ILLINOIS JOURNAL OF MATHEMATICS
Volume 35, Number 2, Summer 1991

A NOTE ON RAMANUJAN COEFFICIENTS OF ARITHMETICAL FUNCTIONS

BY

PAOLO CODECÀ

Introduction. *In* 1984 *H. Delange* ([2]) *proved the following result*:

THEOREM 1. *Let f be an arithmetical function and* $f' = f * \mu$. *Let q be any positive integer. Suppose that*:

(i) $\sum_{n \leq x} |f(n)| = O(x)$.

(ii) *For each positive integer d dividing q, the limit*

$$m_d = \lim_{x \to +\infty} \frac{1}{x} \sum_{\substack{n \leq x \\ (n,q)=d}} f(n)$$

exists. Then the series

$$\sum_{\substack{n=1 \\ n \equiv 0(q)}}^{\infty} \frac{f'(n)}{n}$$

converges and its sum is

$$\frac{1}{\varphi(q)} \lim_{x \to +\infty} \frac{1}{x} \sum_{n \leq x} f(n) c_q(n)$$

where $c_q(n)$ *is the Ramanujan sum*

$$\sum_{\substack{h=1 \\ (h,q)=1}}^{q} e(hn/q).$$

In this note we prove (see Theorem 2 below) a simple result about the existence of the mean value for certain convolutions. This result gives

Received June 20, 1989.
1980 Mathematics Subject Classification (1985 Revision). Primary 11N64; Secondary 11K65.

immediately (under the hypotheses of Theorem 1) the equality

$$(1)\qquad \sum_{\substack{n=1\\ n\equiv 0(q)}}^{\infty} \frac{f'(n)}{n} = \sum_{dq'=q} \frac{\mu(q')}{\varphi(q')} m_d$$

which is the main step in Delange's proof of Theorem 1 (see [2] p. 34, first formula).

As for the other equality of Theorem 1, namely

$$(2)\qquad \sum_{\substack{n=1\\ n\equiv 0(q)}}^{\infty} \frac{f'(n)}{n} = \frac{1}{\varphi(q)} \lim_{x\to+\infty} \frac{1}{x} \sum_{n\le x} f(n)c_q(n)$$

it can be checked directly as in the paragraphs 3.6.1 and 3.6.2. of [2], but it should be noted that this is always the case, in the sense that when both sides of (2) exist for every q then they must also be equal. This is implied by theorem 3 below. To be precise the results we prove are the following.

THEOREM 2. *Let $g:[1,+\infty)\to \mathbf{R}$ be a function of bounded variation on finite intervals and let us suppose that*:

(i) *There is a number $\alpha > 1$ such that $g(x) = O(x/lg^{\alpha}x)$*

(ii) *There is a number $0<\beta<1$ such that $V_g(x^{\beta}) = o(x)$, where $V_g(y)$ is the total variation of g in the interval $[1, y]$.*

Let $f(n)$ be a sequence of real numbers such that

(iii) *$\Sigma_{n\le x}|f(n)| = O(x)$ and $\lim_{x\to+\infty} 1/x\Sigma_{n\le x} f(n) = m$ exists.*

Then we have

$$(3)\qquad \lim_{x\to+\infty} \frac{h(x)}{x} = m\int_1^{+\infty} (g(x))x^{-2}\,dx$$

where $h(x) = \Sigma_{n\le x} f(n)g(x/n)$.

THEOREM 3. *Suppose that, for all d dividing q, the series*

$$\sum_{\substack{n=1\\ n\equiv 0(d)}}^{\infty} \frac{f'(n)}{n}$$

converges, and that the series

$$\sum_{n=1}^{\infty} \frac{f(n)c_q(n)}{n^s}$$

converges for $\operatorname{Re} s = \sigma > 1$. *Then, for* $\operatorname{Re} s = \sigma > 1$, *we have*

(4)
$$F_q(s) = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{f(n)c_q(n)}{n^s}$$
$$= \sum_{\delta|q} \frac{1}{\sigma^2} \left(\sum_{d'|\delta} d'\mu\left(\frac{q}{d'}\right) \right) \left(\sum_{d^*|q/\delta} \frac{\mu(d^*)}{d^{*s}} \right) \left(\sum_{\gamma|\delta} \gamma^s \left(\sum_{\substack{d=1 \\ (d,q)=\gamma}}^{\infty} \frac{f'(d)}{d^s} \right) \right)$$

from which it follows that

$$\lim_{\sigma \to 1^+} F_q(\sigma) = \varphi(q) \sum_{\substack{d=1 \\ d \equiv 0(q)}}^{\infty} \frac{f'(d)}{d} \tag{5}$$

This implies equality (2) *if both sides exist for every* $q \geq 1$.

Before proving Theorems 2 and 3 we want to show how equality (1) can be obtained from Theorem 2.

Under the hypothesis of Theorem 1 let

$$H_q(x) = \sum_{\substack{n \leq x \\ n \equiv 0(q)}} \frac{f'(n)}{n}.$$

A simple calculation gives

$$H_q(x) = \sum_{d \leq x} \frac{f(d)}{d} \left(\sum_{\substack{n \leq x/d \\ n \equiv 0(q/(q,d))}} \frac{\mu(n)}{n} \right) \tag{6}$$
$$= \sum_{\delta|q} \left(\frac{1}{x} \sum_{\substack{d \leq x \\ (d,q)=\delta}} f(d) \frac{x}{d} \left(\sum_{\substack{n \leq x/d \\ n \equiv 0(q/\delta)}} \frac{\mu(n)}{n} \right) \right) = \sum_{\delta|q} \Sigma_\delta(x)$$

If we remember the estimate

$$M_{q/\delta}(x) = \sum_{\substack{n \leq x \\ n \equiv 0(q/\delta)}} \frac{\mu(n)}{n} = O\left(\frac{1}{lg^\alpha x}\right) \text{ with } \alpha > 1 \tag{7}$$

we see that we can apply Theorem 2 to each term of the sum in (6) and obtain

$$(8)\qquad \lim_{x\to+\infty} H_q(x) = \sum_{\delta|q} \lim_{x\to+\infty} \textstyle\sum_\delta(x) = \sum_{\delta|q} m_\delta \int_1^{+\infty} \left(M_{q/\delta}(x)\right) x^{-1}\,dx$$

This proves (1) since

$$(9)\qquad \int_1^{+\infty} \left(M_{q/\delta}(x)\right) x^{-1}\,dx = \frac{\mu(q/\delta)}{\varphi(q/\delta)}$$

In order to justify (7) and (9) we remember that if χ_0 is the principal character mod q we have

$$(10)\qquad \sum_{n=1}^{\infty} \frac{\mu(n)\chi_0(n)}{n^s} = \frac{1}{L(s,\chi_0)} = \frac{1}{\zeta(s)} \prod_{p|q} (1-p^{-s})^{-1}$$

From (10) we obtain, by standard techniques, the estimate

$$(11)\qquad \sum_{\substack{n\le x\\(n,q)=1}} \frac{\mu(n)}{n} = \sum_{n\le x} \frac{\mu(n)\chi_0(n)}{n} = O\left(\frac{1}{lg^\alpha x}\right), \quad \alpha > 1,$$

which gives (7) with q/δ replaced by q, since

$$(12)\qquad \sum_{\substack{n\le x\\ n\equiv 0(q)}} \frac{\mu(n)}{n} = \frac{\mu(q)}{q} \sum_{\substack{n\le x/q\\(n,q)=1}} \frac{\mu(n)}{n}$$

From (11) we see, by partial summation, that the series

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{\mu(n)\,lgn}{n}$$

is convergent, and if we differentiate both sides of (10) and take the limit for $\sigma \to 1^+$ we get

$$(13)\qquad \sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{\mu(n)\,lgn}{n} = -\frac{q}{\varphi(q)}$$

This gives (9), with q/δ replaced by q, again by partial summation.

Let us now prove Theorems 2 and 3.

Proof of Theorem 2. We have

$$h(x) = \sum_{n\leq x} (f(n) - m) g\left(\frac{x}{n}\right) + m \sum_{n\leq x} g\left(\frac{x}{n}\right) = \sum\nolimits_1(x) + \sum\nolimits_2(x) \tag{14}$$

Because of hypotheses (i) and (iii) of Theorem 2, a variant of Axer's theorem (see for example, [1], p. 127, problem 52) implies that

$$\sum\nolimits_1(x) = o(x) \quad \text{as } x \to +\infty \tag{15}$$

so that we can write

$$h(x) = m \sum_{x^{1-\beta}<n\leq x} g\left(\frac{x}{n}\right) + o(x) \quad \text{as } x \to +\infty. \tag{16}$$

Now we use Euler's summation formula with Stieltjes integral. If $F(t)$ is a function of bounded variation on the interval $[n, n+1]$ where n is a positive integer, we have

$$\int_n^{n+1} \left(t - n - \tfrac{1}{2}\right) dF(t) = \tfrac{1}{2}(F(n+1) + F(n)) - \int_n^{n+1} F(t)\, dt \tag{17}$$

Summing over n we get

$$\sum_{n=m}^{N} F(n) = \int_m^N F(t)\, dt + \tfrac{1}{2}(F(m) + F(N)) + O\left(V_{F_{[m,N]}}\right) \tag{18}$$

where $V_{F_{[m,N]}}$ is the total variation of F in the interval $[m, N]$. If we choose $F(t) = g(x/t)$, $m = [x^{1-\beta}] + 1$, $N = [x]$ and put $x/t = u$ in the integral in (18) we obtain

$$\sum_{x^{1-\beta}<n\leq x} g\left(\frac{x}{n}\right) = x\int_1^{x^\beta} (g(u))u^{-2}\, du + o(x) \quad \text{as } x \to +\infty \tag{19}$$

because of hypotheses (i) and (ii). Equation (3) follows from (16) and (19), and this proves Theorem 2.

Proof of Theorem 3. For the proof of (4) we may suppose Res > 2, for the hypotheses imply that both sides exist for Res > 1 and are analytic functions of s in this half-plane.

Since the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{f(n)c_q(n)}{n^s}$$

is convergent for Re$s > 1$ it is absolutely convergent for Re$s > 2$. This permits to write for Re$s > 2$

$$(20) \qquad \sum_{n=1}^{\infty} \frac{f(n)c_q(n)}{n^s} = \sum_{\delta|q} \left(\sum_{\substack{n=1 \\ (n,q)=\delta}}^{\infty} \frac{f(n)c_q(n)}{n^s} \right)$$
$$= \sum_{\delta|q} \left(\sum_{d'|\delta} d'\mu\left(\frac{q}{d'}\right) \right) \sum_{\substack{n=1 \\ (n,q)=\delta}}^{\infty} \frac{f(n)}{n^s}$$

Now

$$(21) \qquad \sum_{\substack{n=1 \\ (n,q)=\delta}}^{\infty} \frac{f(n)}{n^s} = \sum_{\substack{n=1 \\ (n,q)=\delta}}^{\infty} \left(\frac{1}{n^s} \sum_{d|n} f'(d) \right)$$
$$= \sum_{d=1}^{\infty} f'(d) \left(\sum_{\substack{n=1 \\ (n,q)=\delta \\ n \equiv 0(d)}}^{\infty} \frac{1}{n^s} \right)$$

Here the change of the order of summations is justified by the fact that the series $\sum_{d=1}^{\infty} f'(d)\, d^{-s}$ is absolutely convergent for Re$s > 2$ and

$$(22) \qquad \sum_{\substack{d \geq 1 \\ n \geq 1 \\ (n,q)=\delta \\ n \equiv 0(d)}} \left| \frac{f'(d)}{n^s} \right| = \sum_{\substack{d \geq 1 \\ m \geq 1 \\ (md,q)=\delta}} \left| \frac{f'(d)}{m^s d^s} \right| \leq \zeta(\sigma) \sum_{d=1}^{\infty} \left| \frac{f'(d)}{d^s} \right|.$$

It is not difficult to see that

$$(23) \qquad \sum_{\substack{n=1 \\ (n,q)=\delta \\ n \equiv 0(d)}} \frac{1}{n^s} = \begin{cases} 0 & \text{if } (d,q) \nmid \delta, \\ \zeta(s) \displaystyle\sum_{d^*|q/\delta} \mu(d^*) \left(\frac{(d,q)}{dd^*\delta} \right)^s & \text{if } (d,q)|\delta \end{cases}$$

This gives

(24)

$$\sum_{d=1}^{\infty} f'(d)\left(\sum_{\substack{n=1\\(n,q)=\delta\\n\equiv 0(d)}}^{\infty} \frac{1}{n^s}\right) = \zeta(s) \sum_{\substack{d=1\\(d,q)\mid\delta}}^{\infty} f'(d)\left(\sum_{d^*\mid q/\delta} \mu(d^*)\left(\frac{(d,q)}{dd^*\delta}\right)^s\right)$$

$$= \frac{\zeta(s)}{\delta^s} \sum_{d^*\mid q/\delta} \frac{\mu(d^*)}{d^{*s}}\left(\sum_{\substack{d=1\\(d,q)\mid\delta}}^{\infty} \frac{f'(d)(d,q)^s}{d^s}\right)$$

$$= \frac{\zeta(s)}{\delta^s}\left(\sum_{d^*\mid q/\delta} \frac{\mu(d^*)s26}{d^{*s}}\right)\left(\sum_{\gamma\mid\delta}\gamma^s\left(\sum_{\substack{d=1\\(d,q)=\gamma}}^{\infty} \frac{f'(d)}{d^s}\right)\right)$$

and the announced result follows from (20), (21) and (24). To prove (5) we first note that

(25)

$$\lim_{\sigma\to 1^+} F_q(\sigma) = \sum_{\delta\mid q}\left(\sum_{d'\mid\delta}\mu\left(\frac{q}{d'}\right)\frac{d'}{\delta}\right)\left(\sum_{d^*\mid q/\delta}\frac{\mu(d^*)}{d^*}\right)\left(\sum_{\gamma\mid\delta}\gamma\left(\sum_{\substack{d=1\\(d,q)=\gamma}}^{\infty}\frac{f'(d)}{d}\right)\right)$$

since the hypothesis

$$\sum_{\substack{n=1,\\n\equiv 0(d)}}^{\infty} \frac{f'(n)}{n}$$

convergent for every d dividing q implies

$$\sum_{\substack{d=1\\(d,q)=\gamma}}^{\infty} \frac{f'(d)}{d}$$

convergent for every γ dividing q. Now we observe that

$$(26)\quad \left(\sum_{d'\mid\delta}\mu(q/d')\frac{d'}{\delta}\right)\left(\sum_{d^*\mid q/\delta}\frac{\mu(d^*)}{d^*}\right) = \mu\left(\frac{q}{\delta}\right)\left(\sum_{\substack{d'\mid\delta\\(d',q/\delta)=1}}\frac{\mu(d')}{d'}\right)$$

$$\times\left(\sum_{d^*\mid(q/\delta)}\frac{\mu(d^*)}{d^*}\right)$$

$$= \mu\left(\frac{q}{\delta}\right)\varphi(q)\frac{1}{q}$$

where the last equality in (26) is justified since every square-free divisor d of q admits a unique representation as $d = d'd^*$ with $d^* | q/\delta$, $d' | \delta$ and $(d', q/\delta) = 1$.

Equation (5) follows immediately from (25) and (26) by the Möbius inversion formula.

If the limit

$$\lim_{x \to +\infty} \frac{1}{x} \sum_{n \leq x} f(n) c_q(n) = l(q)$$

exists we must also have

$$\lim_{\sigma \to 1^+} F_q(\sigma) = l(q) \tag{27}$$

This proves that when both sides of equation (2) exist for every q they must be equal. This concludes the proof of Theorem 3. It should be noted that the existence of one side of equation (2) does not, in general, imply the existence of the other one. In fact, for the case $q = 1$ for example, since

$$\frac{1}{x} \sum_{n \leq x} f(n) = \sum_{d \leq x} \frac{f'(d)}{d} \left[\frac{x}{d} \right] \frac{d}{x}$$

this is equivalent to saying that there exist Ingham-summable series which are not convergent and vice versa, and this is well known (see for example, [4], p. 376; [6], p. 10–13; [3], p. 98; [5], p. 180).

The author wishes to thank the referee for his very valuable suggestions.

References

1. R. Ayoub, *An introduction to the analytic theory of numbers*, Math. Surveys, Number 10, Amer. Math. Soc., 1963.
2. H. Delange, *On a formula for almost-even arithmetical functions*, Illinois J. Math., vol. 31 (1987), pp. 24–35.
3. P. Erdös and S.L. Segal, *A note on Ingham's summation method*, J. Number Theory, vol. 10 (1978), pp. 95–98.
4. G.H. Hardy, *Divergent series*, Oxford Univ. Press, London, New York, 1947.
5. A.E. Ingham, *Some tauberian theorems connected with prime number theorem*, J. London Math. Soc., vol. 20 (1945), pp. 171–180.
6. A. Wintner, *Eratosthenian averages*, Baltimore, 1943.

Università degli Studi di Ferrara
Ferrara, Italy