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DISTANCE SPHERES AND MYERS-TYPE THEOREMS FOR MANIFOLDS WITH LOWER BOUNDS ON THE RICCI CURVATURE

BY

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Introduction

Let M be a complete connected riemannian manifold of class C^r, $r \geq 3$, and dimension $d \geq 2$. One of the results still viewed by many to be one of the most important as well as the loveliest concerning the global properties of such a space is the following work of S. Myers (1941).

THEOREM. *Suppose that the Ricci curvature of M is bounded from below by a positive constant m. Then the diameter of M is no larger than*

$$\pi\sqrt{(d-1)/m}\,.$$

In particular, M is compact.

Here, the Ricci curvature is viewed as a function on the unit tangent bundle of M. Attempts at generalizing and refining this theorem have received considerable attention. Most notably, there are the works of W. Ambrose [1], E. Calabi [4, 5], A. Avez [2], S.T. Yau [18, 19], K. Shiohama [17], G.J. Galloway [9], S. Markvorsen [13], and J. Cheeger, M. Gromov, and M. Taylor [7]. In the present paper, our purpose is to prove

MAIN RESULTS (Theorems 3.3 and 3.5). *Let m be any given constant, not necessarily positive. Assume that the Ricci curvature of M is bounded below by (resp. strictly greater than) m. Suppose that there exists a point $p \in M$ and a number $r \in \mathbb{R}_+$ such that the distance sphere in M with center p and radius r has mean curvature away from its singularities greater than (resp. greater than or equal to $\sqrt{|m|}$. Then the diameter of M has a finite upper bound, and hence M is compact. In the first case, the upper bound on the diameter can be explicitly estimated in terms of the supremum of the mean curvature.*

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In the case where the supremum of the mean curvature is strictly greater than $\sqrt{|m|}$, if, in addition, we know the area of the distance sphere, we can also improve the previously known estimate on the volume of such M (Corollary 3.4). Perhaps the most interesting applications are for the case $m = 0$. We shall show that we can weaken the assumption in original Myers' theorem so that the Ricci curvature need not be bounded away from 0 on all of M and still get the compactness (Theorem 4.1). The last result in turn allows us to recover a theorem of E. Calabi on the estimate on the decay of the radial Ricci cuvrature near infinity for noncompact manifolds (Corollary 4.2).

Myers-type theorems have found important applications in the field of relativistic cosmology (cf. [9], [13]). In a 4-dimensional space-time, the condition of nonnegative timelike Ricci curvature is implied by the assumption of attracting gravitation and hence is a natural one. In this context, we feel that our result is of particular use. In fact, in this case, the distance sphere in our theorem can be replaced by a distance tube about a Cauchy hypersurface. Thus, on many occasions, compactness of certain hypersurfaces or nonmaximality of certain geodesics can be derived from natural assumptions and calculations within bounded regions.

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1. Notations

Throughout this paper, M will be as described in the first paragraph. The distance between two points $p, q \in M$ will be denoted by $\delta(p, q)$. We set

$$\delta(p) := \sup\{\delta(p, x) | x \in M\},$$

while $\delta(M)$ is the diameter of M. From the triangle inequality, it is easily seen that

$$\delta(M) \leq 2\delta(p)$$

for any $p \in M$.

For $p \in M$, we let UT_pM denote the unit tangent sphere at p. If $\mathbf{u} \in UT_pM$, the *cut length* of $\mathbf{u}$ is defined to be the number

$$\delta(\mathbf{u}) := \sup\{s | \text{the geodesic } s \mapsto \exp_p s\mathbf{u} \text{ is minimal}\};$$

i.e.,

$$\{\exp_p(\delta(\mathbf{u}) \cdot \mathbf{u}) | \mathbf{u} \in UT_pM\}$$

is the *cut locus* of M.

We denote by $Ric : TM \to \mathbb{R}$ the quadratic form obtained by polarizing the Ricci tensor of M. Thus, if $M = S^d$ with its metric g^d_{can} as the unit euclidean sphere in $\mathbb{R}^{d+1}$,

$$Ric(\mathbf{v}) \equiv (d-1)|\mathbf{v}|^2$$

for all tangent vector $\mathbf{v}$. Fixing $p \in M$, we obtain a function defined on $\mathbb{R}_+ \times UT_pM$ by

$$Ric_p(s, \mathbf{u}) := Ric\big((T_{s\mathbf{u}} \exp_p)(\mathbf{u})\big).$$

We shall call this the radial Ricci curvature function at p.

For $r \in \mathbb{R}_+$, $S_r(p)$ denotes the distance sphere in M with center p and radius r. Let $\tilde{S}_r(p)$ be the set of all $q \in S_r(p)$ off the cut locus of p. Then, $\tilde{S}_r(p)$ is a smooth hypersurface of M and we can define the normal bundle $\perp \tilde{S}_r(p)$.

Let S be a smooth local hypersurface of M. Let $q \in S$ and $\mathbf{u} \in \perp_q S$. For us, the *second fundamental form* of S will be defined by

$$\beta_{\mathbf{n}}: T_qS \otimes T_qS \to \mathbb{R},$$
$$\beta_{\mathbf{n}}(\mathbf{v}, \mathbf{w}) := \langle \nabla_{\mathbf{v}}(\mathbf{w}), \mathbf{n} \rangle,$$

For the special case $S = \tilde{S}_r(p)$, we take $\mathbf{n} := \mathbf{n}_q$ to be the outward pointing normal $\mathbf{n}_q := (T_{r\mathbf{u}} \exp_p)(\mathbf{u})$ where $\mathbf{u} \in UT_pM$ with $\exp_p r\mathbf{u} = q$, and in this case, we write β_r for $\beta_{\mathbf{n}_q}$. As usual, the *mean curvature* H of S is the trace of the matrix of $\beta_{\mathbf{n}}$. We put

$$H_p(r) := \inf\{H(q) | q \in \tilde{S}_r(p)\}.$$

Let $r \in \mathbb{R}$, and let $c: [0, \infty) \to M$ be the geodesic

$$c(s) := \exp_q(s - r)\mathbf{n}.$$

Following Bishop and Crittenden [3], a Jacobi field along c satisfying $\mathbf{Y}(r) \in T_qS$ and

$$\beta(\mathbf{Y}(r), \mathbf{w}) = -\langle \mathbf{Y}'(r), \mathbf{w} \rangle$$

for all $\mathbf{w} \in T_q S$ will be called an *S-Jacobi field*. It is well known that the geometric interpretation of this condition is that $\mathbf{Y}$ is the linearization of some variation of c through geodesics normal to S. By the Gauss' lemma, it is seen that for $S = \tilde{S}_r(p)$, $\mathbf{Y}$ is an $\tilde{S}_r(p)$-Jacobi field if and only if $\mathbf{Y}$ is normal to c and extends to a Jacobi field which vanishes at the center $p = c(0)$. It follows that for $t > r$, if $q \in \tilde{S}_r p)$, $q' \in \tilde{S}_t(p)$, then, for any $\mathbf{y} \in T_{q'} S_t(p)$,

$$\beta_t(\mathbf{y},\mathbf{y}) = -I(\mathbf{Y},\mathbf{Y}) + \beta_r(\mathbf{Y}(r),\mathbf{Y}(r))$$

where $\mathbf{Y}$ is the $\tilde{S}_r(p)$-Jacobi field with $\mathbf{Y}(t) = \mathbf{y}$ and I is the *index integral*

$$I(\mathbf{Y},\mathbf{Y}) = \int_r^t \langle \mathbf{Y}'(s),\mathbf{Y}'(s)\rangle - \langle R(\dot{c}(s),\mathbf{Y}(s))\dot{c}(s),\dot{c}(s)\rangle \, ds;$$

cf. Gromoll [10]. Consequently,

$$H(q') = -\sum_{i=1}^{d-1} \{I(\mathbf{Y}_i,\mathbf{Y}_i) - \beta_r(\mathbf{Y}_i(r),\mathbf{Y}_i(r))\},$$

where and $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{d-1}$ are $\tilde{\mathbf{S}}_r(p)$-Jacobi fields that are orthonormal at $q' = c(t)$.

For each S-Jacobi field $\mathbf{Y}$ along $c\colon [r,\infty) \to M$ normal to S, we construct a *modified field* $\mathbf{Y}^\#$ defined by

$$\mathbf{Y}^\#(s) := \begin{cases} \mathbf{Y}(s) & \text{if } \mathbf{Y} \text{ does not vanish on } (r,s), \\ \mathbf{O} & \text{if } \mathbf{Y}(t) = \mathbf{O} \text{ for some } t,\ r < t \le s. \end{cases}$$

The convenience of these modified fields are as follows. For each $q \in \tilde{S}_r(p)$, take $\mathbf{Y}_{q1}, \mathbf{Y}_{q2}, \ldots, Y_{qd-1}$ to be linearly independent Jacobi fields along the geodesic $s \mapsto \exp_q(s-r)\mathbf{n}_q$. Let $A_p(s)$ denote the area of $S_s(p)$, $s > r$. Then, the formula

$$A_p(s) = \int_{\tilde{S}_r(p)} \frac{G^{1/2}\left[\mathbf{Y}^\#_{qi}(s)\right]}{G^{1/2}\left[\mathbf{Y}_{qi}(r) + (s-r)(\mathbf{Y}_{qi})'(r)\right]} \, d\,\mathrm{Area}_{\tilde{S}_r(p)}(q),$$

where $G[\mathbf{v}_i]$ is the Gram determinant made from the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d-1}$, remains valid for all s even beyond the focal points of $\tilde{S}_r(p)$. We remark that if $A_p(s) = 0$ for some $s > r$, then $\delta(p) \le r + s$. Finally, we denote the volume of the unit euclidean sphere $(S^n, g^n_{\mathrm{can}})$ by $\theta(n)$ for convenience.

2. Darboux-Cheeger-Yau models

In this section, we construct certain model spaces which will be used for comparison purposes in the next section.These examples are due to J.G. Darboux in the case $d = 2$ and similar but more special examples where the hypersurface S degenerates to a point were considered by Cheeger and Yau [8].

We take r and R so that $0 < r < R \leq \infty$. Let f: $[r, R) \to \mathbb{R}$ be a continuous function. Let $a = a(s)$ be the solution to the differential equation

$$a'' + f(s)a = 0 \tag{2.1}$$

with initial value

$$a(r) = a_0, \quad a'(r) = a_1. \tag{2.1.1.}$$

Assume that $a(s)$ stays positive for $s \in [r, R)$. We take a space which is diffeomorphic to

$$S^{d-1} \times [r, R)$$

and give it the metric

$$g_{(x,s)} = a(s) g_{\text{can}}^{d-1} \otimes ds^2$$

at each $x \in S^{d-1}$ and $s \in [r, R)$. We denote the resulting incomplete riemannian manifold with boundary by $\mathbb{M}(r, R, f, a_0, a_1/a_0)$. Let us denote by S_s, the *parallel sphere* $S^{d-1} \times \{s\}$ with the normal vector pointing in the positive s direction and by $A^*(s)$, its area. Then, we have:

(2.2) PROPOSITION. *Let $p \in \mathbb{M}(r, R, f, a_0, a_1/a_0)$, $p = (x, s)$ where $x \in S^{d-1}$ and $s \in [r, R)$. Suppose that $\mathbf{u}$ is normal to S_s^* and $\mathbf{v}$, $\mathbf{w}$ are tangent to S_s^*. Then:*

(2.2.1) *The sectional curvature of the plane containing $\mathbf{u}$ and $\mathbf{v}$ is $f(s)$. In particular,*

$$Ric(\mathbf{u}) \equiv (d-1) f(s) |\mathbf{u}|^2.$$

(2.2.2) *Each S_s^* is totally umbilic and has parallel second fundamental form*

$$\beta(\mathbf{v}, \mathbf{w}) \equiv -\frac{a'(s)}{a(s)} \langle \mathbf{v}, \mathbf{w} \rangle.$$

Hence,

$$H \equiv -(d-1)a'(s)/a(s).$$

(2.2.3) *Let* $\mathbf{Y}^*_{x1}, \mathbf{Y}^*_{x2}, \ldots, \mathbf{Y}^*_{x(d-1)}$ *be a linearly independent set of* S^*_r*-Jacobi fields along the geodesic*

$$s \mapsto (x, s) \in \mathbb{M}(r, R, f, a_0, a_1/a_0).$$

Then

$$\begin{aligned} A^*(s) &= \int_{S^*_r} \frac{G^{1/2}[\mathbf{Y}^*_{xi}(s)]}{G^{1/2}[\mathbf{Y}^*_{xi}(r) + (s-r)(\mathbf{Y}^*_{xi})'(r)]}\, d\,\mathrm{Area}_{S^*_r}(x) \\ &= a^{d-1}(s)\theta(d-1). \end{aligned}$$

Proof. Straightforward calculations; cf. O'Neill [15] for the case $d = 2$. Q.E.D.

Of special importance to us will be the case $f(s) \equiv -k^2$, $k \geq 0$ constant. If $k = 0$, the solution is of course

$$a(s) = a_0 + a_1(s - r).$$

Thus, if $a_1 < 0$, $a(s) > 0$ only for

$$s < R := r - a_0/a_1.$$

If $k \neq 0$, the solution to (2.1) is

$$a(s) = \frac{a_0}{2k}\left[(k + a_1/a_0)e^{(s-r)k} + (k - a_1/a_0)e^{(r-s)k}\right].$$

Thus, setting $h := a_1/a_0$, we have:

(2.3) Proposition. *Let* $\mathbb{M}(r, R, -k^2, a_0, h)$ *be as described above where* $k > 0$. *Then, if* $h < -k$,

$$R \leq r + \frac{\log[(h-k)/(h+k)]}{2k},$$

and $A^*(s)$ *tends to* 0 *as* s *approaches the value on the right side. If* $h = -k$, *then* R *can be extended to* ∞, *but* $A^*(s)$ *still tends to* 0 *as* s *goes off to* ∞.

3. Main results

The purpose of this section is to prove our theorems as described in the first section. We start with the following simple observation.

(3.1) LEMMA. *Let $p \in M$, $r \in R_+$, and let x be any point in $M \setminus B_r(p)$. Then there is a point $q \in \tilde{S}_r(p)$ and a geodesic*

$$c\colon [r, s) \to M$$

so that $\delta(q, x) = \delta(S_r(p), x)$, $c(r) = q$, $c(x) = x$, and $\dot{c}(r) \in \perp_q \tilde{S}_r(p)$.

Proof. There is a normal minimal geodesic c: $[0, s] \to M$ connecting p with x. By assumption, $s = \delta(p, x) > r$, and so c intersects S_r at a point $q \in \tilde{S}_r(p)$. The rest of the assertion follows from the Gauss' lemma and the first variation formula for arclength. Q.E.D.

(3.2) LEMMA. *Let S be a smooth oriented local hypersurface in M with mean curvature $H(q) \geq -h^*$ at $q \in S$. Let $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{d-1}$ be a linearly independent set of S-Jacobi fields along the geodesic*

$$s \mapsto \exp(s - r)\mathbf{n}_q.$$

Let $\mathbb{M} := \mathbb{M}(r, R, f, a^*, h^*)$ be a Darboux-Cheeger-Yau model as defined in Section 2 for some R, f:$[r, R) \to \mathbb{R}$, and a^*. Let $S^* := S_r^*$. Let $\mathbf{Y}_1^*, \mathbf{Y}_2^*, \ldots, \mathbf{Y}_{d-1}^*$ be a linearly independent set of S^*-Jacobi fields along the geodesic $s \mapsto (q^*, s)$ for some $q^* \in S^{d-1}$. Assume that

$$Ric\Big(\big(T_{s\mathbf{n}_q} \exp_q\big)(\mathbf{n}_q)\Big) \geq (d-1)f(s)$$

for all $s \in [r, R)$. Then, for all such s,

$$\frac{G^{1/2}\big[\mathbf{Y}_i^\sharp(s)\big]}{G^{1/2}[\mathbf{Y}_i(r) + (s - r)\mathbf{Y}_i'(r)]}$$
$$\leq \frac{G^{1/2}[\mathbf{Y}_i^*(s)]}{G^{1/2}[\mathbf{Y}_i^*(r) + (s - r)(\mathbf{Y}_i^*)'(r)]}.$$

Moreover, if equality holds for some $s = s_1$, then $H(q) = -h^*$ and for all $s \in [r, s_1)$, $Ric((T_{s\mathbf{n}_q} \exp_q)(\mathbf{n}_q)) \equiv (d-1)f(s)$.

Proof. Since the right side of the inequality is strictly positive on the admissible interval, if $\mathbf{Y}_i^\sharp(s) = \mathbf{O}$ for any i, then the inequality follows trivially. If, on the other hand, $\mathbf{Y}_i^\sharp(s) \neq \mathbf{O}$ for all i, then $\mathbf{Y}_i^\sharp(s) = \mathbf{Y}_i(s)$. In this case, the inequality is a special case of the well-known Comparison Theorem of E. Heintze and H. Karcher [11] Q.E.D.

Now, we are ready to prove:

(3.3) THEOREM. *Let $m \leq 0$ be given. Suppose that for some $p \in M$ and some $r \in \mathbb{R}_+$, the radial Ricci curvature $Ric_p(s, \mathbf{u}) \geq m$ for all $\mathbf{u} \in UT_pM$ and $s > r$. Suppose also that $H_p(r) > \sqrt{-m}$. Let*

$$h := -H_p(r)/(d-1) \quad \text{and} \quad k := \sqrt{-m/(d-1)}\,.$$

Then

$$\delta(p) < r + \frac{\log[(h-k)/(h+k)]}{2k}$$

if $m < 0$ and $\delta(p) < r - 1/h$ if $m = 0$. Consequently,

$$\delta(M) < 2r + \frac{\log[(h-k)/(h+k)]}{k}$$

for $m < 0$ and $\delta(M) < 2(r - 1/h)$ for $m = 0$. In particular, M is compact.

Proof. Set

$$R := r + \frac{\log[(h-k)/(h+k)]}{2k} \quad \text{if } m < 0$$

and

$$R := r - 1/h \quad \text{if } m = 0.$$

Let $x \in M$. If suffices to assume that $x \notin B_r(p)^-$ and show that $\delta(x, S_r(p)) < R$.

Take $\mathbb{M} := \mathbb{M}(r, R, -k^2, a_0, h)$ for some $a_0 \in \mathbb{R}_+$. By Lemma (3.1), there exists $c\colon [r, t] \to M$, the minimal unit-speed geodesic joining a point $q := c(r) \in \tilde{S}_r(p)$ with $c(t) = x$. Let $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{d-1}$ be any linearly independent set of $\tilde{S}_r(p)$-Jacobi fields along c. Let $\mathbf{Y}_1^*, \mathbf{Y}_2^*, \dots, \mathbf{Y}_{d-1}^*$ be any linearly independent set of S_r^*-Jacobi along the geodesic $c^*(s) := (q^*, s)$ in $\mathbb{M}$ where $q^* \in$

S^{d-1} is any point and $s \in [r, R)$. Then, by Lemma (3.2),

$$\frac{G^{1/2}\left[\mathbf{Y}_i^\#(s)\right]}{G^{1/2}[\mathbf{Y}_i(r) + (s-r)\mathbf{Y}_i'(r)]} < \frac{G^{1/2}[\mathbf{Y}_i^*(s)]}{G^{1/2}[\mathbf{Y}_i^*(r) + (s-r)(\mathbf{Y}_i^*)'(r)]}.$$

But, by Propositions (2.2.3) and (2.3),

$$G^{1/2}[\mathbf{Y}_i^*(s)]/G^{1/2}[\mathbf{Y}_i^*(r) + (s-r)(\mathbf{Y}_i^*)'(r)] \to 0 \quad \text{as } r \to R.$$

Hence, for some $s_1 < R$, $G^{1/2}[\mathbf{Y}_i^\#(s_1)] = 0$. But, this means that there is a focal point of $\tilde{S}_r(p)$ along c before s_1. Since $c|[r, s]$ minimizes arclength from $S_r(p)$ to x, s must be before the focal point, and hence

$$\delta(x, S_r(p)) = s \le s_1 < R. \qquad \text{Q.E.D.}$$

(3.4) COROLLARY. *Under the same notations as Theorem* (3.3), *assume that* $Ric_p(s, \mathbf{u}) \ge m$ *for all* s *and that* $H_p(r) > \sqrt{-m}$. *Set* $A_0 :=$ Area$(S_r(p))$. *Then the volume of* M *must be strictly less than*

$$\int_0^r \sinh^{d-1} ks\, ds + \int_r^R a^{d-1}(s) \cdot \theta(d-1)\, ds$$

where $a(s)$ *is the solution to* (2.1) *with*

$$a_0 := \sqrt{A_0/\theta(d-1)}, \quad a_1 := ha_0,$$

and

$$R := r + \frac{\log[(h-k)/(h+k)]}{2k} \quad \text{if } m < 0,$$
$$R := r - 1/h \quad \text{if } m = 0.$$

Remark. The estimate here is sharp in the sense that given any m, r, and $\varepsilon > 0$, there is an A_0 so that one can construct a riemannian manifold M with a point p so that the radial Ricci curvature at p is $\ge m$, $H_p(r) > \sqrt{-m}$, $A_p(r) = A_0$ and whose volume is ε-close to the integral above. On the other hand, these examples and the equality discussion in Lemma (3.2) show that in fact the bound can never be achieved by a smooth manifold.

Proof. By Bishop's comparison theorem [3],

$$\mathrm{Vol}(B_r(p)) \leq \int_0^r \sinh^{d-1} ks\, ds.$$

On the other hand, setting $N := M \setminus B_r(p)$,

$$\begin{aligned}
\mathrm{Vol}(N) &= \int_r^{\delta(p)} A_p(s)\, ds \\
&\leq \int_r^{\delta(p)} \int_{\tilde{S}_r(p)} \frac{G^{1/2}\left[\mathbf{Y}^{\#}_{qi}(s)\right]}{G^{1/2}\left[\mathbf{Y}_{qi}(r) + (s-r)\mathbf{Y}'_{qi}(r)\right]}\, d\,\mathrm{Area}_{\tilde{S}_r(p)}(q)\, ds \\
&< \int_r^{\delta(p)} \int_{\tilde{S}_r(p)} \frac{G^{1/2}[\mathbf{Y}_i^*(s)]}{G^{1/2}[\mathbf{Y}_i^*(r) + (s-r)(\mathbf{Y}_i^*)'(r)]}\, d\,\mathrm{Area}_{\tilde{S}_r(p)}(q)\, ds
\end{aligned}$$

from Theorem (3.3) and its proof. Now, take $\mathbb{M} := \mathbb{M}(r, R, -k^2, a_0, h)$. Then $A^*(r) = \mathrm{Area}(S_r(p))$. Since the integrand equals 1 when $s = r$, the quantity above is

$$\begin{aligned}
&< \int_r^R \int_{S_s^*} \frac{G^{1/2}[\mathbf{Y}_i^*(s)]}{G^{1/2}[\mathbf{Y}_i^*(r) + (s-r)(\mathbf{Y}_i^*)'(r)]}\, d\mathrm{Area}_{S_s^*}\, ds \\
&= \int_r^R A^*(s)\, ds = \int_r^R a^{d-1}(s) \cdot \theta(d-1)\, ds.
\end{aligned}$$

Since $\mathrm{Vol}(N) = \mathrm{Vol}(B_r(p)) + \mathrm{Vol}(N))$, our assertion follows. Q.E.D.

(3.5) THEOREM. *Let $m < 0$ be given. Suppose that for some $p \in M$ and some $r \in \mathbb{R}_+$, $Ric_p(s, \mathbf{u}) > m$ for all $\mathbf{u} \in UT_pM$ and almost all $s > r$, and that $H_p(r) \geq \sqrt{-m}$. Then M is compact.*

Proof. For each $x \in M$, let $\mathbf{u}(x)$ be the vector in UT_pM such that

$$x = \exp_p(s\mathbf{u}(x)) \quad \text{for some } s \in \mathbb{R}_+.$$

Note that $\mathbf{u}(x) = \mathbf{u}(y)$ if and only if there is a minimal geodesic segment on which p, x, and y all lie. Let Q be the complement of $\tilde{S}_r(p)$ in $S_r(p)$. Then if $x \in Q$, $\delta(\mathbf{u}(x)) = r$. Consequently, by the continuity of δ as a function on UT_pM, if we choose ε small enough, there is an open neighborhood N of Q in $S_r(p)$ so that if $x \in N$, then

$$\delta(\mathbf{u}(x)) < r + \varepsilon.$$

Now, let $x \in M$ be arbitrary. Let c_x be the minimal geodesic joining x with $\tilde{S}_r(p)$ as given in Lemma (3.1). By the discussion above, if $c_x(r) \in N$, then $\delta(x, p) < r + \varepsilon$. If $q := c_x(r) \in S_r(p) \setminus N$ and $H(q) > \sqrt{-m}$, then, by repeating the argument in Theorem (3.4), we see that $\delta(x, p) < R$ where

$$R = r + \frac{\log\{[-H(q)/(d-1) - k]/[-H(q)/(d-1) + k]\}}{2k}, \quad k = \frac{}{\sqrt{-m/(d-1)}}$$

if $m < 0$ and $R = r + (d-1)/H(q)$ if $m = 0$ as before. Hence, assume that $c_x(r) \notin N$ and $H(c_x(r)) = \sqrt{-m}$. Then, by Lemma (3.2), if we take $\tilde{S}_r(p)$ – Jacobi fields $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{d-1}$ along c_x, for all s, $\delta(x, p) \geq s > r$,

$$\frac{G^{1/2}[\mathbf{Y}_i(r)]}{G^{1/2}[\mathbf{Y}_i(r) + (s - r)\mathbf{Y}_i'(r)]} \leq \frac{G^{1/2}[\mathbf{Y}_i^*(r)]}{G^{1/2}[\mathbf{Y}_i^*(r) + (s - r)(\mathbf{Y}_i^*)'(r)]}.$$

Moreover, if equality held for all s in the above, then $Ric_p(s, \mathbf{u}(x)) \equiv m$, contradicting our assumption that the radial Ricci curvature $\neq m$ for almost all s. Hence, we have a strict inequality in the above at some $s = t > r$. But, by the formula for the mean curvature of distance spheres in Section 2 and a standard index comparison argument (which is also used in [11]), this implies that for some s, $t > s > r$, $c_x(s) \in \tilde{S}_s(p)$ and $H(c_x(s)) = (d-1)h$ where $h < -\sqrt{-m}$. Comparison with the Darboux model $\mathbb{M}(s, R, m/(d-1), a_0, h)$ where a_0 is chosen suitably and

$$R = s + \frac{\log[(h - k)/(h + k)]}{2k}$$

for $m < 0$, $R = s - 1/h$ for $m = 0$ then shows that $\delta(x, p) < R$.

Clearly, the positive number R in the last two cases corresponds to points in $S_r(p) \setminus N$ in a continuous manner. Since the set $S_r(p) \setminus N$ is compact, there is an uniform upper bound for such an R, and hence also for $\delta(x, p)$. Accordingly, $\delta(p)$ is finite, and therefore M is compact. Q.E.D.

4. Nonnegative Ricci curvature case

In this section, we specialize to the case $m = 0$ and describe two applications of the results in Section 3. The first of these applications has an obvious generalization to other m.

(4.1) THEOREM. *Let p be a point in M. Suppose that for all $\mathbf{u} \in UT_pM$, and some $m > 0$, $Ric_p(s, \mathbf{u}) \geq 0$ for all $s > 0$ and > 0 for*

$$s \geq \pi\sqrt{(d-1)/4m}\,.$$

Then M is compact.

Remark. The assumption in this theorem is satisfied if M has Ricci curvature nonnegative everywhere and $> m$ on $B_{\pi\sqrt{(d-1)/4m}}(p)$ for some p. In this sense, this theorem is a sharpening of the compactness statement in the original Myers' theorem. In fact, it is possible to construct, for arbitrary $\varepsilon > 0$, a noncompact surface of positive Gauss curvature with a point p so that the curvature is > 1 on $B_{\pi\sqrt{d-1}/2-\varepsilon}(p)$.

Proof. Let $r := \pi\sqrt{(d-1)/4m}$. Then, for $q \in \tilde{S}_r(p)$,

$$H(q) = -\sum_{i=1}^{d-1} I(\mathbf{Y}_i, \mathbf{Y}_i)$$

where $\mathbf{Y}_1(r), \mathbf{Y}_2(r), \ldots, \mathbf{Y}_{d-1}(r)$ are orthonormal in $T_q\tilde{S}_r(p)$. Comparison with the euclidean sphere in $\mathbb{R}^{d+1}$ of radius $\sqrt{(d-1)/m}$ then shows that $H(q) > 0$ and hence $H_p(r) > 0$. Therefore, by Theorem (3.5), M is compact. Q.E.D.

As an easy corollary of the above, we can deduce the following well-known result of E. Calabi [4].

(4.2) COROLLARY. *Let M be a noncompact manifold of positive Ricci curvature. Then the quantity*

$$\rho(s) := \inf_{\substack{\mathbf{u} \in UT_pM,\\ r<s}} Ric_p(r, \mathbf{u})$$

decays at least quadratically in s.

Remark. Besides Calabi's original proof, this result has also been proven by R. Schneider [16] who also showed that this decay is optimal. It also follows from Theorem 4.8 in Cheeger, Gromov, and Taylor [7].

We are hopeful that the techniques we have described herein have other applications. We have concentrated on using Darboux models where the longitudinal curvature was constant. To study more arbitrary Ricci curvature conditions, we can use Darboux models in the full generality described in Section 2. By analyzing the initial condition (2.1.1), it is possible to give compactness critieria for variable conditions bounding the radial Ricci curvature from below.

Using more refined techniques, we can also study what happens in the extreme case where $Ric \geq m$ and $H_p(r) = \sqrt{-m}$ for some $p \in M$ and $r > 0$. In that case, unless M is compact, a strong splitting phenomenon is observed. The most interesting case is probably the case $m = 0$. However, this technique extends to a much more general situation for branched minimal hypersurfaces. See our forthcoming paper [12] for details.

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