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COMPLETION REGULAR MEASURES ON PRODUCT SPACES WITH APPLICATION TO THE EXISTENCE OF BAIRE STRONG LIFTINGS[1]

BY

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1. Introduction

In [5] and [13] it is proved that, subject to the continuum hypothesis (CH), every (positive) Radon measure supported on a space with a topology basis of card $\leq c$ admits a Baire lifting as well as a Borel strong lifting.

On the other hand, relatively little is known about the existence of Baire liftings that are, at the same time, strong. On the positive side, Baire strong liftings have been shown to exist, when CH is assumed, in a very restricted class of measures: on any product of less than or equal to $\aleph_2$ supported Radon measures, each on a compact metric space ([11]—see also [15]). We also mention that D. Maharam in [12] proved that existence of a (completion) Baire strong lifting for the product measure, if each compact space is either a closed unit interval or two point space, without assuming CH and for any number of factors. On the negative side, D.H. Fremlin has exhibited a completion regular measure on $[0,1]^{\aleph_2}$ that admits no strong lifting [6].

The question that naturally arises is: what (completion regular) measures on $[0,1]^{\aleph_1}$ admit a Baire strong lifting?

We prove that, under CH, the answer is always positive. The proof, in the spirit of [11], is based on a special characterization of a class of "maximal" open sets, for a completion regular measure on an arbitrary product of compact spaces (Lemma 2).

In the sequel, trying to find Baire strong liftings for a product measure, we investigate -in connection with some questions posed in [3] and [2]-which products of two compact completion regular measure spaces (X, μ), (Y, ν) are completion regular. We establish that, subject to Martin's Axiom and the negation of the continuum hypothesis, such a product is completion regular, provided that one of these topological factors is of the form $\Pi_{j \in J} Y_j$, with the Y_j compact metric spaces (Theorem 2).

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This results extends a theorem of Choksi and Fremlin [3, Th. 3] and shows that a question posed in [3] (see note on p. 121; see also [2]) can be answered in the affirmative under quite general assumptions.

In the particular case that both X and Y are products of compact metric spaces, we prove that $\mu \times \nu$ is completion regular without set-theoretical assumptions (this is an earlier result of one of us—see [8]). The proof here (theorem 3) is an order of magnitude simpler that the one given in [8].

Moreover, in this particular case, provided that each X, Y is a product of less than or equal to c compact metric spaces, subject to CH, the measure $\mu \times \nu$ admits a Baire strong lifting; i.e., the Baire strong lifting property is productive (Corollary 2).

2. A lifting theorem

All measure spaces are assumed to be finite.

Let X be a (completely regular Hausdorff topological) space. Let $\mathscr{B}_0(X)$, resp. $\mathscr{B}(X)$, denote the σ-algebra of Baire, resp. Borel, sets in X. Let μ be a (positive Radon) measure on X. Let $\mathscr{B}_\mu(X)$ denote the σ-algebra of μ-measurable sets (i.e., the completion of $\mathscr{B}(X)$ with respect to μ). If $A, B \in \mathscr{B}_\mu(X)$ we write $A \sim B$, if $\mu((A - B) \cup (B - A)) = 0$.

Let Σ be a Boolean subalgebra of $\mathscr{B}_\mu(X)$. A Boolean algebra homomorphism $r\colon \Sigma \to \mathscr{B}_\mu(X)$ is called a lifting if $A \sim rA$ for every $A \in \Sigma$ and $A \sim B$ implies $rA = rB$. r is called strong, if V is a subset of rV for all open sets $V \varepsilon \Sigma$. If there is a strong lifting for $\mathscr{B}_\mu(X)$, we say that μ admits a strong lifting and then, necessarily, the support of μ is the whole space X [9].

The measure μ is called completion regular, if every Borel subset of X is measurable with respect to the completion of the Baire restriction of μ, that is, to every Borel set E, there correspond two Baire sets A and B such that $A \subset E \subset B$ and $\mu(B - A) = 0$.

REMARK 1. If μ admits a Baire strong lifting, one can immediately verify that μ must be completion regular.

THEOREM 1.[2] *We assume the continuum hypothesis* ($c = \aleph_1$).

Let X_i, $i \in I$, *with* card $I \leq c$, *be compact metric spaces and* μ *a completion regular measure on* $X = \Pi_{i \in I} X_i$ *with* supp $\mu = X$. *Then* μ *admits a Baire strong lifting.*

First, we need the following auxiliary notion (see [11, p. 157]). Let Σ be a subalgebra of $\mathscr{B}_\mu(X)$ and a map $d\colon \Sigma \to \mathscr{B}_0(X)$. We shall say that d has the

[2] This theorem has been proved in a much more general context in [7]

property (P), if it satisfies the following conditions:
(1) $A \sim B$ implies $d(A) = d(B)$,
(2) $d(A \cap B) = d(A) \cap d(B)$,
(3) $\mu(d(A) - A) = 0$,
(4) $A^\circ \subset d(A)$(A° denotes the interior of A).
This is essentially the definition of a strong lower density—see [9, p. 36, 64] —with the exception that we do not require $d(A) \sim A$.
The following lemma is from [11] (Lemma 3)—see also [5] or [10].

LEMMA 1. *Let Σ, d be as above with* card $\Sigma \leq \aleph_1$. *Then there exists a Baire strong lifting r on Σ such that $d(A) \subset rA$, for all $A \in \Sigma$.*

The following lemma is of independent interest.

LEMMA 2. *Let $\langle Y_j \rangle_{j \in J}$ be a family of compact spaces and ν a completion regular measure on $Y = \Pi_{j \in J} Y_j$. Then*:
For every ν-measurable subset A of Y, the set

$$U\{V \text{ open in } Y, \text{ such that } \nu(A \cap V) = 0\}$$

depends on a countable subset of J.

Proof. We can assume that A is (compact and) supporting, i.e., A is nonempty and $\nu(A \cap U) > 0$, for every open set U that meets A.

Claim 1. There exists a countable subset J_A of J such that, for every $W = \Pi_{j \in I} W_j \times \Pi_{j \in J-I} Y_j$, $I \subset J$ finite with $I \cap J_A = \varnothing$ and $W_j \subset Y_j$ open, we have $\nu(A \cap W) > 0$.

Let D be a Baire subset of A, with $\nu(D) = \nu(A)$ and J_A a countable subset of J, on which D depends. Suppose, if possible, that the claim is not true, that is, there is an open set W of the above form such that $\nu(A \cap W) = 0$. Then, since A is supporting, $D \subset A \subset Y - W$. But, $Y - W$ depends on $J - J_A$, while D depends on J_A, contradiction.

Claim 2. Let $W = \Pi_{j \in I} W_j \times \Pi_{j \in J-I} Y_j$, $I \subset J$ finite and $W_j \subset Y_j$ open, such that $\nu(A \cap W) = 0$. Then, there is an open set W_1 containing W that depends on J_A and $\nu(A \cap W_1) = 0$.

Express W as $W_1 \cap W_2$, where W_1, W_2 are basic open sets, W_1 depends on J_A and W_2 depends on $J - J_A$. We shall prove that $\nu(A \cap W_1) = 0$.
Suppose, if possible, that $\nu(A \cap W_1) > 0$. We apply claim 1, with $A \cap W_1$ in place of A and we find a countable subset $J_{A \cap W_1}$ of J satisfying the claim. Since $D \cap W_1 \subset A \cap W_1$ and $\nu(D \cap W_1) = \nu(A \cap W_1)$, it is clear, from the

proof of claim 1, that we can choose $J_{A \cap W_1} = J_A$. But, $\nu((A \cap W_1) \cap W_2) = \nu(A \cap W) = 0$, contradiction. So, we have $\nu(A \cap W_1) = 0$.

Now, the conclusion of the lemma follows from claim 2.

REMARK 2. It is clear that, if each Y_j is a compact metric space,

$$U\{V \text{ open in } Y: \nu(A \cap V) = 0\}$$

is a Baire subset of Y. This follows also from Lemma 2, p. 121 of [3].

The next lemma is the key of our argument.

LEMMA 3. *We have the assumptions of Theorem* 1. *For* $A \in \mathscr{B}_\mu(X)$, *set*

$$C_A := \{V \textit{ open in } X: \nu(V - A) = 0\} \quad \textit{and} \quad d(A) := \bigcup_{V \in C_A} V.$$

Then, d satisfies (P).

Proof. Properties (1), (2), (3) and (4) follow easily from the definition of d and the regularity of μ. By Remark 2, we have $d(A) \in \mathscr{B}_0(X)$, for all $A \in B_\mu(X)$.

Proof of Theorem 1. We consider the measure algebra Ω of μ. For $a \in \Omega$ set

$$Q(a) := \{V \text{ open}: V \in a\}$$

and take $\hat{a} = U_{V \in Q(a)} V$, if $Q(a) \neq \varnothing$ and $\hat{a}$ an arbitrary element of a, otherwise. Clearly, $\hat{a} \in a$. Let Σ be the subalgebra of $\mathscr{B}_\mu(X)$ generated by the set $\{\hat{a}: a \in \Omega\}$. Then, card $\Sigma \leq c$. We use Lemma 3 to define a map $d: \Sigma \to \mathscr{B}_0(X)$ with the property (P). Then, by Lemma 1, there is a Baire strong lifting on Σ which extends naturally on $\mathscr{B}_\mu(X)$.

REMARK 3. The proof of Theorem 1 shows that every compact measure space of topological weight $\leq c$ that has the property that every open set has an open Baire cover of the same measure admits a Baire strong lifting. On the other hand, it is known (see e.g. [1, p. 84]) that the Haar measure on a (compact) group has this property. So, we have the following:

COROLLARY 1. *Subject to the continuum hypothesis, the Haar measure on a compact group of topological weight $\leq c$ admits a Baire strong lifting.*

3. On products of completion regular measures

We begin with two topological notions.

Let X be a Hausdorff topological space. We say that X has the countable chain condition (ccc), if every indexed set $\{U_i, i \in I\}$ of nonempty pairwise disjoint open sets in X is at most countable. We also say that X has caliber $\aleph_1$, if for every uncountable set $\{U_i, i \in I\}$ of nonempty open sets in X there is an uncountable subset J of I such that $\cap_{i \in J} U_i \neq \varnothing$.

We shall need the following well known theorems concerning the above notions.

THEOREM A. *(We assume Martin's Axiom and the negation of the continuum hypothesis). Every compact* ccc *space has caliber* $\aleph_1$.

Proof. See [4, Theorem on p. 201].

THEOREM B. *Let* $\langle X_i \rangle_{i \in I}$ *be a family of compact metric spaces. Then* $\Pi_{i \in I} X_i$ *has caliber* $\aleph_1$.

Proof. See [4], [14].

Assume now that (X, μ) is a compact measure space, $\langle Y_j \rangle_{j \in J}$ an uncountable family of compact metric spaces, ν a completion regular measure on $Y = \Pi_{j \in J} Y_j$ and A a subset of $X \times Y$, measurable with respect to the Radon product measure $\lambda = \mu \times \nu$, with $\lambda(A) > 0$.

LEMMA 4. *(We assume Martin's Axiom and the negation of* CH*). Let* (X, μ), (Y, ν), A *be as above. Then there is a countable subset* M *of* J *such that, for every nonempty open set* V *in* Y *that depends on* $J - M$, *we have* $\lambda(A \cap (B \times V)) > 0$ *for all open sets* B *in* X *with* $\lambda(A \cap (B \times Y)) > 0$.

Proof. We need the following result.

Claim. For every λ-measurable set C with $\lambda(C) > 0$, there is a countable subset N_C of J such that $\lambda(C \cap (X \times V)) > 0$ for all nonempty open sets V in Y that depend on $J - N_C$.

Without loss of generality, we can assume that C is supporting (with respect to λ). By Fubini's theorem, $0 < \lambda(C) = \int \nu(C_t) d\mu(t)$ (where $C_t = \{u \in Y: (t, u) \in C\}$). So, there is an $s \in X$ such that $\nu(C_s) > 0$. Also, by claim 1 in the proof of Lemma 2, there is a countable subset N_C of J such that $\nu(C_s \cap V) > 0$, for every basic open set V in Y depending on $J - N_C$. This N_C satisfies the claim. In fact, let V be any nonempty open set in Y that depends on $J - N_C$. Then, $\nu(C_s \cap V) > 0$, so, $C_s \cap V \neq \varnothing$. Thus, $C \cap$

$(X \times V) \neq \varnothing$. Since C is supporting, $\lambda(C \cap (X \times V)) > 0$. This completes the proof of the claim.

Suppose now, if possible, that the conclusion of the lemma is not true. Without loss of generality, we can assume that A is supporting. Then, for every countable subset M of J we can find a basic open set V_M in Y that depends on a finite subset I_M of $J - M$ and an open set U_M in X such that

$$\lambda\big(A \cap (U_M \times Y)\big) > 0 \quad \text{and} \quad \lambda\big(A \cap (U_M \times V_M)\big) = 0.$$

From this (starting with an arbitrary countable subset M_0 of J), using a transfinite induction argument, we can construct a strictly in creasing family $\langle M_\gamma \rangle_{\gamma < \omega_1}$ of countable subsets of J satisfying the conditions:

(I) $\lambda(A \cap (U_{M_\gamma} \times Y)) > 0$ and $\lambda(A \cap (U_{M_\gamma} \times V_{M_\gamma})) = 0$ for $\gamma < \omega_1$;

(II) $I_{M_\gamma} \subset M_\delta$ and $I_{M_\gamma} \cap I_{M_\delta} = \varnothing$ for $\gamma < \delta < \omega_1$.

Thus, there is an uncountable family $\langle V_\gamma \rangle_{\gamma \in \Gamma}$ of basic open sets in Y such that:

(i) For $\gamma, \delta \in \Gamma$, $\gamma \neq \delta$, V_γ and V_δ depend on pairwise disjoint sets of coordinates;

(ii) For every $\gamma \in \Gamma$ there is an open set U_γ in X such that

$$\lambda\big(A \cap (U_\gamma \times Y)\big) > 0 \quad \text{and} \quad \lambda\big(A \cap (U_\gamma \times V_\gamma)\big) = 0.$$

We have the following two alternative cases.

Case 1. The set $\{U_\gamma : \gamma \in \Gamma\}$ is countable. Then, since Γ is uncountable, we can find $\gamma_0 \in \Gamma$ such that the set $\Delta = \{\gamma \in \Gamma : U_\gamma = U_{\gamma_0}\}$ is uncountable. But then, by (ii), we have

$$(*) \qquad \gamma\big(A \cap (U_{\gamma_0} \times V_\gamma)\big) = \lambda\big(A \cap (U_{\gamma_0} \times Y) \cap (X \times V_\gamma)\big) = 0 \quad \text{for all } \gamma \in \Delta.$$

Now, since $\lambda(A \cap (U_{\gamma_0} \times Y)) > 0$, by the claim, there exists a countable subset N of J such that, for every nonempty open set V in Y that depends on $J - N$ we have

$$\lambda\big(A \cap (U_{\gamma_0} \times Y) \cap (X \times V)\big) > 0.$$

Since Δ is uncountable, by (i), there exists an uncountable subset Δ' of Δ such that V_γ depends on $J - N$, for all $\gamma \in \Delta'$. So,

$$\lambda\big(A \cap (U_{\gamma_0} \times Y) \cap (X \times V_\gamma)\big) > 0 \quad \text{for all } \gamma \in \Delta'.$$

But, this contradicts $(*)$. This contradiction ends the proof of the lemma in case 1.

Case 2. The set $\{U_\gamma\colon \gamma \in \Gamma\}$ is uncountable. Then, by (ii), since A is supporting (with respect to λ), $A \cap (U_\gamma \times V_\gamma) = \varnothing$, for all $\gamma \in \Gamma$ and so

$$(**) \qquad \left(A \cap (U_\gamma \times V_\gamma)\right)_t = A_t \cap \left(X \times V_\gamma\right)_t = A_t \cap V_\gamma = \varnothing$$

for every $\gamma \in \Gamma$ and $t \in U_\gamma$.

Now consider the set $F := \{t \in X\colon \nu(A_t) > 0\}$. Clearly, $\mu(F) > 0$. Also, since

$$\lambda\left(A \cap (U_\gamma \times Y)\right) > 0 \quad \text{for all } \gamma \in \Gamma$$

and

$$\lambda\left(A \cap (U_\gamma \times Y)\right) = \int U_\gamma \nu(A_t)\, d\mu(t),$$

we have $\mu(F \cap U_\gamma) > 0$ for every $\gamma \in \Gamma$.

By the regularity of μ, we can find a compact supporting (with respect to μ) subset K of F and an uncountable subset Γ' of Γ such that $\mu(K \cap U_\gamma) > 0$, for all $\gamma \in \Gamma'$. In particular, $K \cap U_\gamma \neq \varnothing$ for all $\gamma \in \Gamma'$.

Now, because K admits a strictly positive measure, it has the countable chain condition. Thus, since Martin's Axiom and the negation of CH are assumed, Theorem A yields that K has $\aleph_1$ caliber. So, there is an uncountable subset H of Γ' such that $E := \bigcap_{\gamma \in H}(K \cap U_\gamma) \neq \varnothing$.

Take $s \in E$. Then we have $\nu(A_s) > 0$ and so, by claim 1 in the proof of Lemma 2, there exists a countable subset N of J such that $\lambda(A_s \cap V) > 0$, for all basic open sets V in Y that depend on $J - N$. Since H is uncountable, by (i) there is an uncountable subset H' of H such that V_γ depends on $J - N$, for every $\gamma \in H'$. So,

$$\lambda\left(A_s \cap V_\gamma\right) > 0 \quad \text{for all } \gamma \in H'.$$

But this contradicts $(**)$. This contradiction ends the proof of the Lemma in case 2.

LEMMA 5. *Let $X = \Pi_{i \in I} X_i$, where each X_i is compact metric and μ, (Y, ν), A as in Lemma 4. Suppose that μ is completion regular. Then there exists a countable subset M of J, as in the conclusion of Lemma 4.*

Proof. Following the proof of Lemma 4 and using Remark 2, we can choose K to be Baire. Since a Baire set depends on a countable set of coordinates, K is a product of compact metric spaces. Therefore, by theorem B, it has caliber $\aleph_1$ and we complete the proof of Lemma 5, exactly as in Lemma 4.

Now, we are ready to prove the main theorem of this section (compare with Choksi and Fremlin [3, Theorem 3]).

THEOREM 2. *Let X be a compact space, $\langle Y_j \rangle_{j \in J}$ an uncountable family of compact metric spaces and μ, ν completion regular measures on X, $Y = \Pi_{j \in J} Y_j$ respectively. Then, subject to the Martin's Axiom and the negation of the continuum hypothesis, the Radon product measure $\lambda = \mu \times \nu$ is completion regular.*

Proof. Let A be a λ-measurable set in $X \times Y$.

Claim. There is a countable subset P of J such that for every basic open set W in $X \times Y$, with $\lambda(A \cap W) = 0$ we have $\lambda(A \cap pr_1^{-1}(pr_1(W))) = 0$, where pr_1 denotes the canonical projection from $X \times Y$ onto $X \times \Pi_{j \in P} Y_j$.

Suppose, if possible, that the claim is not true. Then (taking an arbitrary countable subset P_0 of J), we can find, by transfinite induction,

(1) a strictly increasing family $\langle P_\alpha \rangle_{\alpha < \omega_1}$ of countable subsets of J,

(2) a family $\langle U_\alpha \rangle_{\alpha < \omega_1}$ of open set in X and

(3) two families $\langle V_\alpha^1 \rangle_{\alpha < \omega_1}$, $\langle V_\alpha^2 \rangle_{\alpha < \omega_1}$ of basic open sets in Y satisfying the following conditions:

(a) for each $\alpha < \omega_1$, if V_α^1, V_α^2 depend on I_α^1, I_α^2 respectively, then, $I_\alpha^1 \subset P_\alpha$, $I_\alpha^2 \cap P_\alpha = \varnothing$ and $U_{\beta < \alpha} I_\beta^2 \subset P_\alpha$;

(b) $\lambda(A \cap (U_\alpha \times (V_\alpha^1 \cap V_\alpha^2))) = 0$ and $\lambda(A \cap (U_\alpha \times V_\alpha^1)) > 0$, for all $\alpha < \omega_1$. Now, applying Erdös-Rado's theorem [4, Theorem 1.4, p. 5] to the uncountable family of sets $\langle I_\alpha^1 \rangle_{\alpha < \omega_1}$, we find an $L \subset \omega_1$ uncountable and a finite subset I of J such that $I_\alpha^1 \cap I_\beta^1 = I$, for $\alpha, \beta \in L$, $\alpha \neq \beta$. Since $\Pi_{j \in I} Y_j$ has a countable basis for its topology, there exist a basic open set G in Y that depends on I and an uncountable subset L' of L such that, for each $\alpha \in L'$, $V_\alpha^1 = G \cap W_\alpha^1$, where W_α^1 is an open set in Y that depends on $I_\alpha^1 - I$.

On the other hand, by (b), we have $\lambda(A \cap (U_\alpha \times (G \cap W_\alpha^1 \cap V_\alpha^2)) = 0$ for $\alpha \in L'$. But, this contradicts Lemma 4, for the set $A \cap (X \times G)$, since the sets $W_\alpha^1 \cap V_\alpha^2, \alpha \in L'$ depend on pairwise disjoint sets of coordinates and $\lambda(A \cap (X \times G) \cap (U_\alpha \times Y)) > 0$. This completes the proof of the claim.

The above claim clearly implies that for any λ-measurable subset A of $X \times Y$, the set $U\{O$ an open subset of $X \times Y$: $\lambda(A \cap O) = 0\}$ depends on a set of coordinates disjoint from $J - P$. Therefore, for every open subset U of $X \times Y$ there exist a countable subset R of J and an open set U' containing U, that depends on a set of coordinates disjoint from $J - R$, with $\lambda(U') = \lambda(U)$.

In order to verify the completion regularity of λ it suffices to prove that for every nonempty open set U in $X \times Y$ there is a Baire set B containing U, such that $\lambda(B) = \lambda(U)$. Indeed, let pr_2: $X \times Y \to X \times \Pi_{j \in R} Y_j$ denote the canonical projection and $\lambda' := pr_2(\lambda)$. Since λ' is a (Radon) product measure and the space $\Pi_{j \in R} Y_j$ has a countable basis for its topology, it is easy to prove that λ' is completion regular [3, Th. 3]. Hence, there is Baire set D in $X \times \Pi_{j \in R} Y_j$ containing $pr_2(U')$ with $\lambda'(D) = \lambda'(pr_2(U'))$. But this means that the (Baire) set $B = D \times \Pi_{j \in J-R} Y_j$ contains U' and $\lambda(B) = \lambda(U)$. This ends the proof of Theorem 2.

THEOREM 3. *Let $\langle X_i \rangle_{i \in I}$, $\langle Y_j \rangle_{j \in J}$ be uncountable families of compact metric spaces and μ, ν completion regular measures on $\Pi_{i \in I} X_i$, $\Pi_{j \in J} Y_j$ respectively. Then the Radon product measure $\mu \times \nu$ is completion regular.*

Proof. The proof is exactly the same as that of Theorem 2, where instead of Lemma 4, we use Lemma 5.

Now, combining Theorem 1 and Theorem 3, we obtain the following:

COROLLARY 2. (*Assume the continuum hypothesis*). *Let μ, ν be completion regular measures supported on $\Pi_{i \in I} X_i$, $\Pi_{j \in J} Y_j$ respectively, where X_i, Y_j are compact metric spaces. Suppose that I, J are both of cardinality $\leq c$. Then the Radon product measure $\mu \times \nu$ admits a Baire strong lifting.*

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