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ON COANALYTIC FAMILIES OF SETS IN HARMONIC ANALYSIS

BY

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I. Introduction

In the last few years many natural families of sets, or of functions, from harmonic analysis were shown to be Π_1^1-hard; the reader will find references in the recent book of A. Kechris and A. Louveau [7]. The goal of the present work is to invite another family to join the club.

Let us recall that a subset Λ of a discrete abelian group Γ is called a *Rosenthal set* [9, Def. 2.1] if $L^\infty_\Lambda(\hat{\Gamma}) = \mathscr{C}_\Lambda(\hat{\Gamma})$. It was shown by H. P. Rosenthal ([12]) that there are (what we call now) Rosenthal sets which are not Sidon. It follows from our main result that if Γ is a countably infinite abelian discrete group, then the family $\mathrm{Ros}(\Gamma)$ of Rosenthal subsets of Γ is a Π_1^1-hard subset of $\mathscr{P}(\Gamma)$. This result means in particular that there is no hope to obtain "positive" characterizations of Rosenthal sets, or that any characterization will be at least as complex as the definition.

Our proofs combine a result of F. Lust-Piquard which enables us to construct Rosenthal sets [10, Th. 3], together with a technique of V. Tardivel [13]; actually, our method provides a proof of Tardivel's result which is slightly simpler than the original one. Let us mention however that our proof uses a delicate result on spectral synthesis due to Loomis [8], which depends on a theorem of Bohr about almost-periodic functions.

Notation. Throughout this paper, Γ denotes an abelian discrete group and $G = \hat{\Gamma}$ its compact dual group. $\mathscr{P}(\Gamma)$ is the power set of Γ; if $\Lambda \in \mathscr{P}(\Gamma)$, $L^\infty_\Lambda(G)$ denotes the space of bounded measurable functions on G, with respect to the Haar measure dm of G, whose Fourier transforms vanish outside Λ. The spaces $\mathscr{M}_\Lambda(G)$, $L^1_\Lambda(G)$ and $C_\Lambda(G)$ are defined similarly. A subset Λ of Γ is called a Rosenthal set if $C_\Lambda(G) = L^\infty_\Lambda(G)$; $\mathrm{Ros}(\Gamma)$ denotes the family of Rosenthal sets. Λ is called a Riesz set if $\mathscr{M}_\Lambda(G) = L^1_\Lambda(G)$, and the family of Riesz sets is $\mathscr{R}(\Gamma)$. A subset A of a Polish space P is Σ_1^1 (i.e. analytic) if it is a continuous image of the Polish space $\mathbf{N}^{\mathbf{N}}$; a subset C of P

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such that $P \setminus C$ is Σ_1^1 is said to be Π_1^1 (i.e., coanalytic); it is Π_1^1-hard if it is Π_1^1 but not Borel—equivalently, Π_1^1 but not Σ_1^1.

The other notations are classical or will be defined before use. Our references about Rosenthal sets are [9] and [10], while we refer to [5], [7], [13] about connections between harmonic analysis and descriptive set theory. The results on the Radon-Nikodym property in Banach spaces which we use can be found in [3].

II. The Results

Our proof will crucially use a result of F. Lust-Piquard [10, Th. 3] which we recall now. We outline a proof for completeness.

THEOREM 1. [10]. *Let Γ be a discrete countable abelian group, and Λ be a subset of Γ. We denote by Δ a countable dense subgroup of $G = \hat{\Gamma}$. Let i^* be the canonical map of restriction to Δ, from Γ into the compact dual group $\hat{\Delta}$ of Δ equipped with its discrete topology. If the closure of $i^*(\Lambda)$ in $\hat{\Delta}$ is countable, then $C_\Lambda(G) = L_\Lambda^\infty(G)$; i.e., Λ is a Rosenthal set.*

Proof. If the Banach space $C_\Lambda(G)$ has a Radon-Nikodym property then $C_\Lambda(G) = L_\Lambda^\infty(G)$ [10, Th. 2.1]. Indeed for every $f \in L_\Lambda^\infty(G)$ we consider the operator $T_f(g) = f * g$ from $L^1(G)$ to $C_\Lambda(G)$. If $C_\Lambda(G)$ has the R.N.P. this operator is representable; that is, there exists $h \in L^\infty(G, C(G))$ such that for every $g \in L^1(G)$,

$$T_f(g) = \int_G h(u) g(u)\, dm(u)$$

but since T_f commutes with the translations, h can be written $h(u) = f_0(. - u)$ for some function $f_0 \in C(G)$; and this implies $f_0 = f$ and $f \in C_\Lambda(G)$.

We denote by E the countable closure of $i^*(\Lambda)$ in $\hat{\Delta}$. Since E is countable, E is a set of spectral synthesis; that is, every pseudo-measure ν carried by E belongs to $I(E)^\perp$, where

$$I(E) = \left\{u \in l^1(\Delta) | \hat{u}_{|E} = 0\right\}.$$

Moreover, by Loomis [8], every such ν is almost periodic; that is, every $\nu \in I(E)^\perp$ is in the closure of the linear span of $\{\varepsilon_x | x \in E\}$ for the norm of the pseudo-measures—that is, for the norm of $l^\infty(\Delta)$; in particular $I(E)^\perp$ is a separable dual and therefore it has the Radon-Nikodym property.

It remains now to observe that the canonical map i^* of restriction to Δ induces an isometry from $\mathscr{C}(G)$ into $l^\infty(\Delta)$, and that $i^*(\mathscr{C}_\Lambda(G)) \subseteq I(E)^\perp$.

Since the Radon-Nikodym property is hereditary, this implies that $\mathscr{C}_\Lambda(G)$ shares this property, and concludes the proof.

Before stating our main result, let us define some notation: if $\{n_i | i \geq 1\}$ is a sequence in an abelian discrete group Γ, we denote by $W(\{n_i\})$ the subset of Γ consisting of the "words" that can be written with the sequence $\{n_i\}$; that is,

$$W(\{n_i\}) = \{n \in \Gamma | n = \Sigma \varepsilon_i n_i\}$$

where $\varepsilon_i \in \{-1, 0, 1\}$ and all the ε_i's are zero but a finite number. It is classical that if $\{n_i | i \geq 1\}$ is an infinite sequence, there is a singular measure μ on $G = \hat{\Gamma}$ (a "Riesz product") whose Fourier transform is supported by $W(\{n_i\})$—see e.g. [13, Th. 7]. In particular, $W(\{n_i\})$ is not a Riesz set and a fortiori not a Rosenthal set (See Appendix).

We denote by $\mathscr{W}$ the set of subsets Λ of Γ which are such that there exists an infinite sequence $\{n_i\}$ with $W(\{n_i\}) \subseteq \Lambda$; that is, Λ contains all the words written with the sequence $\{n_i\}$. Finally, $\mathrm{Ros}(\Gamma)$ is the set of Rosenthal subsets of Γ. Our result now reads:

THEOREM 2. *Let Γ be a countable infinite abelian discrete group. Let $\mathscr{A}$ be a Σ_1^1 subset of $\mathscr{P}(\Gamma)$ which contains* $\mathrm{Ros}(\Gamma)$. *Then $\mathscr{A} \cap \mathscr{W} \neq \varnothing$.*

Throughout the proof, Δ denotes a fixed countable dense subgroup of G, and $\hat{\Delta}$ its dual, equipped with the compact topology of pointwise convergence on Δ.

Proof. In the notation of theorem 1, the group $i^*(\Gamma)$ is dense in $\hat{\Delta}$, since i^* extends continuously to the Bohr compactification $b\Gamma$ of Γ and $\overline{\Gamma} = b\Gamma$. Therefore there exists a sequence $\{x_i | i \geq 1\}$ in Γ such that for every $i \geq 1$,

$$0 < d_{\hat{\Delta}}(0, i^*(x_i)) < 3^{-i}$$

where $d_{\hat{\Delta}}$ is a translation invariant distance which defines the topology of $\hat{\Delta}$. We recall that a subset T of $\mathbf{N}^{[\mathbf{N}]}$ is called a *tree* if

$$\begin{aligned}&(n(1), n(2) \dots n(k)) \in T\\ &\qquad \Rightarrow \forall\, i, 1 \leq i \leq k, (n(1), n(2), \dots, n(i)) \in T.\end{aligned}$$

We call $\mathscr{T}$ the set of trees, and we define a map Φ from $\mathscr{T}$ into $\mathscr{P}(\Gamma)$ in the following way: for every $T \in \mathscr{T}$, we let

$$\Phi(T) = \left\{ \sum_{i=1}^{k} \varepsilon_i x_{(\Sigma_{j=1}^{i} n(j))} \right\}$$

where $k \in \mathbf{N}$, $\varepsilon_i \in \{-1, 0, 1\}$ and $(n(1), n(2) \dots, n(i)) \in T$ for every $i \in \{1, \dots, k\}$. We will prove:

Claim. If the closure of $i^*[\Phi(T)]$ in $\hat{\Delta}$ is not countable, then T is not well-founded; that is, there is a sequence $\{n(i) | i \geq 1\}$ in $\mathbf{N}^{\mathbf{N}}$ such that

$$(n(1), n(2), \dots, n(i)) \in T$$

for every $i \geq 1$.

We proceed by induction. For every finite sequence $\sigma = (k_1, k_2, \dots, k_l) \in \mathbf{N}^{[\mathbf{N}]}$, we denote by T_σ the subtree of T which "starts with σ", that is

$$T_\sigma = \{(n(i)) \in T | n(1) = k_1, n(2) = k_2, \dots, n_l = k_l\}$$

Of course $T_\sigma = \varnothing$ if $\sigma \notin T$. With this notation we have

$$T = \bigcup_{k=1}^{\infty} T_{\{k\}}$$

Observe now that if $\sigma = (n(j)) \in T_{\{k\}}$ then $\Sigma n(j) \geq k - 1 + |\sigma|$, where $|\sigma|$ denotes the length of σ. Since $d_{\hat{\Delta}}(0, x_l) < 3^{-l}$, it follows that

$$d_{\hat{\Delta}}(0, u) < 3^{-k} \left(\sum_{i=0}^{\infty} 3^{-i} \right) = 3^{-k+1}/2$$

for every $u \in i^*(\Phi(T_{\{k\}}))$; and thus

$$\overline{\lim_{k \to \infty}} \overline{\{i^*(\Phi(T_{\{k\}}))\}} = \{0\} \tag{1}$$

where the $\overline{\lim}$ is taken in the space of compact subsets of $\hat{\Delta}$. But since

$$i^*(\phi(T)) = \bigcup_{k=1}^{\infty} i^*(\phi(T_{\{k\}}))$$

it follows from (1) that

$$\overline{i^*(\phi(T))} = \overline{\sum_{k=1}^{\cup} i^*(\phi(T_{\{k\}}))} \cup \{0\} = \bigcup_{k=1}^{\infty} \overline{i^*(\phi(T_{\{k\}}))}$$

If we assume now that $\overline{i^*(\phi(T))}$ is not countable, the above equality shows that there exists $k_1 \geq 1$ such that the closure of $i^*(\phi(T_{\{k\}}))$ is not countable. Observe now that the same argument shows that

$$\overline{\lim_{k\to\infty}} \overline{\{i^*(\Phi(T_{\{k_1,k\}}))\}} = i^*(x_{k_1})$$

and thus

$$\overline{i^*(\Phi(T_{\{k_1\}}))} = \bigcup_{k=1}^{\infty} \overline{i^*(\Phi(T_{\{k_1,k\}}))}$$

we can therefore find $k_2 \geq 1$ such that $i^*(\Phi(T_{\{k_1,k_2\}}))$ has an uncountable closure. Continuing this process, we construct inductively an infinite sequence $\{k_i | i \geq 1\}$ such that $i^*(\Phi(T_{\{k_1,k_2,\ldots,k_n\}}))$ has an uncountable closure for every $n \geq 1$. This implies in particular that $(k_1, k_2, \ldots, k_n) \in T$ for every $n \geq 1$. and this proves the claim.

Let us come back now to the proof of the theorem. By the claim, if T is a well-founded tree, $i^*(\Phi(T))$ has a countable closure, and then Theorem 1 shows that $\Phi(T)$ is a Rosenthal set. On the other hand, if T is not a well-founded tree then it is clear that $\Phi(T) \in \mathscr{W}$.

The map $\Phi: \mathscr{T} \to \mathscr{P}(\Gamma)$ is of the first Baire class; indeed we may write

$$\Phi(T) = \bigcup_{\sigma \in \mathbf{N}^{[\mathbf{N}]}} \varphi_\sigma(T)$$

where $\varphi_\sigma = \mathscr{P}(\mathbf{N}^{[\mathbf{N}]}) \to \mathscr{P}(\Gamma)$ is defined in the following way:

If $\sigma = (n(1), n(2), \ldots, n(k))$, then

$$\varphi_\sigma(X) = \begin{cases} \varnothing & \text{if } \sigma \notin X, \\ \left\{ \sum_{i=1}^{k} \varepsilon_i x_{\sum_{j=1}^{i} n(j)} | \varepsilon_i \in \{-1, 0, 1\} \right\} & \text{if } \sigma \in X. \end{cases}$$

The map φ_σ is clearly continuous and since Φ is the limit (for the Frechet filter) of the sequence $\{\varphi_\sigma\}$, if follows that Φ is of the first Baire class.

We conclude now the proof by observing that if $\mathscr{A}$ is a Σ_1^1 subset of $\mathscr{P}(\Gamma)$ containing $\mathrm{Ros}(\Gamma)$, then $\Phi^{-1}(\mathscr{A})$ is a Σ_1^1 subset of $\mathscr{T}$ which contains every well-founded tree. But the set of well-founded trees is a Π_1^1-complete subset of $\mathscr{T}$ (see [2]) and thus $\Phi^{-1}(\mathscr{A})$ contains a tree T_0 which is not well-founded; therefore $\Phi(T_0) \in \mathscr{A} \cap \mathscr{W}$ and this concludes the proof. □

COROLLARY 3. *Let Γ be a countably infinite abelian discrete group. Then* $\mathrm{Ros}(\Gamma)$ *is a Π_1^1-hard subset of* $\mathscr{P}(\Gamma)$.

Proof. By definition, $\Lambda \notin \mathrm{Ros}(\Gamma)$ if and only if there exists $f \in L^{\infty}(G) \setminus \mathscr{C}(G)$. The space $L^{\infty}(G)$ equipped with its w^*-topology is a countable union of metrizable compact sets, and its norm-separable subspace $\mathscr{C}(G)$ is a $K_{\sigma\delta}$ set for the w^*-topology; if we now let

$$\mathscr{H} = \{(\Lambda, f) \in \mathscr{P}(\Gamma) \times L^{\infty}(G) | f \in L^{\infty}_{\Lambda}(G)\}$$

we have

$$(\Lambda, f) \in \mathscr{H} \Leftrightarrow (n \in \Lambda) \text{ or } (\hat{f}(n) = 0)$$

and thus $\mathscr{H}$ is a closed subset of $(\mathscr{P}(\Gamma) \times (\mathrm{L}^{\infty}(\mathrm{G}), \mathrm{w}^*))$. But we have

$$\mathscr{P}(\Gamma) \setminus \mathrm{Ros}(\Gamma) = \pi_1(\mathscr{H} \cap (L^{\infty}(G) \setminus \mathscr{C}(G)))$$

and therefore $\mathscr{P}(\Gamma) \setminus \mathrm{Ros}(\Gamma)$ is Σ^1_1, and $\mathrm{Ros}(\Gamma)$ is $\mathbf{\Pi}^1_1$. On the other hand it follows from Theorem 2 that $\mathrm{Ros}(\Gamma)$ is not Σ^1_1 since $\mathrm{Ros}(\Gamma) \cap \mathscr{W} = \varnothing$; hence $\mathrm{Ros}(\Gamma)$ is Π^1_1-hard.

Let us mention that the proof of Theorem 2 actually shows that $\mathrm{Ros}(\Gamma)$ is Π^1_1-complete, since $\Phi^{-1}(\mathrm{Ros}(\Gamma))$ is exactly the set of well-founded trees.

COROLLARY 4. [13, Cor. 12]. *Let Γ be a countably infinite abelian discrete group. Then the set $\mathscr{R}(\Gamma)$ of Riesz subsets of Γ is a Π^1_1-hard subset of $\mathscr{P}(\Gamma)$.*

Proof. By [13, Th. 11], the set $\mathscr{R}(\Gamma)$ is a Π^1_1 subset of $\mathscr{P}(\Gamma)$. On the other hand, $\mathrm{Ros}(\Gamma) \subset \mathscr{R}(\Gamma) \subset \mathscr{P}(\Gamma) \setminus \mathscr{W}$ and thus $\mathscr{R}(\Gamma)$ is not Σ^1_1. □

Remarks 5. (1) Our proof of Corollary 4 is very similar to V. Tardivel's original proof [13]. The only significative difference is that Loomis's theorem allows us to dispense with the localization technique—for the Bohr topology—that was used in [13]. It is actually fortunate that we don't need to use localization; indeed the class $\mathrm{Ros}(\Gamma)$ is not localizable in the sense of [13, Th. 4] since, for instance, the set $\mathbf{P}$ of prime numbers is not Rosenthal [11].

(2) By [9], the class

$$\mathscr{L}-\mathscr{P}(\Gamma) = \{\Lambda \in \mathscr{P}(\Gamma) | \mathscr{C}_{\Lambda}(G) \text{ does not contain an isomorphic copy of } c_0(\mathbf{N})\}$$

is such that $\mathrm{Ros}(\Gamma) \subset \mathscr{L}-\mathscr{P}(\Gamma) \subset \mathscr{R}(\Gamma)$ and thus $\mathscr{L}-\mathscr{P}(\Gamma)$ is not Σ^1_1. It can be shown by standard techniques that $\mathscr{L}-\mathscr{P}(\Gamma)$ is Π^1_1 and thus $\mathscr{L}-\mathscr{P}(\Gamma)$ is Π^1_1-hard.

(3) Similarly, if we denote by $\mathrm{Sch}(\Gamma)$ the class

$$\mathrm{Sch}(\Gamma) = \{\Lambda \in \mathscr{P}(\Gamma) | \mathscr{C}_{\Lambda}(G) \text{ has the Schur property}\}$$

then by [9. Th. 3], with the notation of theorem 2 we have:

$$T \text{ well-founded} \Leftrightarrow \Phi(T) \in \mathrm{Sch}(\Gamma).$$

Therefore the class Sch(Γ) is not Σ_1^1. Let us mention at this point that it is easily seen that the class Sch(Γ) is Σ_2^1 but I don't know whether Sch(Γ) is actually Π_1^1; indeed the weak Cauchy sequences form a Π_1^1-hard subset of $\mathscr{C}(G)^{\mathbf{N}}$ (see [1]) and thus Sch(Γ) is Σ_2^1 "at first sight".

(4) In connection with the technique which leads to theorem 1, let us mention that the subset of $\mathscr{K}(G)$ consisting of the compacts E which are of spectral synthesis is Π_1^1-hard in $\mathscr{K}(G)$ [7].

(5) It is shown in [5] that the Szlenk index is a Π_1^1-rank on the set $\mathscr{R}(\mathbf{Z})$ of Riesz subsets of $\mathbf{Z}$; in particular, for every countable ordinal α, the set

$$R_\alpha(\mathbf{Z}) = \{\Lambda \in \mathscr{P}(\mathbf{Z}) | S_z(\Lambda) \le \alpha\}$$

is a Borel subset of $\mathscr{P}(\mathbf{Z})$. It follows now from Theorem 2 and its proof that the Szlenk index is bounded on Ros(**Z**) ∩ Sch(**Z**). More "concretely", this means that one can find sets $\Lambda \in$ Ros(**Z**) ∩ Sch(**Z**), which will actually be of the form $\Lambda = \Phi(T)$ for some well-founded tree T, and such that the convergence of the Fejer convolution operators $T_n(f) = f * \sigma_n$ towards the identity of $L^1_\Lambda(\mathbf{T})$ is "arbitrarily slow".

We refer to [7] for much more about Π_1^1-ranks and their connections with harmonic analysis.

(6) It is easy to deduce from the definition of the Sidon sets that the family Sid(Γ) of Sidon subsets of Γ is a K_σ subset of $\mathscr{P}(\Gamma)$; therefore Sid(Γ) is a "very small" subset of Ros(Γ) for any countably infinite abelian discrete group Γ.

Appendix

We mentioned before Theorem 2 that every Rosenthal set is a Riesz set. More generally, Dressler and Pigno have shown [4] that the union of a Riesz set and of a Rosenthal set is a Riesz set; the proof relies on a result of Heard [6]. We will present here an easy and self-contained argument which allows to extend Dressler and Pigno's result; this approach was suggested to me by F. Lust-Piquard and is included here with her kind permission. With the above notation, we have:

Proposition 5. *If $\Lambda \subseteq \Gamma$ is such that $\mathscr{M}_\Lambda = L^1_\Lambda \oplus (\mathscr{M}_s)_\Lambda$ and if Λ_0 is a Rosenthal set, then*

$$\mathscr{M}_{\Lambda \cup \Lambda_0} = L^1_{\Lambda \cup \Lambda_0} \oplus (\mathscr{M}_s)_\Lambda.$$

Of course, if Λ is Riesz then $(\mathscr{M}_s)_\Lambda = \{0\}$ and thus $(\Lambda \cup \Lambda_0)$ is Riesz.

Proof. Let μ be in $\mathscr{M}_{\Lambda \cup \Lambda_0}$. As in [4], we consider the elements

$$g_n = k_n * \mu$$

of $L^1(G)$, where (k_n) is an approximation of the identity in $L^1(G)$. If we let $\tilde{f}(x) = f(-x)$, we have by Fubini,

$$\int_G g_n f\,dm = \int_G \tilde{k}_n(\mu * \tilde{f})\,dm.$$

If we now let $\Lambda' = \Gamma \setminus (-\Lambda)$, we have for every $f \in L^\infty_{\Lambda'}$ that

$$(\mu * \tilde{f}) \in L^\infty_{\Lambda_0} = \mathscr{C}_{\Lambda_0}$$

since Λ_0 is Rosenthal and thus $\lim_{n\to\infty}(\int g_n f\,dm)$ exists for every $f \in L^\infty_{\Lambda'}$.

Since $\mathscr{M}_\Lambda = L^1_\Lambda \oplus (\mathscr{M}_s)_\Lambda$ we have

$$C^*_{\Lambda'} = L^1/L^1_\Lambda \oplus \mathscr{M}_s/(\mathscr{M}_s)_\Lambda. \tag{2}$$

Denote by $\dot{\lambda}$ the class of a measure λ in $C^*_{\Lambda'}$, and let $\dot{\nu}$ be a cluster point to $\dot{g}_n$ in $(C^*_{\Lambda'}, w^*)$. We write $\dot{\nu} = \dot{g} + \dot{\sigma}$ $(g \in L^1, \sigma \in \mathscr{M}_s)$. It is easily seen from (2) that

$$\|\dot{f} + \dot{\sigma}\| = \|\dot{f}\| + \|\dot{\sigma}\|, \quad \forall\, f \in L^1. \tag{3}$$

Of course, $\dot{\sigma}$ is a w^*-cluster point of $(\dot{g}_n - \dot{g})$. If $\dot{\sigma} \neq 0$, it is easy to deduce from (3) and the w^*-lower semi-continuity of the norm that $(\dot{g}_n - \dot{g})$ has a subsequence which is equivalent to the canonical basis of l^1.

But on the other hand $\lim_{n\to\infty}(\int(g_n - g)f\,dm)$ exists for every $f \in L^\infty_{\Lambda'}$. Since

$$\left(L^1/L^1_\Lambda\right)^* = L^\infty_{\Lambda'},$$

this means that the sequence $(\dot{g}_n - \dot{g})$ is weak-Cauchy in L^1/L^1_Λ; thus it cannot contain a subsequence equivalent to l^1 and $\dot{\sigma} = 0$.

We found $g \in L^1$ such that

$$\lim_{n\to\infty} \int (k_n * \mu) f\,dm = \int g f\,dm, \quad \forall\, f \in \mathscr{C}_{\Lambda'}$$

If, in particular, $f = \tilde{\alpha}(\alpha \notin \Lambda)$, we get

$$\hat{g}(\alpha) = \lim_{n\to\infty} \hat{k}_n(\alpha) \cdot \hat{\mu}(\alpha) = \hat{\mu}(\alpha)$$

and thus $(g - \mu) \in \mathscr{M}_\Lambda$ and since $\mu \in \mathscr{M}_{\Lambda \cup \Lambda_0}$ it follows that $g \in L^1_{\Lambda \cup \Lambda_0}$. The result follows by another application of the equation $\mathscr{M}_\Lambda = \mathrm{L}^1_\Lambda \oplus (\mathscr{M}_s)_\Lambda$. $\square$

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