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INTERPOLATION SETS FOR LIPSCHITZ FUNCTIONS ON CURVES OF THE UNIT SPHERE

BY

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Introduction and statement of results

Let B denote the unit ball in $\mathbf{C}^n$, and S its boundary. For $\alpha \in (0,1]$, $Lip_\alpha(B)$ will denote the space of holomorphic functions in B satisfying a Lipschitz condition of order α with respect to the Euclidean distance. For a closed set $I \subset \mathbf{R}$, and $0 < \alpha < 1$, $\Lambda_\alpha(I)$ will denote the space of Lipschitz functions on I, and $\Lambda_1(I)$ the Zygmund class.

We also consider the Koranyi pseudodistance $d(z,w) = |1 - \bar{z}w|$, for $z, w \in S$, where

$$\bar{z}w = \sum_{i=1}^{n} \bar{z}_i w_i.$$

This defines a pseudodistance only on S, but we will as well consider it when one of the two variables is on $\overline{B}$.

We will work with a simple (without intersections) periodic curve of class C^3 γ: $\mathbf{R} \to S$. With a suitable parametrization (arc-length plus a dilatation) we will suppose from now on that γ is 2π-periodic and that there exists $\lambda > 0$ such that for each t, $|\gamma'(t)|^2 = \lambda$. We will write $I = [-\pi, \pi]$ and $\Gamma = \gamma(I)$. We will not distinguish between $\gamma(t)$ and its corresponding parameter on I.

Related to γ we introduce the *index of transversality*, T: $I \to \mathbf{R}$, given by

$$(1) \qquad -iT(x) = \overline{\gamma'(x)}\,\gamma(x), \quad x \in I.$$

Complex-tangential curves (i.e. $\gamma'(t) \in P_{\gamma(t)}$ where $P_{\gamma(t)}$ is the complex-tangential space at $\gamma(t)$) correspond to $T = 0$ and transverse curves to $|T(x)| \geq M$. We introduce the set E of *complex-tangential* points of Γ, given by

$$(2) \qquad E = \gamma(\{x \in I / T(x) = 0\}) = \{\zeta \in I / T_\zeta \Gamma \subset P_\zeta\}$$

where $T_\zeta \Gamma$ is the tangent space of Γ at ζ.

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As it is well known, complex-tangential directions are, in certain sense, twice as regular as the others. In fact (see [R] and [S1]), it can be proved that the restriction to E of each function in $Lip_\alpha(B)$, $\alpha < 1$ is in $\Lambda_{2\alpha}(E)$. Thus it is natural to consider the following definition.

DEFINITION. A closed set $E \subset S$ is an interpolation set for $Lip_\alpha(B)$ $(0 < \alpha < 1)$, if for any $f \in \Lambda_{2\alpha}(E)$, there exists $F \in Lip_\alpha(B)$ such that $F_{|E} = f$.

Now we can state our main result.

THEOREM A. *If E is the set of complex-tangential points of Γ and $\alpha \in (0, 1)$, $\alpha \neq \frac{1}{2}$ then there exists S: $\Lambda_{2\alpha}(E) \to Lip_\alpha(B)$ a linear operator, satisfying $Sf_{|E} = f$ for each $f \in \Lambda_{2\alpha}(E)$. In particular, E is an interpolation set for $Lip_\alpha(B)$.*

In case γ is of class C^∞, we deduce from the proof of theorem A, that E is also a peak set for $A^\infty(B)$, the algebra of holomorphic functions in B, of class $C^\infty(\overline{B})$. This result is also a consequence of [F-H], where it has been proved that the peak sets and the local peak sets for $A^\infty(B)$ coincide, and from the fact (see [R]) that E is locally included in complex-tangential manifolds.

As an immediate corollary to theorem A, we obtain:

COROLLARY A. *For each $\alpha < 1$, $\alpha \neq \frac{1}{2}$, $Lip_\alpha(B)_{|E} = \Lambda_{2\alpha}(E)$.*

These results have already been proved by [B-O] for complex-tangential curves. Our approach in proving them is completely different from the one used in [B-O], and permits for complex-tangential curves to obtain the extreme case $\alpha = \frac{1}{2}$, which was not covered by the methods of [B-O]. More precisely, the following holds:

COROLLARY B. *If Γ is complex-tangential, there exists S: $\Lambda_1(\Gamma) \to Lip_{1/2}(B)$ a linear operator satisfying $Sf_{|\Gamma} = f$ for each $f \in \Lambda_1(\Gamma)$. In particular, E is an interpolation set for $Lip_{1/2}(B)$, and $Lip_{1/2}(B)_{|\Gamma} = \Lambda_1(\Gamma)$.*

The proof of theorem A is essentially different for $\alpha < \frac{1}{2}$ and $\alpha > \frac{1}{2}$ and we will use some of the techniques introduced in [N].

In case $\alpha > \frac{1}{2}$, the function $Sf \circ \gamma$, where $f \in \Lambda_{2\alpha}(E)$ is differentiable at each point of E (see [R]), and we obtain an explicit formula for the derivative, in terms of the index of transversality T already introduced.

Finally, Corollary B will be deduced from the proof of Theorem A, using real interpolation methods.

As final remarks on notation, we denote C all the constants, that may change from one occurrence to another, we write $x \preceq y$ or $x = O(y)$ if there exists $M > 0$ such that $x \leq My$ and $x \simeq y$ if $x \preceq y$ and $y \preceq x$.

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1. Preliminary results. The case $\alpha < \frac{1}{2}$

In this section we will give the principal tools we need, as well as the proof of Theorem A, when $\alpha < \frac{1}{2}$. We omit the proof of the following elementary lemma.

LEMMA 1. $|\gamma(x) - \gamma(y)| \simeq |x - y|$, *provided* $x \in I$, *and* $|x - y| \leq \pi$.

The next result gives an estimate of the Koranyi pseudodistance in a neighborhood of the curve Γ, in terms of a suitable projection.

LEMMA 2. *There exists* $\varepsilon > 0$ *so that for each* z *in* $U = \{z \in \overline{B}/d(z, \Gamma) < \varepsilon\}$ *there exists a unique* $\gamma(x_z) \in \Gamma$, $|x_z| \leq \pi$, *with* $\operatorname{Re} \overline{\gamma'(x_z)}\, z = 0$, *and such that*

$$(3) \quad \left|1 - \overline{\gamma(x)}\, z\right| \simeq \operatorname{Re}\left(1 - \overline{\gamma(x_z)}\, z\right) + \left|\operatorname{Im}\left(1 - \overline{\gamma(x_z)}\, z\right) + T(x_z)(x - x_z)\right| + |x - x_z|^2,$$

provided $|x - x_z| \leq \pi$. *In particular,*

$$\left|1 - \overline{\gamma(x)}z\right| \preceq \left|1 - \overline{\gamma(x_z)}\, z\right| + |T(x_z)|\, |x - x_z| + |x - x_z|^2$$

Proof of Lemma 2. Given $z \in \overline{B}$, let $\gamma(x_z)$ be a point in Γ where the Euclidean distance from z to Γ is attained. Then

$$\operatorname{Re} \overline{\gamma'(x_z)}\, (z - \gamma(x_z)) = 0,$$

and in particular, $\operatorname{Re} \overline{\gamma'(x_z)}z = 0$. Taking z closed enough to Γ, we also obtain the uniqueness of $\gamma(x_z)$.

Using Taylor's development,

$$(4) \qquad 1 - \overline{\gamma(x)}\, z = 1 - \overline{\gamma(x_z)}\, z - \left(\overline{\gamma'(x_z)}\, z\right)(x - x_z) - \left(\overline{\gamma''(x_z)}\, z\right)\frac{(x - x_z)^2}{2} + O\left(|x - x_z|^3\right).$$

Taking modules we get

(5)

$$\begin{aligned}\left|1-\overline{\gamma(x_z)}\,z\right| &\simeq \operatorname{Re}\left(1-\overline{\gamma(x_z)}\,z\right)-\operatorname{Re}\left(\overline{\gamma''(x_z)}\,z\right)\frac{(x-x_z)^2}{2}+O(|x-x_z|^3)\big|\\ &\quad+\Big|\operatorname{Im}\left(1-\overline{\gamma(x_z)}\,z\right)+T(x_z)(x-x_z)\\ &\quad-\operatorname{Im}\left(\overline{\gamma'(x_z)}\,(z-\gamma(x_z))\right)(x-x_z)\\ &\quad-\operatorname{Im}\left(\overline{\gamma''(x_z)}\right)\frac{(x-x_z)^2}{2}+O(|x-x_z|^3)\Big|\end{aligned}$$

This estimate and the fact that

$$\begin{aligned}\left|\operatorname{Im}\left(\overline{\gamma(x_z)}\,(z-\gamma(x_z))\right)(x-x_z)\right| &= O\left(|z-\gamma(x_z)|^2+|x-x_z|^2\right)\\ &= O\left(\operatorname{Re}\left(1-\overline{\gamma(x_z)}\,z\right)+|x-x_z|^2\right),\end{aligned}$$

give the upper estimate of (3).

By differentiating (1), there exists $\varepsilon > 0$ with $\sup_{d(z,\Gamma)\le\varepsilon} -\operatorname{Re}\overline{\gamma''(x)\,z} > 0$. Hence

(6)

$$\begin{aligned}\left|1-\overline{\gamma(x)}\,z\right| &\succeq \operatorname{Re}\left(1-\overline{\gamma(x_z)}\,z\right)+|x-x_z|^2\\ &\quad-O(|x-x_z|^3)+s\left|\operatorname{Im}\left(1-\overline{\gamma(x_z)}\,z\right)+T(x_z)(x-x_z)\right|\\ &\quad-s\Big|\operatorname{Im}\left(\overline{\gamma'(x_z)}\,(z-\gamma(x_z))\right)(x-x_z)\\ &\quad-\operatorname{Im}\left(\overline{\gamma''(x_z)}z\right)\frac{(x-x_z)^2}{2}+O(|x-x_z|^3)\Big|\\ &\succeq \operatorname{Re}\left(1-\overline{\gamma(x_z)}\,z\right)+|x-x_z|^2\\ &\quad-O(|x-x_z|^3)+s\Big|\operatorname{Im}\left(1-\overline{\gamma(x_z)}\,z\right)\\ &\quad+T(x_z)(x-x_z)\Big|-s\left(\frac{\|\gamma'\|^2}{2}|z-\gamma(x_z)|^2+\frac{|x-x_z|^2}{2}+\frac{\|\gamma''\|_\infty}{2}|x-x_z|^2\right),\end{aligned}$$

where $0 < s \le 1$. Choosing s conveniently, we may thus obtain the estimate from below, whenever $|x - x_z| \le \delta$ ($\delta > 0$ small enough). And this finishes the lemma, since in $|x - x_z| \ge \delta$ both quantities that appear in (3) do not vanish (see Lemma 1). ◇

For $p > 0$, let h_p be as in [N] the holomorphic function in B given by

$$h_p(z) = \int_I \frac{dx}{\left(1 - \overline{\gamma(x)}\, z\right)^p}, \quad z \in \bar{B} \setminus \Gamma. \tag{7}$$

Then the following estimate of h_p holds:

Lemma 3. *Let $p \in (0, 1)$. Then*

$$\operatorname{Re} h_p(z) \simeq \left|h_p(z)\right| \simeq \int_I \frac{dx}{\left|1 - \overline{\gamma(x)}\, z\right|^p}. \tag{8}$$

Proof of Lemma 3. If $x \in I$ and $z \in B$, then $\operatorname{Re}(1 - \overline{\gamma(x)}\, z) > 0$. Hence

$$\operatorname{Re}\left(1 - \overline{\gamma(x)}\, z\right)^p \simeq \left|1 - \overline{\gamma(x)}\, z\right|^p,$$

for each $p \in (0, 1)$. ◇

We can now state a result concerning the boundary behavior of h_p.

Proposition 1. *If $p \in (\frac{1}{2}, 1)$, then*

$$\left|h_p(z)\right| \simeq \left(r_z + T(x_z)^2\right)^{1/2-p}, \quad z \in \bar{B} \setminus E \tag{9}$$

where $r_z = |1 - \overline{\gamma(x_z)}\, z|$.

Proof of Proposition 1. The estimate from below of (9) follows from Lemma (3) of Lemma 2, whereas the upper estimate is a consequence of Lemma 2 and the following technical result concerning real integrals (putting $a_z = \operatorname{Re}(1 - \overline{\gamma(x_z)}\, z)$, $b_z = \operatorname{Im}(1 - \overline{\gamma(x_z)}\, z)$, $T = T(x_z)$).

Lemma 4. *Let $p \in (\frac{1}{2}, 1)$ and $a \geq 0$, $b, T \in \mathbf{R}$, satisfying $a + |b| + T^2 > 0$. Then*

$$\int_{-\infty}^{+\infty} \frac{dx}{\left(a + |b + Tx| + x^2\right)^p} = O\left(\left(a + |b| + T^2\right)^{1/2-p}\right). \tag{10}$$

Proof of Lemma 4. Denoting by $I_p(a, b, T)$ the integral of (10), it is immediate to verify that for each $\lambda > 0$, $I_p(\lambda a, \lambda b, \lambda^{1/2} T) = \lambda^{1/2-p} I_p(a, b, T)$. Hence, it suffices to prove that the function I_p is continuous on the compact set

$$K = \{(a,b,T)/0 \le a \le 1, b, T \in [-1,1], a + |b| + |T|^2 = 1\}.$$

And this continuity is a consequence of the finiteness of the integral near zero ($p < 1$), and near ∞ ($p > \frac{1}{2}$). $\diamond$

Remark 1. Defining

$$S_p f(z) = \int_I K_p(x,z) f(x)\, dx, \quad f \in C(\Gamma), \tag{11}$$

where

$$K_p(x,z) = \frac{1}{h_p(z)} \frac{1}{\left(1 - \overline{\gamma(x)\, z}\right)^p},$$

Lemma 3 and Proposition 1, give that $S_p f \in A(B)$ and interpolates f on E, so that E is an interpolation set for $A(B)$. This could be also deduced from [N], since the set of complex tangential points of Γ is locally included in complex-tangential manifolds of the unit sphere (see [R]).

We need to obtain estimates of the radial derivative Rh_p of the function h_p. However, the methods used until now, based on the projection already obtained in Lemma 2, do not permit to get the desired bounds (essentially because the exponent of $(1 - \overline{\gamma(x)\, z})$ in Rh_p is greater than one). The following lemma gives a more accurate projection onto the curve.

LEMMA 5. *There exists $\varepsilon > 0$ so that for each z in $U = \{z \in \overline{B}/d(z,\Gamma) < \varepsilon\}$, there exists $\gamma(x_z^*) \in \Gamma$, $|x_z^*| \le \pi$, and $m_z = m(z) \in \mathbf{R}$, such that*

$$\left|1 - \overline{\gamma(x)\, z}\right| \simeq m_z + \left|T(x_z^*)\right| |x - x_z^*| + |x - x_z^*|^2, \tag{12}$$

provided $|x - x_z| \le \pi$, where

$$m_z \simeq \inf_{x \in I}\left|1 - \overline{\gamma(x)\, z}\right| \quad \text{and} \quad \left(m_z + T(x_z^*)^2\right) \simeq \left(r_z + T(x_z)^2\right).$$

Proof of Lemma 5. In Lemma 2 we have obtained a point $\gamma(x_z) \in \Gamma$, such that

$$\left|1 - \overline{\gamma(x)\, z}\right| \simeq \operatorname{Re}\left(1 - \overline{\gamma(x_z)\, z}\right) + \left|\operatorname{Im}\left(1 - \overline{\gamma(x_z)\, z}\right) + T(x_z)(x - x_z)\right| + |x - x_z|^2 \tag{13}$$

Without loss of generality, we can suppose that $x_z = 0$. We let

$$a_z = \operatorname{Re}\left(1 - \overline{\gamma(0)}\, z\right), \quad b_z = \operatorname{Im}\left(1 - \overline{\gamma(0)}\, z\right),$$

and distinguish two possibilities:

(i) $$T(0)^2 \leq Cr_z,$$

(ii) $$T(0)^2 > Cr_z$$

where $C > 0$ is a constant that will be fixed later. In case (i), it suffices to choose $x_z^* = 0$, $m_z = r_z$. Writing

$$\left|1 - \overline{\gamma(x)}\, z\right| \preceq m_z + |T(0)|\, |x| + x^2,$$

we get the upper estimate of (12). For the estimate from below, it suffices to choose $\varepsilon > 0$ and $k \geq 1$ such that

$$\varepsilon\left(r_z + |T(0)|\, |x|\right) \leq k\left(x^2 + a_z + |b_z + T(0)x|\right), \tag{14}$$

since $a_z + |b_z + T(0)| + x^2 \geq x^2$. And (14) will follow if there exists $\varepsilon > 0$ and $k \geq 1$ with

$$kx^2 + (1 - \varepsilon)r_z \geq (1 + \varepsilon)|T(0)|\, |x| \tag{15}$$

If ε and η are chosen satisfying $(1 + \varepsilon)C = \eta(1 - \varepsilon)$, then from (i) we obtain

$$(1 + \varepsilon)|T(0)|\, |x| \leq (1 + \varepsilon)\eta x^2 + (1 - \varepsilon)r_z,$$

estimate that gives (15) with $k = (1 + \varepsilon)\eta$. That finishes the case (i). For (ii), let x_z^* be a real number satisfying

$$b_z + T(0)x_z^* = 0 \tag{16}$$

Choosing C sufficiently big, formula (16) gives that x_z^* is in I. From (16) we also get that

$$a_z + |b_z + T(0)x| + x^2 = a_z + |T(0)|\, |x - x_z^*| + x^2 \tag{17}$$

If $g(x)$ is the function on the right hand side of (17), and $C \geq 2$, then $\min_{x \in I} g(x) = g(x_z^*)$. We then define

$$m_z = g(x_z^*) = a_z + x_z^{*2}. \tag{18}$$

Since $T(0)$ and $T(x_z^*)$ differs in a term which is $O(|x_z^*|)$, the estimate (12) will follow once we have proved

$$\left|1-\overline{\gamma(x)}\,z\right| \simeq m_z + |T(0)|\,|x-x_z^*| + |x-x_z^*|^2,$$

which is a consequence (using (13) and (16)) of

$$(19)\quad a_z + |T(0)|\,|x-x_z^*| + x^2 \simeq m_z + |T(0)|\,|x-x_z^*| + |x-x_z^*|^2.$$

The definition of m_z already gives the upper estimate of (19), whereas the estimate from below is also a consequence of the definition of m_z, since

$$m_z + |x-x_z^*|^2 \preceq m_z + x^2 + x_z^{*2} \preceq m_z + x^2 \preceq a_z + |T(0)|\,|x-x_z^*| + x^2,$$

and that finishes the proof of (12).

From (12) we also deduce that $m_z \simeq \inf_{x\in I}|1-\overline{\gamma(x)}\,z|$. For the second relation we write (notice that we are in the case $T(0)^2 > Cr_z$),

$$m_z + T(x_z^*)^2 \preceq a_z + x_z^{*2} + T(0)^2 \preceq a_z + \frac{r_z^2}{|T(0)|^2} + |T(0)|^2 \preceq r_z + T(0)^2.$$

The converse estimate is proved in an analogous way. ◇

Remark 2. From the last proof we obtain in particular, that if Γ is complex-tangential then Lemma 5 is valid with $x_z^* = x_z$, $m_z = r_z$.

Now we can state the result concerning the behavior of the radial derivatives of the function h_p.

PROPOSITION 2. *Suppose $p \in (\frac{1}{2}, 1)$. Then for $z \in U \setminus E$,*

$$(20)\qquad Rh_p(z) = O\left(\left(r_z + T(x_z)^2\right)^{-1/2-p}\right).$$

Proof of Proposition 2. We need a technical lemma concerning real integrals.

LEMMA 6. *Let $\alpha > 0$, $q > 0$, $m \geq 0$, $T \in \mathbf{R}$, with $m + T^2 \neq 0$. Then*

$$(21)\qquad \int_0^{\pi} \frac{x^\alpha}{(m + |T|x + x^2)^q}\,dx$$

$$= \begin{cases} O\left(m^{\alpha-q+1}(T^2+m)^{-(\alpha+1)/2}\right), & \text{if } \alpha < q-1 \quad \text{(i)} \\ O\left((T^2+m)^{(\alpha+1)/2-q}\right) & \text{if } q-1 < \alpha < 2q-1 \quad \text{(ii)} \end{cases}$$

Proof of Lemma. We will prove (i) first. If

$$\varepsilon = \frac{2m}{|T| + \sqrt{T^2 + 4m}},$$

we break the integral on the left hand side of (21) in two parts, corresponding to $x \le \varepsilon$ and $x \ge \varepsilon$. Hence

$$\begin{aligned}\int_0^{\pi} &\frac{x^\alpha}{(m + |T|x + x^2)^q}\,dx \\ &= \int_0^{\varepsilon} \frac{x^\alpha}{(m + |T|x + x^2)^q}\,dx + \int_{\varepsilon}^{\pi} \frac{x^\alpha}{(m + |T|x + x^2)^q}\,dx \\ &= I + II.\end{aligned}$$

Since

$$\varepsilon \simeq \frac{m}{(m + T^2)^{1/2}}$$

and in $x \le \varepsilon$, $m + |T|x + x^2 \simeq m$, I is bounded as follows:

$$I \preceq \int_0^{\varepsilon} \frac{x^\alpha}{m^q}\,dx \simeq m^{-q}\varepsilon^{\alpha+1} \simeq m^{\alpha-q+1}(T^2 + m)^{\alpha+1/2}, \tag{22}$$

which is an estimate like (i). In $x \ge \varepsilon$, $m + |T|x + x^2 \simeq |T|x + x^2$. Hence, II is bounded as follows:

$$\begin{aligned} II &\preceq \int_{\varepsilon}^{\pi} \frac{x^\alpha}{(|T|x + x^2)^q}\,dx \\ &= |T|^{\alpha-2q+1} \int_{\varepsilon/|T|}^{\pi/|T|} x^{\alpha-q}(1 + x)^{-q}\,dx. \end{aligned} \tag{23}$$

In order to estimate (23), we distinguish two possibilities

(a) $\qquad T^2 \le 4m,$

(b) $\qquad T^2 > 4m.$

In case (a), $\varepsilon \simeq \sqrt{m}$, and in consequence (23) is bounded by

$$|T|^{\alpha-2q+1}\int_{\varepsilon/|T|}^{\pi/|T|} x^{\alpha-2q}\,dx = O(\varepsilon^{\alpha-2q+1}),$$

which also implies (i) (notice that in case (a), $T^2 + m \simeq m$). In case (b), $\varepsilon \simeq m/|T|$. Hence, (23) is bounded by

$$|T|^{\alpha-2q+1}\left\{\int_{\varepsilon/|T|}^{1} x^{\alpha-q}\,dx + \int_{1}^{\pi/|T|} x^{\alpha-2q}\,dx\right\}$$
$$= |T|^{\alpha-2q+1}\left(\left(\frac{\varepsilon}{|T|}\right)^{\alpha-q+1} + O(1)\right) \simeq |T|^{-q}\varepsilon^{\alpha-q+1},$$

an estimate that also gives (i).

For (ii), it suffices to use a similar argument of the one done in Lemma 4 since the condition $q-1<\alpha<1$ gives the finiteness of the integral of (21), near 0 and near ∞. ◇

Following with the proof of Proposition 2, from the definition of h_p we obtain

$$Rh_p(z) = p\int_I \frac{\overline{\gamma(x)}z}{(1-\overline{\gamma(x)}z)^{p+1}}\,dx, \tag{24}$$

Applying (i) of Lemma 6 to $m = m_z$, $T = T(x_z^*)$, we obtain

$$|Rh_p(z)| = O\left(m_z^{-p}\left(m_z + T(x_z^*)^2\right)^{-1/2}\right), \tag{25}$$

and the problem is that in general m_z and $(m_z + T(x_z^*)^2)$ are not of the same type. That is why we distinguish between two possibilities

(i) $\qquad T(x_z)^2 \le Cr_z,$

(ii) $\qquad T(x_z)^2 > Cr_z$

where $C>0$ will be chosen later. In case (i), $r_z \simeq r_z + T(x_z)^2$, and since then $x_z^* = x_z$ and $m_z = r_z$ (see proof of Lemma 5), the estimate (25) gives (20). Hence it suffices to prove (20) in case (ii). Breaking the integral defining h_p in two parts,

$$h_p(z) = \int_{I_z}\frac{dx}{\left(1-\overline{\gamma(x)}\,z\right)^p} + \int_{I\setminus I_z}\frac{dx}{\left(1-\overline{\gamma(x)}\,z\right)^p} = I + II,$$

where $\varepsilon > 0$ will be fixed later, and $I_z = \{x/|x - x_z| \leq \varepsilon|T(x_z)|\}$, we will estimate both parts separately. If ε is sufficiently small and C is chosen conveniently, then in I_z,

$$|\overline{\gamma'(x)\, z}| \simeq |T(x_z)| \tag{26}$$

since

$$\begin{aligned}
\overline{\gamma'(x)\, z} &= \overline{\gamma'(x_z)}\, \gamma(x_z) + \left(\overline{\gamma'(x)} - \overline{\gamma'(x_z)}\right)\gamma(x_z) + \overline{\gamma'(x)}\,(z - \gamma(x_z)) \\
&= -iT(x_z) + O(|x - x_z|) + O\left(r_z^{1/2}\right) \\
&= -iT(x_z) + \left(\varepsilon + \frac{1}{C^{1/2}}\right) O\left(|T(x_z)|\right).
\end{aligned}$$

In particular, $\overline{\gamma'(x)}z \neq 0$ in I_z. Integrating I by parts,

$$\begin{aligned}
&\int_{I_z} \frac{dx}{\left(1 - \overline{\gamma(x)\, z}\right)^p} \\
&\qquad = \frac{1}{p-1}\left[\frac{1}{\left(\overline{\gamma'(x)\, z}\right)\left(1 - \overline{\gamma(x)\, z}\right)^{p-1}}\right]_{x_z - \varepsilon|T(x_z)|}^{x_z + \varepsilon|T(x_z)|} \\
&\qquad\quad + \frac{1}{p-1}\int_{I_z} \frac{\overline{\gamma''(x)\, z}}{\left(\overline{\gamma'(x)\, z}\right)^2} \frac{dx}{\left(1 - \overline{\gamma(x)\, z}\right)^{p-1}} \\
&= \frac{1}{p-1} \frac{1}{\left(\overline{\gamma'\left(x_z \pm \varepsilon|T(x_z)|\right) z}\right)\left(1 - \overline{\gamma\left(x_z \pm \varepsilon|T(x_z)|\right) z}\right)^{p-1}} \\
&\quad + \frac{1}{p-1}\int_{I_z} \frac{\overline{\gamma''(x)\, z}}{\left(\overline{\gamma'(x)\, z}\right)^2} \frac{dx}{\left(1 - \overline{\gamma(x)\, z}\right)^{p-1}} \\
&= h_p^1(z)
\end{aligned}$$

It can be easily checked that there exists $r_0 > 0$ such that for every point rz, $r \geq r_0$, in the radius joining 0 and z, the pseudodistance from z to Γ is attained at $\gamma(x_z)$. Hence, it does make sense to take the radial derivative of

h_p^1, and we obtain

$$\begin{aligned}
Rh_p^1(z) &= \frac{1}{p-1}\Bigg[\mp \frac{1}{\left(\overline{\gamma'\left(x_z \pm \varepsilon|T(x_z)|\right) z}\right)\left(1-\overline{\gamma\left(x_z \pm \varepsilon|T(x_z)|\right) z}\right)^{p-1}} \\
&\quad \pm \frac{(p-1)\overline{\gamma\left(x_z \pm \varepsilon|T(x_z)|\right) z}}{\left(\overline{\gamma'\left(x_z \pm \varepsilon|T(x_z)|\right) z}\right)\left(1-\overline{\gamma\left(x_z \pm \varepsilon|T(x_z)|\right) z}\right)^{p}} \\
&\quad - \int_{I_z} \frac{\overline{\gamma''(x)\, z}}{\left(\overline{\gamma'(x)\, z}\right)^2} \frac{dx}{\left(1-\overline{\gamma(x)\, z}\right)^{p-1}} \\
&\quad + (p-1)\int_{I_z} \frac{\overline{\gamma''(x)\, z}}{\left(\overline{\gamma'(x)\, z}\right)^2} \frac{dx}{\left(1-\overline{\gamma(x)\, z}\right)^{p}} \\
&= (1)+(2)+(3)+(4).
\end{aligned}$$

By Lemma 2 and (26),

$$\begin{aligned}
(1) &\preceq \frac{\left(r_z + T(x_z)^2\right)^{1-p}}{|T(x_z)|} \simeq \left(r_z + T(x_z)^2\right)^{1/2-p}, \\
(2) &\preceq \frac{1}{|T(x_z)|^{2p+1}} \simeq \left(r_z + T(x_z)^2\right)^{-1/2-p}, \\
(3),(4) &\preceq \frac{1}{|T(x_z)|}\left(r_z + T(x_z)^2\right)^{1-p} \simeq \left(r_z + T(x_z)^2\right)^{1/2-p}, \\
(5) &\preceq \frac{1}{T(x_z)^2}\left(r_z + T(x_z)^2\right)^{-1/2-p},
\end{aligned}$$

where in (4) we have also used Lemma 6. Differentiating II and applying Lemmas 2 and 4, we obtain

$$\begin{aligned}
RII &\preceq \int_{I \setminus I_z} \frac{dx}{\left|1-\overline{\gamma(x)\, z}\right|^{p+1}} \\
&\preceq \frac{1}{T(x_z)^2}\int_{\mathbf{R}} \frac{dx}{\left|1-\overline{\gamma(x)\, z}\right|^{p}} \preceq \left(r_z + T(x_z)^2\right)^{-1/2-p},
\end{aligned}$$

an estimate which finishes the proof of the lemma. ◇

Remark 3. If γ is of class C^∞, the same kind of argument gives that

$$\left|R^k h_p(z)\right| = O\left(\left(r_z + T(x_z)^2\right)^{1/2-p-k}\right), \quad k \geq 0$$

and R^k is the k^{th}-iterated radial derivative. Then the function $F_p = e^{-h_p}$ is in $A^\infty(B)$ and (see Lemma 3) satisfies $|F_p| < 1$ on $\overline{B} \setminus E$, $F_p = 1$ on E. That is, E is a (P)-set for $A^\infty(B)$. This is also (as we have already said at the introduction) a consequence of [F-H], where it has been proved that the peak sets and the local peak sets for $A^\infty(B)$ coincide.

Now we can prove the following result.

THEOREM 1. *Let $\alpha \in (0, \frac{1}{2})$, and $p \in (\alpha + \frac{1}{2}, 1)$. Then $S_p f \in Lip_\alpha(B)$ for each $f \in \Lambda_{2\alpha}(\Gamma)$ and if $\gamma(x_0) \in E$, $S_p f(\gamma(x_0)) = f(x_0)$. In particular, if $S^p = S_p \circ \mathscr{E}^{(0)}$: $\Lambda_{2\alpha}(E) \to Lip_\alpha(B)$, where $\mathscr{E}^{(0)}$ is the linear extension operator given by Whitney's extension theorem* (*see* [S1]), *then S^p is a linear operator that gives the interpolation.*

Proof of Theorem 1. If $p \in \Lambda_{2\alpha}(\Gamma)$, and $p \in (\alpha + \frac{1}{2}, 1)$, we will see

$$\left|RS_p f(z)\right| = O\left((1 - |z|)^{\alpha - 1}\right), \tag{27}$$

a condition which implies that $S_p f$ lies in $Lip_\alpha(B)$ (see [R, page 107]). Since $\int_I K_p(x, z)\,dx = 1$ (see Remark 1), (27) is equivalent to

$$\left|\int_I (f(x) - f(x_z^*)) R_z K_p(x, z)\,dx\right| = O\left((1 - |z|)^{\alpha - 1}\right),$$

provided $z \in U$, and x_z^* is as in Lemma 5. But

$$\begin{aligned}
&\int_I (f(x) - f(x_z^*)) R_z K_p(x, z)\,dx \\
&\quad = R\left(\frac{1}{h_p(z)}\right) \int_I (f(x) - f(x_z^*)) \frac{1}{\left(1 - \overline{\gamma(x)}\,z\right)^p}\,dx \\
&\quad + \frac{1}{h_p(z)} \int_I (f(x) - f(x_z^*)) R_z\left(\frac{1}{\left(1 - \overline{\gamma(x)}\,z\right)^p}\right) dx = I + II,
\end{aligned}$$

and it suffices to obtain estimates like (27) for I and II. Propositions 1, 2, and

Lemma 5 give

$$R\left(\frac{1}{h_p(z)}\right) = O\left(\left(m_z + T(x_z^*)^2\right)^{p-3/2}\right).$$

Hence

$$\begin{aligned} I &\preceq \left(m_z + T(x_z^*)^2\right)^{p-3/2} \int_I \frac{|x - x_z^*|^{2\alpha}}{\left|1 - \overline{\gamma(x)}\, z\right|^p}\, dx \\ &\preceq \left(m_z + T(x_z^*)^2\right)^{\alpha-1}, \end{aligned}$$

where the last estimate is a consequence of Lemma 5, and of (ii) of Lemma 6 (notice that $p - 1 < \alpha < 2p - 1$).

Applying Proposition 2 and (i) of Lemma 6, we get that II is bounded as follows

$$\begin{aligned} II &\preceq \left(m_z + T(x_z^*)^2\right)^{p-1/2} \int_I \frac{|x - x_z^*|^{2\alpha}}{\left|1 - \overline{\gamma(x)}\, z\right|^{p+1}}\, dx \\ &\preceq \left(m_z + T(x_z^*)^2\right)^{p-1} m_z^{\alpha-p} \\ &\preceq (1 - |z|)^{\alpha-1}. \end{aligned}$$

Since $m_z \succeq 1 - |z|$ (see Lemma 5), this finishes the theorem. ◇

2. The case $\alpha \geq \frac{1}{2}$

In this section we will deal with functions in $\Lambda_{2\alpha}(E)$, $\alpha \geq \frac{1}{2}$. The techniques that we will use here are more involved than the ones used before. We will consider the functions h_p already introduced in last section, but now the parameter p needs to be greater than one. When $p < 1$, the estimate $\operatorname{Re} h_p(z) \simeq |h_p(z)| \simeq (r_z + T(x_z)^2)^{1/2-p}$ (which implies in particular that $\operatorname{Re} h_p > 0$ on $\overline{B} \setminus E$) was essential for the proof of the interpolation results. However, if $p > 1$ this global result does not hold, and it is necessary to use a different argument based on the ideas of [N].

With the same notations used since now, we introduce the functions

$$F(z, \gamma(x)) = 1 - \overline{\gamma(x)}\, z, \quad z \in \overline{B},\, x \in I, \tag{1}$$

$$G(z, x_z, x) = F(z, \gamma(x_z)) + iA(z)(x - x_z) + B(z)\frac{(x - x_z)^2}{2}, \quad z \in U,\, x \in \mathbf{R}, \tag{2}$$

where $A(z) = i\overline{\gamma'(x_z)}z$, $B(z) = -\overline{\gamma''(x_z)}z$. The following lemma holds.

LEMMA 1.
(i) $|F(z,\gamma(x)) - G(z,x_z,x)| = O(|x-x_z|^3)$,
(ii) $\operatorname{Re} G(z,x_z,x) \succeq |x-x_z|^2$,
(iii) $\exists\delta > 0$ *such that if* $|x-x_z| \le \delta$, $|F(z,\gamma(x))| \ge \frac{1}{2}|G(z,x_z,x)|$,
provided $z \in U$.

Proof of Lemma 1. Part (i) is a consequence of the definition of the functions F and G, since G is the second order Taylor's development of F.

Part (ii) is also a consequence of the definition of G, since $iA(z) = 0$ (see Lemma 1.2) and $\operatorname{Re} B(z)$ is bounded from below, provided z is closed enough to Γ.

Part (iii) follows from (i) and (ii). ◇

We need the following result that can be deduced from [N, Theorem 8], using an argument of analytic continuation and induction on k. We state it without proof.

THEOREM 1 [N]. *Let* $Z, B \in \mathbf{C}$, *with* $\operatorname{Re} Z > 0$, $\operatorname{Re} B > 0$, $A \in \mathbf{R}$, *and* $k \in \mathbf{N}$. *Then*

$$(3)\quad \int_{\mathbf{R}} \frac{x^k}{(Z + 2iAx + Bx^2)^p}\,dx$$
$$= (-1)^k A^k i^k B^{-k-1/2}\left(Z + \frac{A^2}{B}\right)^{1/2-p}$$
$$\times\left\{\sum_{j=0}^{[k/2]} \binom{k}{2j} A^{-2j} B^j i^{-2j}\left(Z + \frac{A^2}{B}\right)^j \delta_j(p) C(p-j)\right\},$$

provided $p > (k+1)/2$, *where*

$$\delta_j(p) = 1 \quad \text{if } j = 0,$$
$$\delta_j(p) = 2^{-j}\frac{(2j-1)\dots 1}{(p-1)\dots(p-j)} \quad \text{if } j \ge 1,$$

and where $C(p-j)$, *are constants only depending on p and j.*

Remark 1. Define the functions

$$H_p(z) = \int_{\mathbf{R}} \frac{dx}{G(z, x_z, x)^p}, \quad p > \frac{1}{2}, \tag{4}$$

$$G_p(z) = \int_{\mathbf{R}} \frac{(x - x_z)}{G(z, x_z, x)^p}\, dx, \quad p > 1, \tag{5}$$

$$J_p(z) = \int_{\mathbf{R}} \frac{(x - x_z)^2}{G(z, x_z, x)^p}\, dx, \quad p > \frac{3}{2}; \tag{6}$$

then, in particular, the last theorem gives the following formulas:

(7)

$$H_p(z) = 2^p C(p) B(z)^{-1/2} \left(2F(z, \gamma(x_z)) + \frac{A(z)^2}{B(z)}\right)^{1/2-p},$$

(8)

$$G_p(z) = -i 2^p C(p) B(z)^{-3/2} A(z) \left(2F(z, \gamma(x_z)) + \frac{A(z)^2}{B(z)}\right)^{1/2-p},$$

(9)

$$J_p(z) = 2^p B(z)^{-5/2} \left(2F(z, \gamma(x_z)) + \frac{A(z)^2}{B(z)}\right)^{1/2-p}$$
$$\times \left\{ \frac{C(p-1)}{2(p-1)} B(z) \left(2F(z, \gamma(x_z)) + \frac{A(z)^2}{B(z)}\right) - C(p) A(z)^2 \right\}$$

Remark 2. With a more careful calculation done in the proof of [N, Theorem 8], also based in analytic continuation, it can be proved that defining H_p^*, G_p^*, J_p^* similarly to H_p, G_p, J_p respectively, interchanging x_z and x_z^*, and defining

$$A^*(z) = i\overline{\gamma'(x_z^*)}z, \quad B^*(z) = -\overline{\gamma''(x_z^*)}z,$$

the corresponding formulas (7), (8) and (9) for H_p^*, G_p^*, J_p^* remain valid.

We can now prove a result concerning the behavior of the function h_p near E.

PROPOSITION 1. *Let* $p \in (1,3)$.
(i) *Then there exists a neighborhood* W *of* E *in* $\mathbf{C}^n$ *such that*

$$h_p(z) = \left(2F(z,\gamma(x_z)) + \frac{A(z)^2}{B(z)}\right)^{1/2-p} \times \left\{2^p C(p) B(z)^{-1/2} + o(1)\right\} \tag{10}$$

for $z \in (W \cap \bar{B}) \setminus E$ *and* $d(z,E) \to 0$, *where* $C(p)$ *is the constant given in* (7).
(ii) *In particular, from* (i) *we obtain*

$$|h_p(z)| \simeq \left(r_z + T(x_z)^2\right)^{1/2-p} \quad \text{if } z \in (W \cap \bar{B}) \setminus E.$$

(iii) *Furthermore, if* $p \in (1,2)$, *then*

$$|Rh_p(z)| = O\left(\left(r_z + T(x_z)^2\right)^{-1/2-p}\right) \quad \text{if } z \in (W \cap \bar{B}) \setminus E.$$

Proof of Proposition 1. We need the following Lemma

LEMMA 2. *Let*

$$Q(z) = \left(2F(z,\gamma(x_z)) + \frac{A(z)^2}{B(z)}\right).$$

Then

$$|Q(z)| \simeq \left(r_z + T(x_z)^2\right), \tag{11}$$

for z *close enough to* E.

Proof of Lemma 2. The upper estimate follows from the fact that $A(z)$ and $T(x_z)$ differ in a term which is $O(r_z^{1/2})$, since then $r_z + A(z)^2 \simeq r_z + T(x_z)^2$. The estimate from below of $|Q(z)|$ is deduced also using this fact and the fact that $\operatorname{Re} B(z) = -\operatorname{Re} \gamma''(x_z)\, z$ is greater than zero, if z is close enough to Γ. ◇

Following the proof of the proposition, formulas (7) and (11) of Lemma 2 show that it suffices to prove that

$$|h_p(z) - H_p(z)| = o\left(\left(r_z + T(x_z)^2\right)^{1/2-p}\right) \quad \text{if } d(z,E) \to 0. \tag{12}$$

Now we can write

$$\begin{aligned}|h_p(z) - H_p(z)| &\le \int_{\{x\in I/|x-x_z|\le\delta\}} \left\{ \frac{1}{\left(1-\overline{\gamma(x)}\,z\right)^p} - \frac{1}{G(z,x_z,x)^p} \right\} dx \\ &\quad + \int_{I\setminus\{|x-x_z|\le\delta\}} \frac{dx}{\left|1-\overline{\gamma(x)}\,z\right|^p} + \int_{\{|x-x_z|\ge\delta\}} \frac{dx}{|G(z,x_z,x)|^p} \\ &= I + II + III,\end{aligned}$$

where $\delta > 0$ is as in Lemma 1. As usual we will see that each of the three integrals satisfies an estimate like (12). The two last integrals II and III, are bounded, since on one hand $|1-\overline{\gamma(x)}z|$ is bounded from below in $I\setminus\{|x-x_z|\le\delta\}$, and on the other hand $p > \frac{1}{2}$ (in fact $p > 1$). For the first summand, an argument like the one done in [U] gives

$$I \le \int_{\{x\in I/|x-x_z|\le\delta\}} \frac{|F(z,\gamma(x)) - G(z,x_z,x)|}{|s(x)F(z,\gamma(x)) + (1-s(x))G(z,x_z,x)|^{p+1}}\,dx,$$

where $|s(x)| \le 1$. Now, by (i) and (iii) of Lemma 1, this integral is bounded by

$$\int_{\mathbf{R}} \frac{|x-x_z|^3}{|F(z,\gamma(x))|^{p+1}}\,dx,$$

which (see Lemma 1.2) is bounded by

$$\int_{\mathbf{R}} \frac{|x-x_z|^3}{\left(a_z + |b_z + T(x_z)(x-x_z)| + (x-x_z)^2\right)^{p+1}}\,dx$$

(here $a_z = \mathrm{Re}(1-\overline{\gamma(x_z)}z)$ $b_z = \mathrm{Im}(1-\overline{\gamma(x_z)}z)$). Finally, by Lemma 1.4, this integral is

$$O\left(\left(r_z + T(x_z)^2\right)^{1-p}\right).$$

The estimate of (ii) is a consequence of (i) and (11) of Lemma 2. Finally, part (iii) is a consequence of (ii), since

$$Rh_p(z) = -p\left(h_p(z) - h_{p+1}(z)\right). \quad \diamond \tag{13}$$

Remark 3. Notice that in (i) we have proved that

$$(14) \qquad |h_p(z) - H_p(z)| = O\Big(\big(r_z + T(x_z)^2\big)^{1-p}\Big).$$

and that the same holds if we replace H_p by H_p^*.

Notice also that if $S_pf(z)$ is defined as in Remark 1.1, but only when $z \in W \cap B$, then $S_pf|_E = f$ for $f \in \Lambda_{2\alpha}(\Gamma)$, and $p > 1$, $\alpha > 1/2$.

We can now state a result concerning the local behavior of the operator S_p.

THEOREM 2. *Let* $\alpha \in (\frac{1}{2}, 1)$ *and* $p \in (\frac{3}{2}, \alpha + 1)$. *Then*

$$(15) \qquad |RS_pf(z)| = O\big((1 - |z|)^{\alpha-1}\big) \quad \textit{if } f \in \Lambda_{2\alpha}(\Gamma),\ z \in W \cap B.$$

Proof of Theorem 2. By (13),

$$(16) \qquad RK_p(x, z) = \frac{ph_{p+1}(z)}{h_p(z)}\{K_p(x, z) - K_{p+1}(x, z)\}.$$

Hence, if $\alpha \in (\frac{1}{2}, 1)$, $p \in (\frac{3}{2}, \alpha + 1)$, $f \in \Lambda_{2\alpha}(\Gamma)$, $\alpha > \frac{1}{2}$, and z in $W \cap B$, equalities (13) and (16) give

$$\begin{aligned}(17) \quad RS_pf(z) &= \int_I (f(x) - f(x_z^*))RK_p(x, z)\,dx \\ &= -pf'(x_z^*)\frac{h_{p+1}(z)}{h_p(z)}\int_I (x - x_z^*)\{K_p(x, z) - K_{p+1}(x, z)\}\,dx \\ &\quad + \frac{h_{p+1}(z)}{h_p(z)}\int_I O(|x - x_z^*|^{2\alpha})K_p(x, z)\,dx \\ &\quad + \frac{h_{p+1}(z)}{h_p(z)}\int_I O(|x - x_z^*|^{2\alpha})K_{p+1}(x, z)\,dx \\ &= I + II + II,\end{aligned}$$

and we will prove that each one of the summands satisfies an estimate like (15). The estimate of I will follow once we prove

(18)

$$\left| h_{p+1}(z)\int_I \frac{(x - x_z^*)}{\big(1 - \overline{\gamma(x)}\,z\big)^p}\,dx - h_p(z)\int_I \frac{(x - x_z^*)}{\big(1 - \overline{\gamma(x)}\,z\big)^{p+1}}\,dx \right| = O\big(|h_p(z)|^2\big)$$

Defining the function

$$g_p^*(z) = \int_I \frac{(x - x_z^*)}{\left(1 - \overline{\gamma(x)}\, z\right)^p}\, dx, \tag{19}$$

Proposition 1.1 gives that (18) is equivalent to

$$\left|h_{p+1}(z) g_p^*(z) - h_p(z) g_{p+1}^*(z)\right| = O\left(\left(m_z + T(x_z^*)^2\right)^{1-2p}\right). \tag{20}$$

Now a similar argument to the one done in part (i) of Proposition 1, gives that

$$g_p^*(z) = G_p^*(z) + O\left(\left(m_z + T(x_z^*)^2\right)^{3/2-p}\right). \tag{21}$$

Putting together this formula and the corresponding one for h_p (see Proposition 1 and Remark 3) we get

$$\begin{aligned} &\left|h_{p+1}(z) g_p^*(z) - h_p(z) g_{p+1}^*(z)\right| \\ &\quad = \left|H_{p+1}^*(z) G_p^*(z) - H_p^*(z) G_{p+1}^*(z)\right| + O\left(\left(m_z + T(x_z^*)^2\right)^{1-2p}\right), \end{aligned}$$

which is $O((m_z + T(x_z^*)^2)^{1-2p})$, since by formulas (7) and (8), and Remark 2,

$$H_{p+1}^*(z) G_p^*(z) - H_p^*(z) G_{p+1}^*(z) = 0.$$

For II, using the estimate of h_p given in Proposition 1, and Lemma 1.6, we obtain

$$II = O\left(\left(m_z + T(x_z^*)^2\right)^{p-1/2}\right) \int_I \frac{|x - x_z^*|^{2\alpha}}{\left|1 - \overline{\gamma(x)}\, z\right|^{p+1}}\, dx,$$

$$\int_I \frac{|x - x_z^*|^{2\alpha}}{\left|1 - \overline{\gamma(x)}\, z\right|^{p+1}}\, dx \preceq \begin{cases} m_z^{2\alpha-p}\left(m_z + T(x_z^*)^2\right)^{-\alpha-1/2} & \text{if } 2\alpha < p \\ \left(m_z + T(x_z^*)^2\right)^{\alpha-1/2-p} & \text{if } p < 2\alpha < 2p+1, \end{cases}$$

estimates which in both cases ($p < \alpha + 1$) give $II = O((1 - |z|)^{\alpha-1})$.

The estimate of III follows from Proposition 1 and (ii) of Lemma 1.6. ◇

In particular we have the following:

COROLLARY 2. *If $\alpha \in (\frac{1}{2}, 1)$ and $p \in (\frac{3}{2}, 1)$ there exists a W neighborhood of E in $\mathbf{C}^n$ and a linear operator*

$$\tilde{S}^p \colon \Lambda_{2\alpha}(E) \to H(W \cap B) \cap C(W \cap \bar{B}) \cap \{h / |Rh(z)| = O((1 - |z|)^{\alpha - 1})\}$$

with $\tilde{S}^p f(\gamma(x_0)) = f(x_0)$ for each $x_0 \in \gamma^{-1}(E)$.

Proof of Corollary 2. If $\alpha \in (\frac{1}{2}, 1)$, it suffices to consider the composition of the operators S_p, $p \in (\frac{3}{2}, \alpha + 1)$ with the linear extension operator E given by Whitney's theorem. ◇

Remark 4. Notice that the regularity of the extension operators E, give that $\tilde{S}^p f$ is of class C^∞ in $W \cap (\bar{B} \setminus E)$.

Now we will see that we can extend these linear operators to the whole $Lip_\alpha(B)$, by solving a suitable $\bar{\partial}$-equation.

THEOREM 3. (i) *If $\alpha \in (\frac{1}{2}, 1)$, and $p \in (\frac{3}{2}, 1)$ then there exists a linear operator*

$$S^p \colon \Lambda_{2\alpha}(E) \to Lip_\alpha(B)$$

such that $S^p f(\gamma(x_0)) = f(x_0)$, for each $x_0 \in \gamma^{-1}(E)$.

(ii) *Furthermore, $S^p f \circ \gamma$ is differentiable on $\gamma^{-1}(E)$ and for each $x_0 \in \gamma^{-1}(E)$,*

$$\frac{d}{dx} S^p f(\gamma(x_0)) = \frac{\lambda}{\lambda + iT'(x_0)} f'(x_0)$$

(*recall that λ is given by the parametrization of γ*).

Proof of Theorem 3. If $\alpha \in (\frac{1}{2}, 1)$, let $p \in (\frac{3}{2}, 1)$ and W be the corresponding neighborhood given in Proposition 1. We will also consider the operator $\tilde{S}^p$ given in Corollary 2. Let V be another neighborhood of E in $\mathbf{C}^n$ with $\bar{V} \subset W$ (without loss of generality we will suppose that both neighborhoods are simply connected). Let χ be a C^∞ function in $\mathbf{C}^n$, such that $\chi = 1$ on V, and $\operatorname{supp} \chi \cap B \subset W \cap B$. Then the function $\chi S f$ is well defined in B and satisfies

$$\chi \tilde{S}^p f_{|V \cap \bar{B}} = \tilde{S}^p f_{V \cap \bar{B}}.$$

At this point, we need the following result.

LEMMA 3. *There exists*

$$\psi \in C^\infty(\overline{B}\setminus E) \cap A^1(B)$$

with $\psi^{-1}(0) = E$, *and* $\nabla\psi_{|E} = 0$, *where*

$$\nabla\psi = \left(\frac{\partial\psi}{\partial z_1}, \dots, \frac{\partial\psi}{\partial z_n}\right).$$

Proof of Lemma 3. It suffices to consider the function $\psi = 1/h_q^k$, where $q < 1$ and $k \geq 1$ satisfy $(k+2)/2k < q$. ◇

Following the proof of the theorem, since (see Remark 3) $\tilde{S}^p f$ is of class C^∞ in $W \cap (\overline{B}\setminus E)$, and $\bar{\partial}\chi = 0$ on $V \cap \overline{B}$, we have that the (0,1)-form $\tilde{S}^p f(\bar{\partial}\chi/\psi)$ is of class C^∞ on $\overline{B}$. Hence (see [R, page 357] and [B]), there exists a linear operator U that solves the $\bar{\partial}$-equation $\bar{\partial}u = \tilde{S}^p f(\bar{\partial}\chi/\psi)$, and such that the function $u = U(\tilde{S}^p f(\bar{\partial}\chi/\psi))$ is of class C^∞ on $\overline{B}$. Now, defining the holomorphic function in B by $v^p = \chi\tilde{S}^p f - \psi u$, it can immediate be verified that v^p is in $Lip_\alpha(B)$ (see [R, page 107]). Hence, it suffices to consider the linear operator given by $S^p f = v$.

For (ii), it is enough to prove (see [R, page 106] and (i)) that if $\gamma(x_0)$ is in E, $f \in \Lambda_{2\alpha}(\Gamma)$ and $p \in (\frac{3}{2}, \alpha + 1)$, then

$$\lim_{r\to 1,\, r\gamma(x_0)\in W\cap B} R_{\nu(z)} S_p f(z) = \frac{\lambda}{\lambda + iT'(x_0)} f'(x_0), \tag{22}$$

where W is the neighborhood of E given in Proposition 1, $\nu(z) = r\gamma'(x_0)$, and for each $F \in H(B)$,

$$R_{\nu(z)}F(z) = \sum_{i=1}^n \frac{\partial F}{\partial z_i}(z)\nu_i.$$

Using Taylor's development on f, we obtain

$$\begin{aligned} R_{\nu(z)} S_p f(z) &= \int_I (f(x) - f(x_0)) R_{\nu(z)}\big(K_p(x,z)\big)\,dx \\ &\quad + \int_I O\big(|x - x_0|^{2\alpha}\big) R_{\nu(z)}\big(K_p(x,z)\big)\,dx \\ &= I + II, \end{aligned}$$

and we will see that

$$\lim_{r\to 1,\, r\gamma(x_0)\in W\cap B} I = \frac{\lambda}{\lambda + iT'(x_0)} f'(x_0),$$

whereas the limit of II is zero. Using the definition of $R_{\nu(z)}$, we obtain

$$I = \frac{pf'(x_0)}{h_p(z)^2}(XY - ZV),$$

where

$$X = h_p(z),$$

$$Y = \int_I \frac{\left(\nu(z)\overline{\gamma(x)}\right)(x - x_0)}{\left(1 - \overline{\gamma(x)}\, z\right)^{p+1}}\, dx,$$

$$Z = \int_I \frac{\left(\nu(z)\overline{\gamma(x)}\right)}{\left(1 - \overline{\gamma(x)}\, z\right)^{p+1}}\, dx,$$

and

$$V = g_p(z)$$

(notice that, shrinking W if necessary, Lemmas 1.2 and 1.5 give $x_0 = x_z = x_z^*$).

Defining

$$j_p(z) = \int_I \frac{(x - x_0)^2}{\left(1 - \overline{\gamma(x)}\, z\right)^{p}}\, dx, \quad p > \frac{3}{2},$$

and using Taylor's development, we obtain

$$Y = \left(\nu(z)\overline{\gamma(x_0)}\right) g_{p+1}(x_0) + \left(\nu(z)\overline{\gamma'(x_0)}\right) j_{p+1}(z) + O\left((1 - r)^{1/2-p}\right)$$

$$Z = \left(\nu(z)\overline{\gamma(x_0)}\right) h_{p+1}(z) + \left(\nu(z)\overline{\gamma'(x_0)}\right) g_{p+1}(z) + O\left((1 - r)^{1/2-p}\right).$$

Hence

$$\begin{aligned}(23)\quad XY - ZV = {} & \left(\nu(z)\overline{\gamma(x_0)}\right)\{h_p(z) g_{p+1}(z) - g_p(z) h_{p+1}(z)\} \\ & + \left(\nu(z)\overline{\gamma'(x_0)}\right)\{h_p(z) j_{p+1}(z) - g_p(z) g_{p+1}(z)\} \\ & + O\left((1 - r)^{3/2-2p}\right).\end{aligned}$$

By (20) and the fact that

$$\nu(z)\overline{\gamma(x_0)} = O\big((1-r)^{1/2}\big) \quad \big(\nu(z)\overline{\gamma(x_0)} = 0\big)$$

we see that the first summand of (23) is

$$O\big((1-r)^{1/2}\big)O\big(|h_p(z)|^2\big) = o\big(|h_p(z)|^2\big).$$

In consequence (notice that $\nu(z)\overline{\gamma'(x_0)} = r\lambda$),

$$XY - ZV = r\lambda\{h_p(z)j_{p+1}(z) - g_p(z)g_{p+1}(z)\} + o\big(|h_p(z)|^2\big).\big)$$

The same kind of argument as in Proposition 1 gives

$$j_{p+1}(z) = J_{p+1}(z) + O\big((1-r)^{1-p}\big).$$

Using (7), (8) and (9), we obtain

$$\begin{aligned}\{h_p(z)j_{p+1}(z) &+ g_p(z)g_{p+1}(z)\}\\ &= \frac{2^{2p}}{p}C(p)^2B(z)^{-2}\left(2F(z,\gamma(x_0)) + \frac{A(z)^2}{B(z)}\right)^{1-2p}\\ &\quad + O\big((1-r)^{3/2-2p}\big).\end{aligned}$$

Consequently, I is equal to

$$\frac{r\lambda f'(x_0)2^{2p}C(p)^2B(z)^{-2}\left(2F(z,\gamma(x_0)) + \frac{A(z)^2}{B(z)}\right)^{1-2p}}{2^{2p}C(p)^2B(z)^{-1}\left(2F(z,\gamma(x_0)) + \frac{A(z)^2}{B(z)}\right)^{1-2p} + O\big((1-r)^{3/2-2p}\big)}$$
$$+ o(1).$$

Letting $r \to 1$, the definition of B, gives

$$\lim_{r\to 1} I = \frac{\lambda}{-\gamma''(x_0)\overline{\gamma(x_0)}} = \frac{\lambda}{\lambda + iT'(x_0)}.$$

Now we will see that $\lim_{r\to 1} II = 0$. We write

$$
\begin{aligned}
|II| &= \left| \int_I o(|x-x_0|^{2\alpha}) R_{\nu(z)}(K_p(x,z))\,dx \right| \\
&\preceq \frac{|R_{\nu(z)}|}{|h_p(z)|^2} \int_I \frac{|x-x_0|^{2\alpha}}{\left|1-\overline{\gamma(x)}\,z\right|^p}\,dx \\
&\quad + \frac{1}{|h_p(z)|} \int_I \frac{|x-x_0|^{2\alpha}\left(\nu(z)\overline{\gamma(x)}\right)}{\left|1-\overline{\gamma(x)}\,z\right|^{p+1}}\,dx \\
&= III + IV,
\end{aligned}
$$

and we will estimate each one separately. For IV, we will use Proposition 1, and the fact that

$$\left|\nu(z)\overline{\gamma(x)}\right| \preceq |z-\gamma(x)| + |x-x_0| \preceq \left|1-\overline{\gamma(x)}\,z\right|^{1/2}.$$

since $T(x_0)=0$, to obtain

$$IV \preceq (1-r)^{p-1/2} \int_I \frac{|x-x_0|^{2\alpha}}{\left|1-\overline{\gamma(x)}\,z\right|^{p+1/2}}\,dx \preceq (1-r)^{\alpha-1/2},$$

where in the last estimate we have used (ii) of Lemma 1.6 (note that $2p-\frac{1}{2}<2\alpha<2p$). And this converges to zero, when $r\to 1$.

For the estimate of III, we will first obtain a bound of $R_{\nu(z)}$. We write

$$
\begin{aligned}
R_{\nu(z)} &= p\int_I \frac{\left(\nu(z)\overline{\gamma(x)}\right)}{\left(1-\overline{\gamma(x)}\,z\right)^{p+1}}\,dx \\
&= p\left(\nu(z)\overline{\gamma(x_0)}\right)\int_I \frac{dx}{\left(1-\overline{\gamma(x)}\,z\right)^{p+1}} \\
&\quad + p\left(\nu(z)\overline{\gamma(x_0)}\right)\int_I \frac{(x-x_0)}{\left(1-\overline{\gamma(x)}\,z\right)^{p+1}}\,dx \\
&\quad + \int_I \frac{O(|x-x_0|^2)}{\left|1-\overline{\gamma(x)}\,z\right|^{p+1}}\,dx \\
&= V + VI + VII.
\end{aligned}
$$

Since $T(x_0) = 0$, we have

$$V \preceq |\nu(z) - \nu(\gamma(x_0))| \, |h_{p+1}(z)| \preceq (1-r)^{-p}.$$

For VI, formula (8) and an argument like the one used in Theorem 2, give

$$VI \preceq |g_{p+1}(z)| \preceq (1-r)^{-p}.$$

And (ii) of Lemma 1.6 (note that $p < 2 < 2p+1$), gives

$$VII \preceq (1-r)^{1/2-p}.$$

Hence $|R_{\nu(z)}| \preceq (1-r)^{-p}$, and

$$III \preceq (1-r)^{p-1} \int_I \frac{|x-x_0|^{2\alpha}}{\left|1-\overline{\gamma(x)}\,z\right|^{p+1}}\,dx \preceq (1-r)^{\alpha-1/2},$$

an expression that also converges to zero, when $r \to 1$. ◇

Theorem 1.1 of the last section, and Theorem 3 finishes the proof of Theorem A stated in the introduction, and also give the corresponding Corollary A.

Now we will give the proof of Corollary B.

Proof of corollary B. Let $\alpha_0 < \frac{1}{2}$ and $\alpha_1 > \frac{1}{2}$. Since Γ is complex-tangential, it is easy to check that the operators S^p, with $\frac{3}{2} < p < 2$ are also interpolation operators for $Lip_{\alpha_0}(B)$ (see Remark 1.2). In particular, there exists a linear interpolation operator S^p, which maps

$$\Lambda_{2\alpha_0}(\Gamma) \to Lip_{\alpha_0}(B) \quad \text{and} \quad \Lambda_{2\alpha_1}(\Gamma) \to Lip_{\alpha_1}(B).$$

It follows from the real interpolation method and the fact that these Lipschitz spaces are Besov spaces (see [B-L, page 153]) that it maps $\Lambda_{2\alpha}(\Gamma) \to Lip_\alpha(B)$, for each $\alpha \in (\alpha_0, \alpha_1)$ and it suffices to take $\alpha = \frac{1}{2}$. ◇

References

[B-L] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, New York, 1976.

[B] J. Bruna, *Nucleos de Cauchy en dominios estrictamente pseudoconvexos y operadores integrales que invierten la ecuacion $\bar\partial$*, Contribuciones matematicas en honor de Luis Vigil, Univ. de Zaragoza, 1984.

[B-O] J. Bruna and J.M. Ortega, *Interpolation by holomorphic functions smooth to the boundary in the unit ball of* $\mathbf{C}^n$, Math. Ann., vol. 274 (1986), 527–575.

[F-H] J.E. FORNAESS and B.S. HENRIKSEN, *Characterization of global peak sets for $A^\infty(D)$*, Math. Ann., vol. 259 (1982), pp. 125–130.

[N] A. NAGEL, *Smooth zero sets and interpolation sets for some algebras of holomorphic functions on strictly pseudoconvex domains*, Duke Math. J., vol. 43 (1976), 323–347.

[R] W. RUDIN, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980.

[S1] E. STEIN, *Singular integrals and differentiability properties of functions*, *Princeton*, 1970.

[S2] ______, *Singular integrals and estimates for the Cauchy-Riemann equations*, Bull. Amer. Math. Soc., vol. 79 (1973), pp. 440–445.

[U] J. UBØE, *Smooth interpolation sets, A study on an article by A. Nagel*, preprint, 1985.

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