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ON THE SPIN BORDISM OF $B(E_8 \times E_8)$

BY

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Let E_8 be the exceptional Lie group; let BE_8 be its universal classifying space. Bott and Samuelson (2) have shown that in dimensions less than 16, the only non-zero homotopy of E_8 is $\pi_3(E_8) = Z$, $\pi_{15}(E_8) = Z$. By the long exact homotopy sequence of the universal E_8-bundle, $\pi_4(BE_8) = Z$, $\pi_{16}(BE_8) = Z$, and $\pi_k(BE_8) = 0$ for all other $k \leq 16$. Let $K(Z,4)$ be the Eilenberg-MacLane space whose only non-trivial homotopy group is infinite cyclic in dimension 4. By the Whitehead theorem (see, e.g., Serre (5)) the map $BE_8 \to K(Z,4)$, sending generator to generator in cohomology, yields an isomorphism in homology through dimension 15. Similarly, the map

$$B(E_8 \times E_8) = BE_8 \times BE_8 \to K(Z,4) \times K(Z,4)$$

induces an isomorphism in homology through dimension 15. We use this isomorphism to compute the spin bordism of $B(E_8 \times E_8)$.

The motivation for this investigation was given by Witten (10), who examined a model for heterotic string theory for which an eleven-dimensional compact spin manifold M has 2 principal E_8-bundles $V_1 \oplus V_2 \to M$. Here the fundamental relations are that homotopy classes of maps of compact spin manifolds $g\colon M^n \to B(E_8 \times E_8)$ are in one-to-one correspondence with principal $E_8 \times E_8$-bundles $V_1 \oplus V_2 \to M^n$, but pairs (M^n, g) are elements of $\Omega_n^{\mathrm{Spin}}(B(E_8 \times E_8))$. Corollary 7 shows that $\Omega_{11}^{\mathrm{Spin}}(B(E_8 \times E_8)) = 0$, so that in fact an $E_8 \times E_8$-bundle over an 11-manifold must be trivial. This insures that the global space-time anomaly vanishes in the above mentioned model for string theory.

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Anderson, Brown, and Peterson (1) completed the calculation of the spin bordism ring Ω_*^{Spin}. We note the low-dimensional groups for later reference:

dim	0	1	2	3	4	5	6	7	8	9	10	11
group	Z	Z_2	Z_2	0	Z	0	0	0	$2Z$	$2Z_2$	$3Z_2$	0

Stong (7) has calculated

$$\tilde{\Omega}_n^{\mathrm{Spin}}(K(Z,4)) = \begin{cases} Z, & n = 4, \\ 2Z_2, & n = 8, \\ Z_2, & n = 9, \\ 2Z_2, & n = 10, \\ 0, & \text{all other } n \leq 11. \end{cases}$$

Since $\tilde{\Omega}_*^{\mathrm{Spin}}(X)$ is a homology theory,

$$\tilde{\Omega}_n^{\mathrm{Spin}}(K(Z,4) \times K(Z,4))$$
$$\cong \tilde{\Omega}_n^{\mathrm{Spin}}(K(Z,4) \wedge K(Z,4)) \oplus \tilde{\Omega}_n^{\mathrm{Spin}}(K(Z,4)) \oplus \tilde{\Omega}_n^{\mathrm{Spin}}(K(Z,4))$$

(where $X \wedge Y$ is the smash product), and the problem reduces to a calculation of $\tilde{\Omega}_n^{\mathrm{Spin}}(K(Z,4) \wedge K(Z,4))$.

LEMMA 1.

$$\tilde{\Omega}_n^{\mathrm{Spin}}(K(Z,4) \wedge K(Z,4)) \cong \begin{cases} Q, & n = 8, \\ 0, & \text{all other } n \leq 11. \end{cases}$$

Proof. $\tilde{H}^*(K(Z,4) \wedge K(Z,4); Q) \cong \tilde{H}^*(K(Z,4); Q) \otimes \tilde{H}^*(K(Z,4); Q)$ by the Kunneth formula, but $H^*(K(Z,4); Q) \cong Q[i]$, where i is the image under the coefficient homomorphism of the standard generator $i \in H^4(K(Z,4); Z)$. Also, $\Omega_*^{\mathrm{Spin}} \otimes Q \cong Q[x_{4i}]$ is a polynomial ring on $4i$-dimensional generators. In the Atiyah-Hirzebruch spectral sequence with

$$E_{p,q}^2 = \tilde{H}_p\big(K(Z,4) \wedge K(Z,4); \Omega_q^{\mathrm{Spin}} \otimes Q\big)$$

converging to a filtration of $\tilde{\Omega}_{p+q}^{\mathrm{Spin}}(K(Z,4) \wedge K(Z,4)) \otimes Q$, for $p + q \leq 11$, the only non-zero element is at $E_{8,0}^2$. This element survives to E^∞.

LEMMA 2.

$$\tilde{H}^k(K(Z,4) \wedge K(Z,4); Z_2) \cong \begin{cases} Z_2, & k = 8, \\ 2Z_2, & k = 10, \\ 2Z_2, & k = 11, \\ 3Z_2 & k = 12, \\ 0, & \text{all other } k \leq 12. \end{cases}$$

Proof. $\tilde{H}(K(Z,4); Z_2)$ is a polynomial ring over Z_2 on admissible classes $Sq^I i$ (see Serre (4)), where $\dim i = 4$. By the Kunneth formula,

$$\tilde{H}^*(K(Z,4) \wedge K(Z,4); Z_2)$$

has the following basis:

dimension	basis elements
8	$i \otimes i$
10	$Sq^2 i \otimes i,\ i \otimes Sq^2 i$
11	$Sq^3 i \otimes i,\ i \otimes Sq^3 i$
12	$i^2 \otimes i,\ i \otimes i^2,\ Sq^2 i \otimes Sq^2 i$

LEMMA 3. *In dimensions ≤ 12, $K(Z,4) \wedge K(Z,4)$ has no p-torsion for any odd prime p except $p = 3$.*

Proof. $\tilde{H}^*(K(Z,4); Z_p)$ is a module over the mod-p Steenrod algebra (Cartan (3)) with generators $i_4, \mathscr{P}^1 i_4$ of dimension $4 + 2(p-1)$, $\beta\mathscr{P}^1 i_4$ of dimension $4 + 2(p-1) + 1$, plus higher dimensional terms. In dimensions ≤ 12, the only non-integral classes in $\tilde{H}^*(K(Z,4) \wedge K(Z,4); Z_p)$ are $\mathscr{P}^1 i_4 \otimes i_4$ and $i_4 \otimes \mathscr{P}^1 i_4$ when $p = 3$.

PROPOSITION 4.

$$\tilde{H}_n(K(Z,4) \wedge K(Z,4); Z) \cong \begin{cases} Z, & n = 8, \\ Z_2 \oplus Z_2, & n = 10, \\ 2Z \oplus 2Z_3 \oplus Z_2, & n = 12, \\ 0, & \text{all other } n \leq 12. \end{cases}$$

Proof. The proof follows easily from the lemmas and the Universal Coefficient Theorem.

COROLLARY 5. *For $n \leq 11$, the torsion in $\tilde{\Omega}_n^{\mathrm{Spin}}(K(Z,4) \wedge K(Z,4))$ is all 2-torsion.*

Proof. In the Atiyah-Hirzebruch spectral sequence with

$$E^2_{p,q} = \tilde{H}_p\left(K(Z,4) \wedge K(Z,4); \tilde{\Omega}^{\text{Spin}}_q\right),$$

the E^2-term is

3	0			
2	Z_2	0		
1	Z_2	0	$2Z_2$	
0	Z	0	$2Z_2$	0
	8	9	10	11

for $p + q \leq 11$.

Rather than further analyze the preceding spectral sequence, we next make use of the generalized Thom construction (see Thom [9], and Stong [8]):

$$\tilde{\Omega}^{\text{Spin}}_*(K(Z,4) \wedge K(Z,4)) \cong \pi_*(K(Z,4) \wedge K(Z,4) \wedge \text{MSpin}),$$

where MSpin denotes the Thom space of the universal bundle over BSpin.

Anderson, Brown, and Peterson [1] have given a decomposition of MSpin into $BO \times BO\langle 8, \dots \rangle \times \dots$. This decomposition implies that

$$\tilde{H}^*(\text{MSpin}; Z_2) \cong \mathscr{A}/(\mathscr{A}Sq^1 + \mathscr{A}Sq^2)U + \mathscr{A}/(\mathscr{A}Sq^1 + \mathscr{A}Sq^2)w_4^2 \cdot U + \text{higher terms},$$

where $\mathscr{A}$ is the mod-2 Steenrod Algebra, U is the Thom class of the universal bundle over BSpin, and w_4 is the image under the Thom isomorphism of the Stiefel-Whitney class $w_4 \in H^4(\text{BSpin}; Z_2)$. Since in low dimensions, $\tilde{H}^*(\text{MSpin}; Z_2)$ is a free $\mathscr{A}/\mathscr{A}_1$-module (where $\mathscr{A}_1$ is the subalgebra generated by Sq^1 and Sq^2), to determine

$$\tilde{H}^*(K(Z,4) \wedge K(Z,4) \wedge \text{MSpin}; Z_2) \cong \tilde{H}^*(K(Z,4) \wedge K(Z,4); Z_2) \otimes \tilde{H}^*(\text{MSpin}; Z_2)$$

as a module over $\mathscr{A}$, it suffices to consider $\tilde{H}^*(K(Z,4) \wedge K(Z,4); Z_2)$ as a module over $\mathscr{A}_1$. The Adem relations (see Steenrod (6)) give us the following basis of $\mathscr{A}_1$:

dim 0	1	2	3	4	5	6
1	Sq^1	Sq^2	Sq^3	Sq^3Sq^1	$Sq^5 + Sq^4Sq^1$	Sq^5Sq^1
			Sq^2Sq^1			

Consider the action of $\mathscr{A}_1$ on $\tilde{H}^*(K(Z,4) \wedge K(Z,4); Z_2)$ in low dimensions. The actions are determined by the Cartan formula, the Adem relations, and the fact that $Sq^1 i = 0$, since i is an integral class.

$i \otimes i$

$$\begin{aligned} Sq^2(i \otimes i) &= Sq^2 i \otimes i + i \otimes Sq^2 i \\ Sq^3(i \otimes i) &= Sq^3 i \otimes i + i \otimes Sq^3 i \\ (Sq^5 + Sq^4 Sq^1)(i \otimes i) &= Sq^2 i \otimes Sq^3 i + Sq^3 i \otimes Sq^2 i. \end{aligned}$$

All others are 0.

$i \otimes Sq^2 i$

$$\begin{aligned} Sq^1(i \otimes Sq^2 i) &= i \otimes Sq^3 i \\ Sq^2(i \otimes Sq^2 i) &= Sq^2 i \otimes Sq^2 i \\ Sq^3(i \otimes Sq^2 i) &= Sq^2 i \otimes Sq^3 i + Sq^3 i \otimes Sq^2 i \\ Sq^2 Sq^1(i \otimes Sq^2 i) &= Sq^2 i \otimes Sq^3 i \\ Sq^3 Sq^1(i \otimes Sq^2 i) &= Sq^3 i \otimes Sq^3 i. \end{aligned}$$

All others are 0.

$i \otimes i^2$

$$\begin{aligned} Sq^2(i \otimes i^2) &= Sq^2 i \otimes i^2 \\ Sq^3(i \otimes i^2) &= Sq^3 i \otimes i^2. \end{aligned}$$

All others are 0.

$i^2 \otimes i$

$$\begin{aligned} Sq^2(i^2 \otimes i) &= i^2 \otimes Sq^2 i \\ Sq^3(i^2 \otimes i) &= i^2 \otimes Sq^3 i. \end{aligned}$$

All others are 0.

This shows that in low dimensions,

$$\tilde{H}^*(K(Z,4) \wedge K(Z,4) \wedge \mathrm{MSpin}; Z_2)$$

is isomorphic to

$$\begin{aligned} &(\mathscr{A}/\mathscr{A}Sq^1) i \otimes i \otimes U + (\mathscr{A}/\mathscr{A}Sq^2 Sq^1 Sq^2) i \otimes Sq^2 i \otimes U \\ &\quad + (\mathscr{A}/\mathscr{A}Sq^1 + \mathscr{A}Sq^5) i \otimes i^2 \otimes U + (\mathscr{A}/\mathscr{A}Sq^1 + \mathscr{A}Sq^5) i^2 \otimes i \otimes U \\ &\quad + \text{higher degree terms}. \end{aligned}$$

Consider the following Eilenberg-MacLane spaces with Z_2-cohomology generators:

space	$K(Z,8)$	$K(Z_2,10)$	$K(Z,12)$	$K(Z,12)$
generator	i_8	i_{10}	i_{12}	j_{12}

Let

$$f\colon K(Z,4) \wedge K(Z,4) \wedge \mathrm{MSpin} \to K(Z,8) \times K(Z_2,10) \times K(Z,12) \times K(Z,12)$$

be a map that induces the following in Z_2-cohomology:

$$\begin{aligned} f^*(i_8) &= i \otimes i \otimes U \\ f^*(i_{10}) &= i \otimes Sq^2 i \otimes U \\ f^*(i_{12}) &= i \otimes i^2 \otimes U \\ f^*(j_{12}) &= i^2 \otimes i \otimes U \end{aligned}$$

Since the Steenrod operations are natural transformations, they commute with the homomorphism f^*. A dimension by dimension examination of

$$\tilde{H}^*(K(Z,8) \times K(Z_2 10) \times K(Z,12) \times K(Z,12))$$

as a module over $\mathscr{A}$ shows that the homomorphism f^* is a bijection through dimension 12, and in dimension 13 f^* is surjective with kernel Z_2. The corresponding map f_* in homology must then be a bijection through dimension 12.

THEOREM 6.

$$\tilde{\Omega}_n^{\mathrm{Spin}}(K(Z,4) \wedge K(Z,4)) \cong \begin{cases} Z, & n = 8, \\ Z_2, & n = 10, \\ 0, & \textit{all other } n \leq 11. \end{cases}$$

Proof. Corollary 5 has shown that for $n \leq 11$, the only torsion in

$$\tilde{\Omega}_n^{\mathrm{Spin}}(K(Z,4) \wedge K(Z,4))$$

is 2-primary. By the Whitehead theorem (Serre (5)), since f induces an isomorphism in homology through dimension 12 (modulo odd torsion), f induces an isomorphism through dimension 11 in homotopy (modulo odd torsion). But there is no odd torsion in the homotopy of

$$K(Z,8) \times K(Z_2,10) \times K(Z,12) \times K(Z,12),$$

and thus

$$\begin{aligned}
&\pi_n(K(Z,8) \times K(Z_2,10) \times K(Z,12) \times K(Z,12)) \\
&\quad \cong \pi_n(K(Z,4) \wedge K(Z,4) \wedge \mathrm{MSpin}) \\
&\quad \cong \tilde{\Omega}_n^{\mathrm{Spin}}(K(Z,4) \wedge K(Z,4)) \quad \text{for } n \leq 11.
\end{aligned}$$

COROLLARY 7. $\Omega_{11}^{\mathrm{Spin}}(B(E_8 \times E_8)) \cong 0$.

Proof. Since Ω_*^{Spin} is a homology theory,

$$\begin{aligned}
\Omega_{11}^{\mathrm{Spin}}(B(E_8 \times E_8)) &\cong \Omega_{11}^{\mathrm{Spin}} \oplus \tilde{\Omega}_{11}^{\mathrm{Spin}}(B(E_8 \times E_8)) \\
&\cong \Omega_{11}^{\mathrm{Spin}} \oplus \tilde{\Omega}_{11}^{\mathrm{Spin}}(BE_8 \wedge BE_8) \\
&\qquad \oplus \tilde{\Omega}_{11}^{\mathrm{Spin}}(BE_8) \oplus \tilde{\Omega}_{11}^{\mathrm{Spin}}(BE_8),
\end{aligned}$$

but by the preceding calculations and the equivalence between $K(Z,4)$ and BE_8, all relevant groups are 0.

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