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ON HYPERSURFACES OF LIE GROUPS

BY

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0. *Introduction.* In his work on the theory of surfaces Gauss introduced what is called today the (normal) Gauss map of an orientable hypersurface in Euclidean space E^n.

Considering E^n as a commutative Lie group, the Gauss map is just the translation of the normal vector of the hypersurfaces at any point to the identity. Reasoning this way, we can also consider such a map in an orientable hypersurface of an arbitrary Lie group. In this work, we use this map to obtain some results on the geometry and topology of a hypersurface in a Lie group which apply, in particular, to S^3 and E^n. To state the results, let us consider a Lie group G with a left invariant metric, an orientable hypersurface M of G, and a map $\gamma: M \to \Gamma$, Γ being the Lie algebra of G, which (left) translate the normal vector of M at any point to the identity of G.

As in Euclidean spaces, the tangent space of M at any point can be identified, up to a left translation, with the tangent space of the unit sphere of the Lie algebra of G. Therefore, the derivative of γ can be considered a tensor on TM. Its determinant, say K, in E^n coincides with the Gauss-Kronecker curvature of M (that is, the determinant of the 2nd fundamental form of M), which is the intrinsic curvature of M when $n = 3$. In S^3, we prove that K is also the intrinsic curvature of M (§5). In general, the derivative of γ is the sum of the shape operator of M plus a tensor, depending essentially on G, which we call here the invariant shape operator of M (Definition 2 and Proposition 3). We will say that a point $x \in M$ is degenerate if $d\gamma(x) \equiv 0$.

We prove that *if M is compact and has no degenerate points, then either M is homeomorphic to a sphere or K is* zero *in a non denumerable subset of M* (Theorem 7).

When G is commutative, we have that x is degenerate if and only if x is totally geodesic. In general, if x is degenerate then the translation of $T_x(M)$ to the identity is a Lie subalgebra of codimension 1 (Proposition 10). It follows, in particular, that if G has finite center and the metric is bi-invariant, then M has no degenerate points.

A classical result due to Hadamard says that a compact hypersurface of E^n with $K > 0$ is a convex hypersurface ([6]). Chern-Lashof obtained the same conclusion, in E^3, assuming just $K \geq 0$, and constructed an example of a compact hypersurface of E^n, $n \geq 4$, with $K \geq 0$ which is not even homeomorphic to a sphere ([1]). It follows from Theorem 7 that *if $M \subset E^n$ compact has no totally geodesic points and $K \geq 0$, then either M is homeomorphic to a sphere or the set in which $K = 0$ is non denumerable.* Theorem 7 also implies that *if a compact dimension 2 Riemannian manifold M can be isometrically immersed in S^3, then either M is diffeomorphic to S^2 or M has a non denumerable set of flat points.*

We obtain a generalization of the Gauss-Bonnet formula: *if M is compact and of even dimension, then*

$$\int_M K = \tfrac{1}{2}c_{n-1}\chi(M) \quad \text{(Theorem 8)}.$$

Considering the case in which G is provided with a bi-invariant metric, we obtain a partial generalization of Hadamard's Theorem: *if the shape operator of M is positive definite, then M is diffeomorphic to a sphere* (Theorem 9). Contrary to what happens in Euclidean spaces, there are cases in which γ is a diffeomorphism but M is not embedded (Remark of p. 9).

In [5], Sebastiani defines the map γ for submanifolds of Lie groups of arbitrary codimension, and some of the above results are generalized. Using the curvature K defined as above, instead of the Lipschitz-Killing curvature, he generalizes a classical result due to R. Langevin ([2]) about the curvature of singularities of analytic maps in complex Lie groups.

The questions discussed here came up in a seminar with Marcos Sebastiani, to whom I am indebted for valuable advice.

1. *Assumptions.* The following notations will be used through the paper:
G n-dimensional Lie group provided with a left invariant metric $\langle\ ,\ \rangle$,
Γ Lie algebra of G with bracket $[\ ,\ ]$,
L_x left translation determined by $x \in G$,
∇ Riemannian connection determined by $\langle\ ,\ \rangle$.

2. *Definition of γ, K and of the invariant shape operator of M.* Let M be an orientable compact hypersurface of G and η an unitary normal vector field of M. We define

$$\gamma\colon M \to S^{n-1}(1) = \{X \in \Gamma \mid \|X\| = 1\},$$
$$x \to (L_{x^{-1}})_*(\eta(x)).$$

Remark. Given $x \in M$, since

$$d\gamma(x)(T_x(M)) \subset T_{\gamma(x)}(S^{n-1}) = \{\gamma(x)\}^{\perp}$$

and L_x is an isometry, we have

$$\begin{aligned}(L_x)_*(d\gamma(x)(T_x(M))) &\subset (L_x)_*(L_{x^{-1}})_*\left(\{\eta(x)\}^{\perp}\right)\\ &= \{\eta(x)\}^{\perp} = T_x(M)\end{aligned}$$

so that $(L_{(\cdot)})_* \circ d\gamma$ is a tensor on TM. Therefore, we define

$$K(x) \equiv \det((L_x)_* \circ d\gamma(x)).$$

The invariant shape operator of M is the section α in the bundle $\operatorname{Hom}(TM, TM)$ given by

$$\alpha_x(X) = \nabla_X \tilde{\eta}\,|_x$$

where $\tilde{\eta}$ is the left invariant vector field of G such that $\tilde{\eta}(x) = \eta(x)$.

We recall that the shape operator of M is the section A in the bundle $\operatorname{Hom}(TM, TM)$ given by

$$A_x(X) = -\nabla_X \eta\,|_x$$

The proposition below establishes a relationship between γ and the extrinsic geometry of M.

3. Proposition. *For any $x \in M$, we have the identity*

$$(L_x)_* \circ d\gamma(x) = -(A_x + \alpha_x) \tag{1}$$

Proof. Let $X_1, \ldots, X_n$ be an orthonormal basis of $T_x(G)$ such that $X_n(x) = \eta(x)$. Let $Z_1, \ldots, Z_n$ be the left translation of $X_1, \ldots, X_n$ and set $W_i = Z_i(e)$, $i = 1, \ldots, n$, e the identity of G.

Given $y \in M$, one can write

$$\eta(y) = \sum_{j=1}^{n} a_j(y) Z_j(y) \tag{2}$$

where $a_j(x) = 0$ if $1 \leq j \leq n-1$ and $a_n(x) = 1$. Therefore,

$$\gamma(y) = (L_{x^{-1}})_*(\eta(y)) = \sum_{j=1}^{n} a_j(y) W_j$$

and hence

$$(L_x)_* \circ d\gamma(x)(X) = \sum_{j=1}^{n-1} X(a_j) W_j. \tag{3}$$

From (2), we obtain

$$\nabla_X \eta|_x = \sum_{j=1}^{n-1} X(a_j) W_j + \nabla_X W_n|_x$$

and this with (3) gives (1).

4. *Remark.* The invariant shape operator α depends essentially on G. It is determined by $(n-1) \times (n-1)$-matrices

$$\alpha_B = \left(\langle \nabla_{W_i} W_n, W_j \rangle\right), \quad 1 \leq i, j \leq n-1$$

where $B = \{W_1, \ldots, W_n\}$ is an orthonormal basis of left invariant vector fields.

Clearly, if F is an isometric automorphism of Γ, then $\alpha_B = \alpha_{F(B)}$. Therefore, if O_n denotes the set of orthonormal n-frames of Γ, the set of matrices α_B is parametrized by $O_n/I(G)$, where $I(G)$ is the group of isometric automorphism of G.

5. *Special cases.* (a) *The commutative case.* When G is commutative, we see that the invariant shape operator α of M is identically zero, and γ is the usual Gauss map of M. Therefore, K is the Gauss-Kronecker curvature of M.

(b) *The sphere S^3.* We consider in S^3 a bi-invariant metric. Then, for any $x \in S^3$, the adjoint representation $\mathrm{Ad}_x: \zeta \to \zeta$, ζ being the Lie algebra of G, is an isometry. It is known that $\mathrm{Ad}_G = O(3) =$ orthogonal group, which acts transitively on O_3. Therefore, by Remark 4, α is constant.

Computations using the quaternionic model for S^3 yields

$$\alpha = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Therefore, given $x \in M$, we have

$$K(x) = \det(L_x)_* d\gamma(x) = \det - (A_x + \alpha_x) = \det - A_x + 1,$$

and, from the Gauss-Codazzi equation of an isometric immersion, K is the intrinsic curvature of M.

The lemma that follows is necessary for proving Theorem 7. Its proof was given by Prof. Sebastiani.

6. Lemma. *Let M be a compact connected differentiable manifold and*

$$f \colon M \to [-1, 1]$$

a differentiable map. Let $C = \{x_1, \dots, x_k, y_1, \dots, y_l\}$ be the set of critical points of f, and assume that $f(x_i) = 1$ and $f(y_j) = -1$, $1 \le i \le k, 1 \le j \le l$. Then M is homeomorphic to a sphere.

Proof. If $n = 1$ there is nothing to proof, so that we assume $n > 1$.

We prove that $k = l = 1$. Therefore, from Theorem 1′ of [4], M is homeomorphic to a sphere. Let us choose a Riemannian metric in M and set $X = \operatorname{grad}(f)$.

Given $x \in M$, let $s_x \colon R \to M$ be the trajectory of X such that $s_x(0) = x$. It is not difficult to prove that the ω-limit of s_x is contained in $\{x_1, \dots, x_k\}$ and that the α-limit of s_x is contained in $\{y_1, \dots, y_l\}$, for any $x \in M$ such that $x \notin \{x_1, \dots, x_k\} \cup \{y_1, \dots, y_l\}$.

Given $1 \le i \le k$, set $U_i = \{x \in M | \lim_{t \to \infty} s_x(t) = x_i\}$. $U_i \neq \varnothing$ since $x_i \in U_i$. We prove that U_i is open. By contradiction, let $z_n \in M - U_i$ be such that $z_n \to z \in U_i$. Set $L = \inf_{i \neq j} d(x_i, x_j)$ and choose $R > 0$. Then, there exists $T > 0$ such that

$$d(s_x(T), x_i) < L/2 \quad \text{and} \quad f(s_x(T)) > 1 - R.$$

There exists N such that if $n > N$, then

$$d\big(s_{z_n}(T), x_i\big) < L/2 \quad \text{and} \quad f\big(s_{z_n}(T)\big) > 1 - R.$$

Since one can also find T' such that $d(s_{z_n}(T'), x_i) > L/2$, there exists T_0 such that

$$d\big(s_{z_n}(T_0), x_i\big) = L/2 \quad \text{and} \quad f\big(s_{z_n}(T_0)\big) > f\big(s_{z_n}(T)\big) > 1 - R.$$

Given $r > 0$, let $u_r = s_{z_n}(T_0)$ be such that

$$f(u_r) > 1/r \quad \text{and} \quad d(u_r, x_i) = L/2.$$

Then, taking the limit for $r \to \infty$, we obtain a contradiction with the choice of L.

Since $n \geq 2$, $M - \{y_1, \ldots, y_l\}$ is connected. So $k = 1$. Similar reasoning shows that $l = 1$, proving the lemma.

7. THEOREM. *Let M be an orientable compact hypersurface of G which has no degenerate points. Then either M is homeomorphic to a sphere or the set in which $K = 0$ is non-denumerable.*

Proof. Set $T = \{x \in M | K(x) = 0\}$, and let us assume that T is denumerable. Given $t \in T$, set

$$R_t = d\gamma(t)(T_t(M))^{\perp} \cap S^{n-1}.$$

Since M has no degenerate points, $\dim(d\gamma(t)(T_t(M))) > 0$, and R_t is a totally geodesic k-sphere of S^{n-1} with $k < n - 1$. Therefore

$$W := S^{n-1} - \left(\bigcup_{t \in T} R_t \cup \gamma(T) \right) \neq \varnothing$$

Choose $v \in W$ and define $f\colon M \to [-1, 1]$ by $f(x) = \langle \gamma(x), v \rangle$. Set

$$U = \gamma^{-1}(v) \cup \gamma^{-1}(-v).$$

Then U is finite since it contains just regular points of γ. Furthermore, $f(x) = \pm 1$, for any $x \in U$.

If y is a critical point of f, then $df(y) = \langle d\gamma(y), v \rangle = 0$ and thus

$$v \in d\gamma(y)(T_y(M))^{\perp}.$$

Hence, $y \in T$, that is, y is a regular point of γ. But then,

$$d\gamma(y)(T_y(M))^{\perp} = \operatorname{span}\{\gamma(y)\}$$

and $v = \pm\gamma(y)$, that is, $y \in U$. We now apply Lemma 6 to conclude the proof.

8. THEOREM. *Let M be a compact orientable hypersurface of even dimension. Let ω be the volume element of M with respect to the induced metric of G. Then*

$$(*) \qquad \int_M K\omega = \tfrac{1}{2} c_{n-1} \chi(M)$$

here $\chi(M)$ is the Euler characteristic of M and $c_{n-1} = \mathrm{vol}(S^{n-1}(1))$.

Proof. It follows from Fubini's Theorem that

$$\int_M K\omega = \int_M \gamma^*(\sigma) = \deg(\gamma)\int_{S^{n-1}} \sigma = c_{n-1}\deg(\gamma).$$

We prove that $\deg(\gamma) = \frac{1}{2}\chi(M)$. For let $\tilde{\gamma}$ be the section in the bundle

$$\mathrm{Hom}(TG, Gx\Gamma)$$

given by

$$\tilde{\gamma}_x(X) = (L_{x^{-1}})_*(X).$$

Hence, $\tilde{\gamma}_x(\eta(x)) = \gamma(x)$, $x \in M$. Since $\tilde{\gamma}_x\colon T_x(G) \to \Gamma$ is an isometry, we have

$$\tilde{\gamma}_x(T_x(M)) = T_{\gamma(x)}(S^{n-1}), \quad \dim(M) = n-1,$$

so that we have a bundle morphism

$$\begin{array}{ccc} TM & \xrightarrow{\tilde{\gamma}} & TS^{n-1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\gamma} & S^{n-1}. \end{array}$$

Therefore, if c_M and c_S are the obstructions to extend to M and S^{n-1} vector fields without critical points, and since $n-1$ is even, we obtain

$$\begin{aligned}\chi(M) &= (c_M, [M]) = (\gamma^*(c_S), [M]) \\ &= (c_S, \gamma^*([M])) = (c_S, \deg(\gamma)[S^{n-1}]) \\ &= \deg(\gamma)(c_S, [S^n]) = \deg(\gamma)\chi(S^n) = 2\deg(\gamma),\end{aligned}$$

where the bracket denotes fundamental class in homology and (,) the duality between homology and cohomology.

Remark. If M is odd dimensional there is no relationship between the degree of γ and $\chi(M)$. In fact, in this case, $\chi(M) = 0$. The best one can assert in the case of arbitrary dimension is that $\int_M K\omega = c_{n-1}\deg(\gamma)$.

9. THEOREM. *Assume that the metric of M is bi-invariant. Let M be an orientable compact hypersurface of G whose second fundamental form is positive definite. Then M is diffeomorphic to a sphere.*

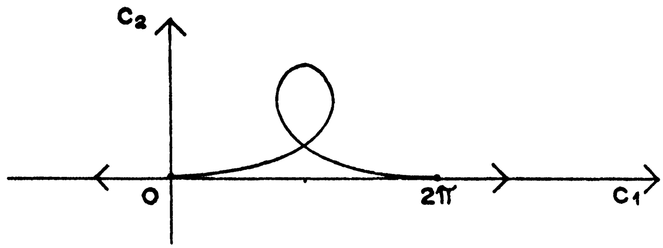

FIG. 1

Proof. The metric of G being bi-invariant, the Riemannian connection of G is given by

$$\nabla_X Z = \tfrac{1}{2}[X, Z] \quad (\text{see [3]}).$$

It follows that the invariant shape operator of M is skew-symmetric.

It is not difficult to prove the sum of a positive definite operator plus a skew-symmetric one is inversible. Hence, we apply Proposition 3 to conclude that

$$\gamma\colon M \to S^{n-1}$$

is a local diffeomorphism. Since M is compact, γ is a global diffeomorphism.

Remark. Contrary to what happens in Euclidean spaces, $\gamma\colon M \to S^{n-1}$ can be a diffeomorphism without M being embedded. The following example was given by Prof. Sebastiani.

Let $M = S^1 = \{x \in \mathbf{C} \mid \|x\| = 1\}$ and $G = S^1 \times \mathbf{R}$. Let $c\colon [0, 2\pi] \to \mathbf{C}$, $c(t) = c_1(t) + ic_2(t)$, be a differentiable curve whose image is

We define $f\colon M \to G$ by $f = (e^{c_1}, c_2)$.

In the proposition that follows we give a characterization of the degenerate points of a hypersurface.

10. PROPOSITION. *Let M be an orientable hypersurface of G. If $x \in M$ is degenerate then $(L_{x^{-1}})_*(T_x(M))$ is a Lie subalgebra of codimension* 1 *of* Γ.

Proof. Let $x \in M$ be a degenerate point. It follows from Proposition 3 that $\alpha_x = -A_x$. Since A_x is a symmetric operator, so is α_x.

Let $\{X_1, \ldots, X_{n-1}\}$ be a basis of $T_x(M)$ and set $W_j = (L_{x^{-1}})_*(X_j)$, $1 \le j \le n-1$ and $W = \gamma(x)$. We must then have

$$\langle \nabla_{W_i} W, W_j \rangle = \langle \nabla_{W_j} W, W_i \rangle, \quad 1 \le i, j \le n-1.$$

In terms of invariant vector fields, the covariant derivative of G is given by

$$\langle \nabla_{W_i} W, W_j \rangle = \tfrac{1}{2}\left\{ \langle [W_i, W], W_j \rangle - \langle [W, W_j], W_i \rangle + \langle [W_j, W_i], W] \rangle \right\} \quad ([3])$$

Computations then show

$$\langle [W_i, W_j], W \rangle = 0, \quad 1 \leq i, j \leq n-1,$$

that is, $[W_i, W_j] \in \operatorname{span}\{W_1, \ldots, W_{n-1}\}$.

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