $P \in I(G) \cong H^*(BG, \mathbf{R})$ is clearly determined by u with $ru = w(P)$, so we shall write

$$S_u = S_{P,u} \in H^*(BG_d, \mathbf{R}/\mathbf{Z}).$$

Notice that for u *odd*-dimensional S_u is simply the primary characteristic class corresponding to $-\beta^{-1}(u) \in H^*(BG, \mathbf{R}/\mathbf{Z})$ (the indeterminacy goes to zero).

We get a cochain description of S_u in the following case (which covers, for example, u a Chern class for $G = U(n)$): Suppose $H \subseteqq G$ is a subgroup with G/H $(q-2)$-connected and suppose that $v \in H^{q-1}(G/H, \mathbf{Z})$ transgresses to $u \in H^q(BG, \mathbf{Z})$ in the fibration $G/H \to BH \to BG$. Then we have:

THEOREM 5.3. *Let ω be the G-invariant $(q-1)$-form on G/H representing rv and choose any $(q-1)$-filling on G/H. Then $S_u = \mathscr{I}(\omega) \in H^{q-1}(BG_d, \mathbf{R}/\mathbf{Z})$.*

Proof. The G-invariant form ω corresponds to an element $\omega_0 \in \Lambda^{q-1}((\mathfrak{g}/\mathfrak{h})^*)^H$ which transgresses in the relative Weil algebra $W(G, H)$ to $P \in I^*(G)$; i.e., there is a $TP \in W(G, H)$ restricting to ω_0 and with $dTP = P$. Using the canonical connection $\bar{\theta}$ in the simplicial bundle $N\bar{G} \to NG$ we obtain the associated forms $TP(\bar{\theta}, \bar{\Omega})$ and $P(\bar{\Omega}^{q/2})$ in $A^*(N\bar{G}/H)$ and $A^*(NG)$ respectively (for q odd, $P = 0$).

Now similar to the construction of a q-filling we construct a "section" ρ of $N\bar{G}/H \to NG$ over the $(q-1)$-skeleton; i.e., for each l, k with $l + k \leq q - 1$

and $\rho\colon \Delta^k \to NG(l)$ a singular simplex we obtain a commutative diagram

$$\begin{tikzcd} & \Delta^l \times (N\bar{G}/H)(l) \arrow[d, "\pi"] \\ \Delta^l \times \Delta^k \arrow[ur, "\rho_\tau"] \arrow[r, "\mathrm{id}\times\tau"] & \Delta^l \times NG(l) \end{tikzcd}$$

such that the family $\{\rho_\tau\}$ satisfies certain obvious compatibility relations. For $l + k = q$ we then have a similar diagram with ρ defined on $\partial(\Delta^l \times \Delta^k)$. In that case the evaluation of the cohomology class $-u$ on $[\tau] \in C_q(\|NG\|, \mathbf{Z})$ is the obstruction to deforming ρ to the trivial section i of the bundle pulled back by $(\mathrm{id} \times \tau)$ to $\Delta^l \times \Delta^k$. That is

$$\begin{aligned} -\langle u, [\tau]\rangle &= \int_{\partial(\Delta^l\times\Delta^k)} \rho^* TP(\bar{\theta}, \bar{\Omega}) - \int_{\partial(\Delta^l\times\Delta^k)} i^* TP(\bar{\theta}, \bar{\Omega}) \\ &= \langle \mathscr{I}(\rho^* TP(\bar{\theta}, \bar{\Omega})), \partial[\tau]\rangle - \int_{(\Delta^l\times\Delta^k)} i^* dTP(\bar{\theta}, \bar{\Omega}) \\ &= \langle \delta(\mathscr{I}(\rho^* TP(\bar{\theta}, \bar{\Omega}))), [\tau]\rangle - \int_{(\Delta^l\times\Delta^k)} (\mathrm{id}\times\tau)^* P(\bar{\Omega}^{q/2}) \\ &= \langle \delta(\mathscr{I}(\rho^* TP(\bar{\theta}, \bar{\Omega}))), [\tau]\rangle - \langle \mathscr{I}(P(\bar{\Omega}^{q/2})), [\tau]\rangle, \end{aligned}$$

and hence in $C^q(\|NG\|, \mathbf{R})$,

$$\mathscr{I}(P(\bar{\Omega}^{q/2})) - ru = \delta(\mathscr{I}(\rho^* TP(\bar{\theta}, \bar{\Omega}))). \tag{5.4}$$

Now taking $l = 0$, $TP(\bar{\theta}, \bar{\Omega})$ reduces to $\omega \in \mathscr{A}^{q-1}(G/H)$ and the family $\{\rho_\tau\}$ corresponds to a $(q-1)$-filling; so for this filling the theorem follows directly from (5.4). But since $\mathscr{I}(\omega)$ is independent of the choice of filling, the proof is finished. ■

Remark. If $q = 2l$, then $ru = w(P)$ for $P \in \ker(I^l(G)) \to I^l(H))$ and ω can be described explicitly in a way similar to $\sigma P(\theta)$ in Theorem 5.2 (for G/H a symmetric space it is exactly the form of $\sigma P(\theta)$).

COROLLARY 5.5. *Let $G = SO(n)$ and $V_{n,n-k} = SO(n)/SO(k)$ the Stiefel manifold with k even. Let ω be the invariant k-form representing the real image of the generator of $H^k(V_{n,n-k}, \mathbf{Z}) \cong \mathbf{Z}$. Then for any choice of k-filling we have that $\mathscr{I}(\omega)$ represents the image of the Stiefel-Whitney class $w_k \in H^k(BSO(n)_d, \mathbf{Z}/2)$ under the map induced by the inclusion $\mathbf{Z}/2 \to \mathbf{R}/\mathbf{Z}$ sending the generator to $\frac{1}{2}$.*

Proof. It follows immediately from Theorem 5.3 that $\mathscr{I}(\omega)$ in this case is $\beta^{-1}(w_{k+1}) = w_k$. ■

Remarks. (1) On $S^{n-1} = SO(n)/SO(n-1)$ it is possible to choose an $(n-1)$-filling consisting of geodesic simplices. In particular for n odd this gives a cochain for w_{n-1} expressed in terms of the volume of such simplices. This is exactly dual to the description of the Euler class for flat $SO(n,1)$-bundles, n even, using the volume of geodesic simplices in hyperbolic space. However, in general for G/H there seems not to be any canonical way of constructing q-fillings as in the dual situation of a non-compact symmetric space.

(2) The principle of Theorem 5.2 can be generalized to give cochain formulas for second order real characteristic classes associated to flat Diff(M)-bundles, M any compact manifold (see e.g. Bott [Bot] or Brooks [Br] for the case of the Godbillon-Vey invariant). We shall elsewhere deal with this and similar formulas for Cheeger-Chern-Simons classes using the principle of Theorem 5.3.

References

[Be1] A.J. Berrick, *An approach to algebraic K-theory*, Research Notes in Math., vol. 56, Pitman, Boston, 1982.

[Be2] ______, *Characterization of plus-constructive fibrations*, Adv. in Math., vol. 48 (1983), pp. 172–176.

[Bor] A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup., vol. 7 (1974), 235–272.

[Bot] R. Bott, "On some formulas for the characteristic classes of group actions" in *Differential topology, foliations and Gelfand-Fuks cohomology*, ed. P. A. Schweitzer, Lecture Notes in Mathematics, no. 652, Springer Verlag, New York, 1978, pp. 25–61.

[Br] R. Brooks, *Volumes and characteristic classes of foliations*, Topology, vol. 18 (1979), 295–304.

[Cr] J. Cheeger, *Invariants of flat bundles*, Proceedings of the International Congress of Mathematicians, Vancouver, 1974, pp. 3–6.

[Cr-Si] J. Cheeger and J. Simons, "Differential characters and geometric invariants" in *Geometry and topology, Proc. Special Year Univ. of Maryland* 1983–84, eds. J. Alexander and J. Harer; Lecture Notes in Mathematics, no. 1167, Springer Verlag, New York, 1985, pp. 50–80.

[Cn-Si] S.-S. Chern and J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. (2), vol. 99 (1974), pp. 48–69.

[Cv] C. Chevalley, *Theory of Lie groups I*, Princeton Univ. Press, Princeton, N.J., 1946.

[D1] J.L. Dupont, *Simplicial DeRham cohomology and characteristic classes of flat bundles*, Topology, vol. 15 (1976), pp. 233–245.

[D2] ______, *Curvature and characteristic classes*, Lecture Notes in Mathematics, no. 640, Springer-Verlag, New York, 1978.

[D-Ha-Z] J.L. Dupont, R. Hain and S. Zucker, *Regulators and characteristic classes of flat bundles*, in preparation.

[F] A. Floer, *An instanton-invariant for 3-manifolds*, Comm. Math. Phys., vol. 118 (1988), pp. 215–240.

[G-M] P.A. GRIFFITHS and J.W. MORGAN, *Rational homotopy theory and differential forms*, Progress in Mathematics, no. 16, Birkhäuser Verlag, Boston, 1981.

[Hae] A. HAEFLIGER, "Whitehead products and differential forms" in *Differential topology, foliations and Gelfand-Fuks cohomology*, ed. P. A. Schweitzer, Lecture Notes in Mathematics, no. 652, Springer Verlag, New York, 1978, pp. 13–24.

[Hal] S. HALPERIN, *Lectures on minimal models*, Univ. de Lille I, preprint, 1977. Also Mém. Soc. Math. France (N.S.) 9/10, 1983.

[Hu1] S.E. HURDER, *Dual homotopy invariants of G-foliations*, Topology, vol. 20 (1981), 365–387.

[Hu2] ______, *On the secondary classes of foliations with trivial normal bundle*, Comment. Math. Helv., vol. 56 (1981), pp. 307–326.

[Hu-K] S.E. HURDER and F.W. KAMBER, "Homotopy invariants of foliations" in *Topology Symposium*, Siegen 1979, eds. U. Koschorke and W. D. Neumann, Lecture Notes in Mathematics, no. 788, Springer Verlag, New York, 1980, pp. 49–61.

[K-T1] F. W. KAMBER and PH. TONDEUR, *Cohomologie des algèbres de Weil relatives tronquées*, C. R. Acad. Sci. Paris, vol. 276 (1973), pp. 459–462.

[K-T2] ______, *Foliated bundles and characteristic classes*, Lecture Notes in Mathematics, vol. 493, Springer-Verlag, New York, 1975.

[K-T3] ______, *Characteristic classes and Koszul complexes*, Proc. Symp. Pure Math., vol. 32 (1978), pp. 159–166.

[K-T4] ______, *Semi-simplicial Weil algebras and characteristic classes*, Tôhoku Math. J., vol. 30 (1978), pp. 373–422.

[Ka] M. KAROUBI, *Homologie cyclique et K-Théorie*, Astérisque, vol. 149, Société Mathématique de France, 1987.

[Sh-Ti] H. SHULMAN and D. TISCHLER, Leaf invariants for foliations and the Van Est isomorphism, *J. Differential Geometry*, *vol.* 11 (1976), *pp.* 535–546.

[So] C. SOULÉ, *Connexions et classes caractéristiques de Beilinson*, Université de Paris VII, prépublication, 1986.

[Su] D. SULLIVAN, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math., vol. 47 (1977), pp. 269–331.

AARHUS UNIVERSITET
AARHUS, DENMARK
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS