ILLINOIS JOURNAL OF MATHEMATICS
Volume 35, Number 4, Winter 1991

# INVERSE REAL CLOSED SPACES

BY

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## Introduction

Building up semi-algebraic geometry, H. Delfs and M. Knebusch relied almost entirely on geometric arguments (cf. [5]). To develop algebraic methods appropriate for the purposes of semi-algebraic geometry the rings of sections of the structure sheaves of semi-algebraic spaces must be studied. First steps in this direction were taken in [13], [14], [15], [16], [17] (see also [2]). An attempt to develop an algebraic version of semi-algebraic geometry with a sufficient degree of generality and at the same time keeping the connections with geometry leads to a category of locally ringed spaces, called real closed spaces [13], [15], [16]. This class of spaces generalizes locally semi-algebraic spaces much in the same way as schemes generalize classical algebraic varieties.

Using weakly semi-algebraic spaces [10], M. Knebusch has been particularly successful developing algebraic topology for semi-algebraic spaces. These spaces are obtained from affine semi-algebraic spaces by glueing them together on closed semi-algebraic subspaces. From a purely algebraic point of view these spaces are nothing new since their rings of sections are also real closed rings [16, Chapter I, §4]. However, to keep close connections between algebra and geometry the development of an algebraic version of the weakly semi-algebraic spaces requires the construction of a new class of spaces: affine real closed spaces have to be glued together on closed constructible subspaces.

Recall that an affine real closed space is a pro-constructible subspace of the real spectrum of a ring together with a sheaf of real closed rings [15], [16]. To glue two such spaces together on closed constructible subspaces and still be able to use the usual sheaf theoretic techniques, the notion of openness is redefined: The *inverse topology* on the pro-constructible subset $K$ of the real spectrum $\mathrm{Sper}(A)$ of the ring $A$ has the closed (in the usual topology) constructible subsets of $K$ as its basis. If $K$ with the inverse topology is denoted by $K^*$, then $K^*$ can be equipped with a structure sheaf of real

Received May 25, 1989.

1980 Subject Classification (1985 Revision). Primary 14G30; Secondary 14A99.

closed rings to become a ringed space, an affine inverse real closed space. These spaces form a category which is equivalent to the category of affine real closed spaces. So, affine phenomena can be studied equally well in both categories. In [15] and [16], affine real closed spaces are glued together along open subspaces to yield real closed spaces. Similarly, if one glues affine inverse real closed spaces together along open subspaces then one obtains inverse real closed spaces. According to the different meaning of openness both procedures lead to different spaces.

The general theory of inverse real closed spaces is developed only so far as is necessary to establish the precise relationship with M. Knebusch's weakly semi-algebraic spaces. Subspaces (Section 2) can be defined exactly as for real closed spaces [15, Chapter V, §2], [16, Chapter II, §2]. Also, for the notions of quasi-compactness and quasi-separatedness of morphisms (Section 3) the most basic formal properties are exactly those familiar from schemes [7, I, §6.1] or real closed spaces [15, Chapter V, §3], [16, Chapter II, §4]. However the geometric meaning of these notions may be different. A case in point is the affineness result of Theorem 3.7: A quasi-compact and quasi-separated space is affine (cf. [15, Theorem V 4.8], [16, Theorem II 5.8]). In Section 4 the equivalence between the categories of affine real closed spaces and affine inverse real closed spaces is extended to a much larger category, containing for example all paracompact spaces. In Sections 5 and 6 the notions of completely quasi-separated spaces and of finitely presented morphisms are discussed. These are exactly the notions required for the characterization of those inverse real closed spaces which correspond to weakly semi-algebraic spaces under a natural functor (Theorem 7.1). In fact, the weakly semi-algebraic spaces over a real closed field $R$ may be considered as a full subcategory of the category of inverse real closed space over $\mathrm{Sper}(R)$.

## 1. The category of inverse real closed spaces

Let $A$ be a real closed ring with real spectrum $\mathrm{Sper}(A)$ [1], [3], [11], [15], [16], $K \subset \mathrm{Sper}(A)$ a pro-constructible subset [7, I 7.2.2], [15, Chapter II, §1], [16, Chapter I, §1]. If $C \subset K$ is a (relatively) constructible subset then $C_A(C)$ denotes the real closure of $A$ on $C$ [15, Definition III 2.6], [16, Definition I 2.8]. In [15] and [16] the locally ringed spaces $(K, C_{(A,K)})$ with $C_{(A,K)}(U) = C_A(U)$ ($U \subset K$ open constructible) are used as affine building blocks of real closed spaces.

If $C \subset \mathrm{Sper}(A)$ is pro-constructible and $C = C_1 \cup \cdots \cup C_r$ is a closed constructible cover and if $a_i \in C_A(C_i)$, $i = 1, \ldots, r$, are such that $a_i|C_i \cap C_j = a_j|C_i \cap C_j$ for all $i, j$, then there is a unique $a \in C_A(C)$ such that $a|C_i = a_i$ for every $i$. For, $a$ is a constructible section [16, p. 8] since every $a_i$ is a constructible section on the constructible subset $C_i \subset C$. Moreover, $a$ is a compatible section [16, Definition I 2.1] since every $C_i$ is closed in $C$ and $a_i$

is a compatible section on $C_i$. Theorem I in [16, 2.7] now proves that $a$ is a semi-algebraic function. This can be reformulated by saying that the abstract semi-algebraic functions have a sheaf property which, however, does not refer to the usual weak topology of the real spectrum. First of all we give a name to the appropriate topology:

DEFINITION 1.1. Let $K \subset \mathrm{Sper}(A)$ be pro-constructible, let $\overline{\mathscr{C}}(K)$ be the lattice of closed constructible subsets of $K$. $\overline{\mathscr{C}}(K)$ is the basis of a topology of $K$ which is called the *inverse topology*.

If $\mathfrak{T}$ denotes the weak topology on $K$ then $\mathfrak{T}^*$ denotes the inverse topology. Instead of $(K, \mathfrak{T}^*)$ we will often write $K^*$. If it is not clear from the context which topology some topological term is referring to we will say, for example, weakly open or inversely open. If $X \subset K^*$ is a subset then the inverse topology on $X$ is the restriction of the inverse topology on $K$. In case $X$ is pro-constructible in $K$ it is clear that the inverse topology on $X$ (as defined in Definition 1.1) and the restriction of the inverse topology of $K$ both coincide. If $K = \mathrm{Sper}(A)$ we will write $\mathrm{Sper}^*(A)$ instead of $K^*$.

According to [9], Proposition 8, $K^*$ is a spectral space. So, in $K^*$ there is the notion of constructible subsets. These are exactly the same as the constructible subsets of $K$. Therefore the pro-constructible subsets of $K$ and $K^*$ are also the same.

If we set $C^*_{(A,K)}(C) = C_A(C)$ for $C \in \overline{\mathscr{C}}(K)$ then the above mentioned sheaf property says that $C^*_{(A,K)}$ is a sheaf of rings on $K^*$, given on a basis of open subsets [7, 0 3.2.1]. Thus, $(K^*, C^*_{(A,K)})$ is a ringed space. In the theory of schemes mostly locally ringed spaces are considered. In our situation we must deal with another special class of ringed spaces:

DEFINITION 1.2. A ringed space $(X, O_X)$ is *integrally ringed* if all the stalks $O_{X,x}$ are integral domains. If $(X, O_X)$, $(Y, O_Y)$ are integrally ringed spaces and

$$(f, f^{\#}): (X, O_X) \to (Y, O_Y)$$

is a morphism of ringed spaces, then $(f, f^{\#})$ is a *morphism of the integrally ringed spaces* if every homomorphism $f^{\#}_x: O_{Y,f(x)} \to O_{X,x}$ of stalks is a monomorphism.

In a sense locally ringed spaces and integrally ringed spaces are dual to each other. If $x$ is a point in a locally ringed space $X$ then the stalk $O_{X,x}$ has a unique maximal ideal. On the other hand, if $y$ is a point in an integrally ringed space $Y$ then the stalk $O_{Y,y}$ has a unique minimal ideal. With $x$ we associate a canonical homomorphism from the stalk into a field, namely from

$O_{X,x}$ to its residue field. Similarly, with $y$ we associate a canonical homomorphism from the stalk at $y$ into a field, namely the injection of $O_{Y,y}$ into its quotient field. So, the quotient field of $O_{Y,y}$ plays the same role for $Y$ as does the residue field of $O_{X,x}$ for $X$. Accordingly the following notation will be used: If $a \in O_X(U)$ (for $U \subset X$ open) and $x \in U$ then the image of $a$ under the canonical homomorphism

$$O_X(U) \longrightarrow O_{X,x} \longrightarrow O_{X,x}/M_{X,x}$$

is denoted by $a(x)$. If $a \in O_Y(V)$ (for $V \subset Y$ open) and $x \in V$ then the canonical image of $a$ under

$$O_Y(V) \longrightarrow O_{Y,y} \longrightarrow qf(O_{Y,y})$$

will be denoted by $a(x)$.

It is immediately clear that the integrally ringed spaces form a category. First we show that the spaces $(K^*, C^*_{(A,K)})$ belong to this category.

Proposition 1.3. *With the notation above, $(K^*, C^*_{(A,K)})$ is integrally ringed.*

*Proof.* By definition,

$$C^*_{(A,K),x} = \lim_{\rightarrow} C^*_{(A,K)}(C) = \lim_{\rightarrow} C_{(A,K)}(C)$$

where $C$ runs over the weakly closed constructible subsets of $K$ containing $x$. The evaluation maps $C_{(A,K)}(C) \to \rho(x)$ form a direct system. (Here, and also throughout the paper, $\rho(x)$ denotes the real closed residue field of the ring $A$ at the point $x \in \mathrm{Sper}(A)$.) Hence there is a natural homomorphism $C^*_{(A,K),x} \to \rho(x)$. Suppose that $a \in C_{(A,K),x}$ is mapped to 0. Let $b \in C^*_{(A,K)}(C)$ be a representative of $a$. The set $D = \{\alpha \in C \mid b(\alpha) = 0\}$ is inversely open constructible and contains $x$. Thus, the restriction $b|D$ of $b$ is 0 in $C^*_{(A,K)}(D)$, and so is the canonical image $a \in C^*_{(A,K),x}$. □

The proof of Proposition 1.3 also shows that, if $x \in K$ and $Y \subset K$ is the closure of $x$, then $C^*_{(A,K),x} = C_{(A,K)}(Y)$. Moreover, $\rho(x)$ is the quotient field of $C^*_{(A,K),x}$.

By [15, Theorem III, 3.2, Theorem III 3.5] or [16, Theorem I 4.5, Theorem I 4.8] the stalks of the space $(K^*, C^*_{(A,K)})$ are real closed integral domains. As such they are also local rings (observe that the specializations of a point in the real spectrum form a chain [1, 7.1.23) and apply [15, Proposition II 4.10] or [16, Theorem I 3.10]. So, $(K^*, C^*_{(A,K)})$ is a locally ringed space and an integrally ringed space. We must consider it as an integrally ringed space to obtain the same kind of correspondence between morphisms of these spaces and homomorphisms of their global rings of sections as we have in the theory of schemes [7, I 1.6.3] or in the theory of real closed spaces [15, Proposition V 2.20, Definition V 2.24], [16, Proposition II 2.17].

DEFINITION 1.4. An integrally ringed space which is isomorphic to a space $(K^*, C^*_{(A,K)})$ is called an *affine inverse real closed space*. An integrally ringed space $(X, O_X)$ is an *inverse real closed space* if there is an open cover $X = \bigcup_{i\in I} X_i$ such that every $(X_i, O_x|X_i)$ is an affine inverse real closed space. The inverse real closed spaces are considered as a full subcategory of the category of integrally ringed spaces.

We return again to the spaces $K = (K, C_{(A,K)})$, $K^* = (C^*_{(A,K)})$ considered above. By [15, Theorem II 4.17] or [16, Theorem I 3.25] we may assume that $A = \Gamma(K)$. By definition of $K^*$, the rings of global sections of $K$ and $K^*$ agree. It was mentioned above that the constructible topologies on $K$ defined by the weak topology and by the inverse topology both coincide. Let $\mathfrak{T}^c$ be the constructible topology and set $K^c = (K, \mathfrak{T}^c)$. Setting $C^c_{(A,K)}(C) = C_A(C)$ we define a presheaf of rings on $K^c$. If $I$: $K^c \to K$, $J$: $K^c \to K^*$ denote the identity maps then the direct images $I_*(C^c_{(A,K)})$ and $J_*(C^c_{(A,K)})$ are exactly the sheaves $C_{(A,K)}$ on $K$ and $C_{(A,K)}$ on $K^*$.

We see that each one of the three spaces $K^c$, $K$ and $K^*$ determines the other two. This gives us a very easy way to pass back and forth between $K$ and $K^*$. The same kind of connection exists between morphisms: First note that the stalk of $C^c_{(A,K)}$ at $x \in K$ is the real closed field $\rho(x)$. Now let

$$f\colon (K, C_{(A,K)}) \to (L, C_{(B,L)})$$

be a morphism of affine real closed spaces. Then $f$ is continuous in the constructible topology. Using the functor defined in [16, p. 10/11] we see that, for $C \subset L$ constructible, there is a canonical homomorphism

$$C_{(B,L)}(C) \to C_{(A,K)}(f^{-1}(C)).$$

Altogether this gives us a morphism

$$f^c\colon (K^c, C^c_{(A,K)}) \to (L^c, C^c_{(B,L)})$$

of pre-ringed spaces. Applying the direct image functors $I_*$ and $J_*$ we get the morphisms

$$\begin{aligned} f = I_*(f^c)\colon (K, C_{(A,K)}) \\ = I_*(K^c, C_{(A,K)}) \longrightarrow (L, C_{(B,L)}) = I_*(L^c, C^c_{(B,L)}), \\ f^* = J_*(f^c)\colon (K^*, C^*_{(A,K)}) \\ = J_*(K^c, C^c_{(A,K)}) \longrightarrow (L^*, C^*_{(B,L)}) = J_*(L^c, C^c_{(B,L)}). \end{aligned}$$

Conversely, we can start from a morphism $g\colon (K^*, C^*_{(A,K)}) \to (L^*, C^*_{(B,L)})$ and repeat the same arguments. This gives us a morphism

$$g^c\colon \left(K^c, C^c_{(A,K)}\right) \to \left(L^c, C^c_{(B,L)}\right)$$

of the pre-ringed spaces and morphisms

$$g = J_*(g^c)\colon \left(K^*, C^*_{(A,K)}\right) \longrightarrow \left(L^*, C^*_{(B,L)}\right),$$
$$g^w = I_*(g^c)\colon \left(K, C_{(A,K)}\right) \longrightarrow \left(L, C_{(B,L)}\right).$$

It is also clear that $f^c = (f^*)^c$ and $g^c = (g^w)^c$. This almost proves:

THEOREM 1.5. *The category of affine real closed spaces is equivalent to the category of affine inverse real closed spaces.*

*Proof.* Let $(X, O_X)$ be an affine real closed space, $\varphi_X\colon (X, O_X) \to (K, C_{(A,K)})$ an isomorphism. We set $F(X) = (K^*, C^*_{(A,K)})$. Let $(Y, O_Y)$ be another affine real closed space, $\varphi_Y\colon (Y, O_Y) \to (L, C_{(B,L)})$ an isomorphism, $f\colon X \to Y$ a morphism. Then there is a unique morphism $f_1\colon K \to L$ such that $f_1\varphi_X = \varphi_Y f$. We set $F(f) = f_1^*$. Clearly, $F$ is a functor from the category of affine real closed spaces to the category of affine inverse real closed spaces. Now we define a functor $G$ in the opposite direction: Let $(X, O_X)$ be an affine inverse real closed space,

$$\psi_X\colon (X, O_X) \to \left(K^*, C^*_{(A,K)}\right)$$

an isomorphism. We define $G(X) = (K, C_{(A,K)})$. If $(Y, O_Y)$ is another space,

$$\psi_Y\colon (Y, O_Y) \to \left(L^*, C^*_{(B,L)}\right)$$

an isomorphism, $g\colon X \to Y$ a morphism, there is a unique morphism $g_1\colon K^* \to L^*$ with $g_1\psi_x = \psi_Y g$. We set $G(g) = g_1^w$. Again this is a functor, and it is clear that $F$ and $G$ are quasi-inverse to each other. □

In Section 4 the equivalence of Theorem 1.5 is extended to larger categories of real closed spaces and inverse real closed spaces.

If we are only interested in affine spaces, Theorem 1.5 tells us that, from the categorical point of view, it does not make any difference if we work with real closed spaces or with inverse real closed spaces. However, if $X$ is an affine real closed space then there are big differences between $X$ and $F(X)$ on the topological level. So, glueing spaces together on open subspaces means completely different things in the two categories. The effect of this is that in the non-affine theory completely different spaces can be constructed.

On the categorical level, Theorem 1.5 in connection with results from [15] or [16] gives us the following information about inverse real closed spaces:

COROLLARY 1.6 (cf. [15, p. 135], [16, p. 40]). *Let* $R_0$ *be the field of real algebraic numbers. Then* $\mathrm{Sper}(R_0) = \mathrm{Sper}^*(R_0)$ *is the final object in the category of inverse real closed spaces.*

COROLLARY 1.7. *Fiber products exist in the category of inverse real closed spaces.*

*Proof.* For affine spaces this is a consequence of Theorem 1.5 and [16, Theorem II, 3.1] or [15, Theorem V 2.26]. By a standard argument this can be extended to the non-affine case (cf. [8, Proof of Theorem II 3.3]. □

## 2. Subspaces

As with real closed spaces (cf. [15, Chapter V, §2], [16, Chapter II, §2]), for inverse real closed spaces there is a very general notion of subspaces. The definition is practically identical to [15, Definition V 2.5] or [16, Definition II 2.2]. However it should be noted that, even in the affine case, the subspaces of $(K, C_{(A,K)})$ are not the same as those of $(K^*, C^*_{(A,K)})$ (see Example 2.25). This is due to the fact that the topology is used in defining the notion of a subspace.

First we define open subspaces:

DEFINITION 2.1. Let $(X, O_X)$ be an inverse real closed space, $X' \subset X$ an open subset. Then $(X', O_X|X')$ is called an *open subspace* of $X$.

Of course we want to prove that an open subspace is an inverse real closed space. For this we need:

LEMMA 2.2. *If* $(X, O_X)$ *is an affine inverse real closed space and* $X' \subset X$ *is an open constructible subset then* $(X', O_X|X')$ *is an affine inverse real closed space.*

*Proof.* We may assume that $(X, O_X) = (K^*, C^*_{(A,K)})$ and $X = L^* \subset K^*$ is open constructible. By definition of the space $K^*$ it is clear that

$$C^*_{(A,K)})|L^* = C^*_{(A,L)}. \qquad \square$$

PROPOSITION 2.3. *An open subspace* $X' \subset X$ *in an inverse real closed space is an inverse real closed space.*

*Proof.* This is an immediate consequence of Lemma 2.2 and the definition of inverse real closed spaces. □

DEFINITION 2.4. An open subspace $X' \subset X$ is an *open affine subspace* if it is an affine inverse real closed space. By Definition 1.4 every inverse real closed space can be covered by open affine subspaces. We call this an *open affine cover*.

In an affine space we can recognize easily if an open subspace is itself affine. This result is extended to the non-affine situation in Theorem 3.7. (The obvious proof is omitted.)

PROPOSITION 2.5. *Let $X$ be an affine inverse real closed space, $X' \subset X$ an open subspace. $X'$ is affine if and only if $X'$ is quasi-compact.*

Now we define the notion of a subspace:

DEFINITION 2.6. (a) Let $X$ be an affine inverse real closed space, $X' \subset X$ a subset. $X'$ is a *subspace* if there is an open cover $X' = \bigcup_{i \in I} X_i'$ with each $X_i'$ pro-constructible in $X$.

(b) Let $X$ be an inverse real closed space, $X' \subset X$ a subset. $X'$ is a *subspace* if for every open affine subspace $X_1 \subset X$, $X' \cap X_1$ is a subspace of the affine inverse real closed space $X_1$.

Observe that an open subspace (Definition 2.1) is a subspace in the sense of this definition.

First we record the following fact about subspaces:

PROPOSITION 2.7. *The set of subspaces of an inverse real closed space is closed under finite intersections.*

*Proof.* Same proof as [15, Proposition V 2.18] or [16, Proposition II 2.8]. □

The corresponding statement about unions is false. (Also, the corresponding part of [15, Proposition V 2.18] and [16, Proposition II 2.8] is false.) See however Proposition 2.16.

There is a natural way to endow subspaces with a structure sheaf so that they become inverse real closed spaces. First we do this for a subspace $X'$ of an affine space $X$: Let $X' = \bigcup_{i \in I} X_i'$ be a cover as in Definition 2.6(a). Since every $X_i'$ is pro-constructible in $X$, $X_i'$ has a natural structure sheaf which makes it an affine inverse real closed space. Now consider $X_i' \cap X_j'$. This is pro-constructible in $X$, hence also in $X_i'$ and $X_j'$. By [16, Theorem I 3.25], the structure sheaves induced on $X_i' \cap X_j'$ by $X$, $X_i'$, $X_j'$ all coincide. So these sheaves can be glued together to give a sheaf on $X'$. $X'$ is an inverse real closed space with $\bigcup_{i \in I} X_i'$ an open affine cover. It is an easy matter to show that this construction is independent of the cover used.

To extend this construction to the non-affine case we need:

LEMMA 2.8. *Let $X$ be an inverse real closed space, $X_1, X_2 \subset X$ open affine subspaces, $Y \subset X$ a subspace such that $Y \subset X_1 \cap X_2$. Then $Y$ is a subspace of $X_1$ and $X_2$. The structure sheaves induced by $X_1$ and $X_2$ agree.*

*Proof.* It is clear from Definition 2.6 that $Y$ is a subspace of both $X_1$ and $X_2$. Pick an open cover $Y = \bigcup_{j \in J} Y_j$ such that $Y_j \subset X_1$ is pro-constructible for every $j$. We choose an open affine cover $X_1 \cap X_2 = \bigcup_{k \in K} X_k$. For $j \in J$ there is a finite subset $K_j \subset K$ such that $Y_j \subset \bigcup_{k \in K_j} X_k = X_j$ (by quasi-compactness of $Y_j$). By Proposition 2.5, $X_j$ is an open affine subspace of $X_1$ and $X_2$ which contains $Y_j$. [16, Theorem I 3.25] shows that $X_1$ and $X_j$ induce the same structure sheaf on $Y_j$, and also $X_2$ and $X_j$ induce the same structure sheaf on $Y_j$. □

LEMMA 2.9. *If $X$ is an inverse real closed space and $X' \subset X$ is a subspace and $X_1, X_2 \subset X$ are open affine subspaces, then $X' \cap X_1 \cap X_2$ is a subspace of $X_1$ and $X_2$.*

*Proof.* This is a trivial consequence of Definition 2.6 and Proposition 2.7. □

Now we can prove:

THEOREM 2.10. *If $X$ is an inverse real closed space and $X' \subset X$ is a subspace then $X'$ carries a natural structure sheaf which makes it an inverse real closed space.*

*Proof.* If $X_1 \subset X$ is open and affine then we have already defined a structure sheaf on $X' \cap X_1$. We must see that these sheaves can be glued together. So, let $X_2 \subset X$ be another open affine subspace. Again, $X' \cap X_2$ already has a structure sheaf. By Lemma 2.9, $X' \cap X_1 \cap X_2$ is a subspace of $X_1$ and $X_2$. Lemma 2.8 shows that the structure sheaves induced by $X_1$ and $X_2$ on $X' \cap X_1 \cap X_2$ both agree. Now it is clear that these sheaves glue together, giving the desired structure sheaf for $X'$. □

Before discussing general properties of subspaces we record a few examples.

*Example* 2.11. We noted already that open subspaces are subspaces.

*Example* 2.12. Let $Y \subset X$ be a subset in an inverse real closed space such that, for all open affine subspaces $X' \subset X$, $Y \cap X' \subset X'$ is a pro-constructible subset. $Y$ is a subspace, called a *locally pro-constructible subspace*. As special cases of this we note the following:

(a) If $Y \subset X$ is closed then $Y$ is a *closed subspace*.

(b) If $Y \subset X$ is finite then $Y$ is a *finite subspace*. In particular, if $B \subset R$ is a convex subring of a real closed field and if $x \in \text{Sper}^*(B)$ is the generic point, $y \in \text{Sper}^*(B)$ the closed point, then $\{x, y\}$ is a finite subspace of $\text{Sper}^*(B)$. $\{x, y\}$ is called a *valuative inverse real closed space*.

(c) If $Y \subset X$ is locally pro-constructible and quasi-compact then $Y$ is called *pro-constructible*. In particular, if $f\colon Z \to X$ is a morphism of affine inverse real closed spaces then $f(Z) \subset X$ is pro-constructible. $f(Z)$ is called the *image* of $f$.

*Example* 2.13. Let $Y \subset X$ be a subset such that, for all open affine $X' \subset X$, the subspace $X' \cap Y \subset X'$ is constructible. Then $Y$ is a *locally constructible subspace*. $Y$ is *constructible* if it is, in addition, quasi-compact.

Both the notion of locally pro-constructible subspaces and the notion of locally constructible subspaces are defined by referring to all open affine covers of a space. However, an easy argument shows that $Y \subset X$ is locally (pro-)constructible if and only if there is an open affine cover $X = \bigcup_{i \in I} X_i$ such that $Y \cap X_i \subset X_i$ is (pro-)constructible for all $i \in I$. Note that similarly in [16, Corollary 4.17] the hypothesis "quasi-separated" is unnecessary.

Now we mention a few general facts about subspaces.

PROPOSITION 2.14. *For a subset $X' \subset X$ of an inverse real closed space the following statements are equivalent*:

*(a) $X'$ is a subspace.*

*(b) There is an open affine cover $X = \bigcup_{i \in I} X_i$ such that $X' \cap X_i$ is a subspace of $X_i$ for all $i$.*

PROPOSITION 2.15. *Let $X' \subset X$ be a subspace, $X'' \subset X'$ a subset. Then $X''$ is a subspace of $X'$ if and only if it is a subspace of $X$.*

It was mentioned above that finite unions of subspaces need not be subspaces. However we have:

PROPOSITION 2.16. *In an inverse real closed space $X$ the locally pro-constructible* (*locally constructible*) *subspaces form a lattice* (*Boolean lattice*).

*Proof.* This follows immediately from the fact that in an affine space the pro-constructible (constructible) subspaces form a lattice (Boolean lattice). □

For the further discussion of subspaces it is useful to have the following characterization of monomorphisms:

PROPOSITION 2.17. *Let $f$: $X \to Y$ be morphism of inverse real closed spaces. Then $f$ is a monomorphism if and only if $f$ is injective on the underlying spaces and if, for every $x \in X$, the homomorphism $\rho(f(x)) \to \rho(x)$ of real closed fields is an isomorphism.*

The proof is identical to the proof of [15, Proposition V 2.17] or [16, Proposition II 2.13]. The same kind of arguments shows:

PROPOSITION 2.18. *A morphism $f$: $X \to Y$ is an epimorphism if it is surjective on the underlying spaces.*

*Example* 2.19. If $X' \subset X$ is a subspace we have the morphism inclusion. This is a monomorphism.

PROPOSITION 2.20. *Let $f$: $X \to Y$ be a morphism of inverse real closed spaces, $Z \subset Y$ a subspace with $f(X) \subset Z$. Then there is a unique morphism $g$: $X \to Z$ with $f = ig$ ($i$: $Z \to Y$ the inclusion).*

*Proof.* Uniqueness is clear since $i$ is a monomorphism. To show existence we may assume that $X, Y, Z$ are all affine. Then the functors $F$ and $G$ of §1 can be used to transfer the corresponding property ([16, Proposition II 2.15] or [15, Proposition V 2.28]) from affine real closed spaces to affine inverse real closed spaces. □

For the rest of this section more examples are discussed.

*Example* 2.21. Let $x \in X$ be a point in an inverse real closed space and let Gen($x$) be the set of generalizations of $x$. Pick some open affine subspace $X_0 \subset X$ with $x \in X_0$. Then Gen($x$) $\subset X_0$ is a pro-constructible subspace. By Proposition 2.15, Gen($x$) is a subspace of $X$, called the *local subspace* of $x \in X$.

*Example* 2.22. If $f$: $X \to Y$ is a morphism of inverse real closed spaces and $Y' \subset Y$ is a subspace, then so is $f^{-1}(Y') \subset X$. This subspace is called the *inverse image* of $Y'$. In particular, the inverse image of the subspace $\{y\} \subset Y$ is denoted by $f^{-1}(y)$ and is called the *fibre* of $f$ at $y$. Inverse images can also be described in terms of fibre products: If $i$: $Y' \to Y$ is the inclusion we consider the cartesian square

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{f} & Y \end{array}$$

Using Proposition 2.20 we can show that there is a canonical isomorphism $f^{-1}(Y') \to X \times_Y Y'$. In particular, we have $f^{-1}(y) \cong X \times_Y \{y\}$.

*Example* 2.23. For any morphism $f$: $X \to Y$ of inverse real closed spaces we have the diagonal morphism $\Delta$: $X \to X \times_Y X$ (by the existence of fibre products—see Corollary 1.7). We will see now that $\Delta$ is an isomorphism onto a subspace. We start with an open affine cover $Y = \bigcup_{i \in I} Y_i$ and set $X_i = f^{-1}(Y_i)$. Then

$$X \times_Y X = \bigcup_{i \in I} X_i \times_{Y_i} X_i$$

is an open cover. For each $i$ let $X_i = \bigcup_{j \in J_i} X_{ij}$ be an open affine cover. Then

$$X \times_Y X = \bigcup_{i \in J} \bigcup_{j, k \in J_i} X_{ij} \times_{Y_i} X_{ik}$$

is an open affine cover. We must show that $\Delta(X) \cap (X_{ij} \times_{Y_i} X_{ik})$ is a subspace of $X_{ij} \times_{Y_i} X_{ik}$ for all $i, j, k$ (Proposition 2.14). We have

$$\Delta(X) \cap (X_{ij} \times_{Y_i} X_{ik}) = \Delta(X_{ij} \cap X_{ik}).$$

If $X_{ij} \cap X_{ik} = \bigcup_{l \in L_{ijk}} X_{ijkl}$ is an open affine cover then

$$\Delta(X_{ij} \cap X_{ik}) = \bigcup_{l} \Delta(X_{kjkl})$$

is a cover with the following properties:

$\Delta(X_{ijkl}) = \Delta(X_{ij} \cap X_{ik}) \cap p_1^{-1}(X_{ijkl})$ is open in $\Delta(X_{ij} \cap X_{ik})$.

$\Delta(X_{ijkl}) \subset X_{ij} \times_{Y_i} X_{ik}$ is pro-constructible, being the image of a morphism of affine spaces (Example 2.12(c)).

This proves that $\Delta(X) \subset X \times_Y X$ is a subspace. We must now show that the restriction $\Delta'$: $X \to \Delta(X)$ (cf. Proposition 2.19) is an isomorphism. Let $i$: $\Delta(X) \to X \times_Y X$ be the inclusion. Since $\mathrm{id}_X = p_1 i \Delta'$ it follows that $\Delta'$ is a monomorphism. Since $\Delta'$ is surjective it is an epimorphism (Proposition 2.18).

We have

$$(p_1 i)\Delta' = p_1\Delta = \mathrm{id}_X.$$

To show that $\Delta'$ is an isomorphism we must prove $\Delta'(p_1 i) = \mathrm{id}_{\Delta(X)}$. Now $\Delta'$ is an epimorphism and we have $(p_1 i)\Delta' = p_1\Delta = p_2\Delta = (p_2 i)\Delta'$, thus, $p_1 i = p_2 i$ holds. This shows that $i = (p_1 i, p_2 i)$: $\Delta(X) \to X \times_Y X$ factors through $\Delta'$:

$$i: \Delta(X) \xrightarrow{p_1 i} X \xrightarrow{\Delta'} \Delta(X) \xrightarrow{i} X \times_Y X.$$

Since $i$ is a monomorphism and $i = i \circ \mathrm{id}_{\Delta(X)}$ we conclude that $\Delta'(p_1 i) = \mathrm{id}_{\Delta(X)}$.

*Example* 2.24. Let $X, Y$ be inverse real closed spaces over an inverse real closed space $Z$, let $f$: $X \to Y$ be a morphism over $Z$. Then the graph morphism $\Gamma$: $X \to X \times_Z Y$ is an isomorphism onto a subspace. For, $\Gamma$ is obtained by base extension from $\Delta$: $Y \to Y \times_Z Y$ [7, 0 1.4.9].

The final example shows that in the real spectrum Sper($A$) of a ring there can be subsets $X$ and $Y$ such that $X$ is a subspace of the real closed space Sper($A$), but not of the inverse real closed space Sper*($A$) and $Y$ is a subspace of Sper*($A$), but not of Sper($A$).

*Example* 2.25. Let $M$ be a totally ordered set with a nontrivial cut $M = M_1 \cup M_2$, $M_1 < M_2$ such that $M_1$ has no largest element and $M_2$ has no smallest element. We order $G = \mathbf{Q}^{(M)}$ lexicographically. If $v$: $G \setminus \{0\} \to M$ maps $g$ to the largest element in the support of $g$, then we have $g > 0$ if and only if $g(v(g)) > 0$. There is an injective order-preserving map from $M$ into the set of convex subgroups of $G$: $x \in M$ is mapped to

$$C_x = \{0\} \cup \{g \in G \mid v(g) \leq x\}.$$

Let $C$ be the convex subgroup

$$\bigcup_{x \in M_1} C_x = \{0\} \cup \{g \in G \mid v(g) \in M_1\} = \bigcap_{x \in M_2} C_x.$$

Now let $H$ be the formal power series field $\mathbf{R}((G))$. Let $w$: $H \to G \cup \{\infty\}$ be the natural valuation, $W \subset H$ the corresponding valuation ring. Thus, $H$ is a real closed field, $W$ a convex subring [12, Chapter II, §5; Chapter III, §4], hence a real closed valuation ring [6], [7]. So, we may identify Sper($W$) = Spec($W$) [15, Proposition II 4.10], [16, Theorem I 3.1.0]. The convex subgroups of $G$ correspond with the (convex) prime ideals of $W$. So, for every convex subgroup $C_x$ there is a prime ideal $P_x \subset W$. Let $P$ be the prime ideal corresponding to $C$. We set

$$X = \{\alpha \in \mathrm{Sper}(W) | \mathrm{supp}(\alpha) \subsetneqq P\}, Y = \{\alpha \in \mathrm{Sper}(W) | \mathrm{supp}(\alpha) \supsetneqq P\}.$$

Then $X$ is a union of open constructible subspaces of Sper($W$), $Y$ is a union of closed constructible subspaces of Sper($W$). By construction $Y$ does not have a generic point. Therefore $Y$ is not a subspace of Sper($W$). On other hand, in Sper*($W$), $Y$ is a union of open constructible subspaces and $X$ is a union of closed constructible subspaces. Again by construction, $X$ does not have a generic point with respect to the inverse topology, hence is not a subspace of Sper*($W$). On the other hand, $X \subset \mathrm{Sper}(W)$ and $Y \subset \mathrm{Sper}^*(W)$ are clearly subspaces.

## 3. Quasi-compactness and quasi-separatedness

We say that a morphism $f$: $X \to Y$ of inverse real closed spaces is *quasi-compact* if $f^{-1}(Y')$ is quasi-compact for every open affine subspace $Y' \subset Y$ [7, I 6.1.1], [15, Definition V 3.2], [16, Definition II 4.1]. It is easy to see that this is equivalent to: There is an open affine cover $Y = \bigcup_{i \in I} Y_i$ such that $f^{-1}(Y_i)$ is quasi-compact for every $i$ [7, p. 290], [15, Proposition V 3.5, [16, Proposition II 4.4]. $f$ is *quasi-separated* if the diagonal morphism $\Delta_f$ is quasi-compact [7, I 6.1.3], [15, Definition V 3.2], [16, Definition V 4.1]. As in [15, Proposition V 3.9] or [16, Proposition II 4.12] one sees that each of the following two statements is equivalent to $f$ being quasi-separated.

For every open affine subspace $Y' \subset Y$ the restriction $f'$: $f^{-1}(Y') \to Y'$ of $f$ is quasi-separated.

There is an open affine cover $Y = \bigcup_{i \in I} Y_i$ such that every restriction

$$f_i: f^{-1}(Y_i) \to Y_i$$

of $f$ is quasi-separated.

We will now collect a few basic properties of quasi-compact and quasi-separated morphisms (cf. [7, I §6.1], [15, Chapter V, §3], [16, Chapter II, §4]). This will be done without proof since the proofs are virtually identical to those in the references.

The following terminology will be used: let $\mathscr{C}$ be a class of morphisms of inverse real closed space. We say that the class $\mathscr{C}$ has property

(A) if it is stable under composition,
(B) if it is stable under base extension,
(C) if it is stable under products.

PROPOSITION 3.1. (a) *Morphisms of affine spaces are quasi-compact.*

(b) *If $X \subset Y$ is a locally pro-constructible subspace then the inclusion is quasi-compact.*

(c) *Quasi-compact morphisms have properties* (A), (B), (C).

(d) *If $f$: $X \to Y$, $g$: $Y \to Z$ are morphisms, $gf$ is quasi-compact and $g$ is quasi-separated, then $f$ is quasi-compact.*

(e) *If $f$: $X \to Y$, $g$: $Y \to Z$ are morphisms, $gf$ is quasi-compact and $f$ is surjective, then $g$ is quasi-compact.*

PROPOSITION 3.2. (a) *Morphisms of affine spaces are quasi-separated.*

(b) *Monomorphisms are quasi-separated.*

(c) *Quasi-separated morphisms have properties* (A), (B), (C).

(d) *If $f$: $X \to Y$, $g$: $Y \to Z$ are morphisms such that $gf$ is quasi-separated, then $f$ is quasi-separated.*

(e) *If $f$: $X \to Y$, $g$: $Y \to Z$ are morphisms, $gf$ is quasi-separated and $f$ is quasi-compact and surjective, then $g$ is quasi-separated.*

In Example 2.12(c) we saw that morphisms of affine spaces have images. This is not true for arbitrary morphisms. But we have:

PROPOSITION 3.3 (cf. [16, Proposition II 4.2], [15, Proposition V 3.4]). *If $f$: $X \to Y$ is quasi-compact then $f(X) \subset Y$ is a locally pro-constructible subspace.*

*Proof.* This follows directly from Example 2.12(c) and Proposition 2.16. □

If $X$ is an affine inverse real closed space and $C \subset X$ is pro-constructible then the closure of $C$ is the set of all specializations of elements $C$ (cf. [15, p. 27], [16, p. 4]). Just as in [15, Proposition V 3.3 or [16, Corollary II 4.3], this can be used to prove:

PROPOSITION 3.4. *For a quasi-compact morphism $f$: $X \to Y$ the following conditions are equivalent*:

(a) *$f$ is closed.*

(b) *For every $x \in X$ and every $y' \in \overline{\{f(x)\}}$ there is some $x' \in \overline{\{x\}}$ with $f(x') = y'$.*

We call a space $X$ *quasi-separated* if the unique morphism $X \to \mathrm{Sper}^*(R_0)$ is quasi-separated. By Proposition 3.2, a morphism $f$: $X \to Y$ into an affine space is quasi-separated if and only if the space $X$ is quasi-separated. So (by the remarks at the beginning of this section) quasi-separatedness of morphisms can be reduced to quasi-separatedness of spaces. Similar to [7, I

6.1.12] and [16, Theorem II 4.16] or [15, Proposition V 3.16] this can be characterized in the following way.

PROPOSITION 3.5. *Let $X = \bigcup_{i \in I} X_i$ be an open affine cover. The following statements are equivalent:*

(a) *$X$ is quasi-separated.*

(b) *For all $i, j \in I$, $X_i \cap X_j$ is quasi-compact.*

(c) *For all quasi-compact open subspaces $U, V \subset X$, $U \cap V$ is quasi-compact.*

As a consequence, quasi-separated spaces can be characterized by the property that every open affine subspace is constructible. Another immediate consequence of the definitions and Proposition 3.2 is that arbitrary subspaces of quasi-separated spaces are quasi-separated. We also see that the quasi-separated spaces with the quasi-separated morphisms form a full subcategory of the category of inverse real closed spaces.

All the properties of quasi-compact and quasi-separated morphisms that have been mentioned so far can be summarized by saying that the formal properties of these notions are the same as in the case of schemes or real closed spaces. But when we look at the geometric meaning of these notions then things are different:

THEOREM 3.6. *A quasi-separated and quasi-compact inverse real closed space $X$ is affine.*

For the proof of Theorem 3.6 we need the following generalization of [16, Theorem I 4.5].

LEMMA 3.7. *If $A$ is a ring, $K \subset \mathrm{Sper}(A)$ is pro-constructible and $C \subset K$ is closed then the canonical restriction map $C_A(K) \to C_A(C)$ is surjective.*

*Proof.* We may assume that $A = C_A(K)$ [16, Theorem I 3.25]. If the statement is true for all closed irreducible subsets $C \subset K$ then the proof of the general statement is identical to the proof of [16, Theorem I 4.5]. So, it remains to prove the claim if $C$ is closed and irreducible. This corresponds to [16, Lemma I 4.4]. Although the proof is similar to the one in the reference we repeat it here since a few changes have to be made.

Suppose that $C = \overline{\{x\}}$, $a \in C_A(C)$, $0 \leq a$. For every $y \in C$ there is a neighborhood $U_y \subset K$ of $y$ and there is some $a_y \in C_A(U_y)$ with $a_y(y) = a(y)$. Then

$$F_y = \{z \in C \cap U_y | a_y(z) = a(z)\}$$

is closed and constructible in $C \cap U_y$ and contains $y$. This shows that $C = \bigcup_{y \in C} F_y$ is a constructible cover. By compactness of the constructible topology there is a finite subcover $C = \bigcup_{i=1}^{r} F_i$, where we set $F_i = F_{y_i}$ and $a_i = a_{y_i}$.

Altogether this shows that there is a finite cover $C \subset \bigcup_{i=1} G_i$ with $G_i = U_i \cap H_i$, $U_i \subset K$ open constructible, $H_i \subset K$ closed constructible, and elements $a_i \in C_A(U_i)$ such that $a_i | G_i \cap C = a | G_i \cap C$. Suppose that such a cover has been chosen with minimal $r$.

Suppose that $r = 1$. There is some $0 \leq u_1 \in A$ such that

$$U_1 = \{\alpha \in K | u_1(\alpha) > 0\}.$$

Let $z \in C$ be the closed point. Then $A = C_A(K) \to \rho(z)$ is surjective. Pick $b \in \rho(z)$, $c \in A$ with $c(z) = b$ and $b > 1 + a(z)u_1^{-1}(z)$. Then $d = \sup(c, 1) \in A^*$, hence

$$U_1 = \{\alpha \in K | (du_1)(\alpha) > 0\}$$

and

$$(du_1)(y) > u_1(y) + a_1(y) > a_1(y) \quad \text{for all} \quad y \in C.$$

The set $D = \{\alpha \in U_1 | a_1^2(\alpha) \geq (du_1)^2(\alpha)\}$ is closed in $U_1$ and has empty intersection with $C$. [16, Corollary I 3.28] or [15, Corollary II, 4.16] shows the existence of some $0 \leq v \leq 1$, $v \in C_A(U_1)$ such that $v|D = 0$, $v|C = 1$. For $va_1 \in C_A(U_1)$ this implies $va_1|C = a_1|C = a$, $(va_1)(\alpha) \leq (du_1)(\alpha)$ if $\alpha \in U_1$. If we define $e(\alpha) = (va_1)(\alpha)$ for $\alpha \in U_1$, $e(\alpha) = 0$ for $\alpha \in K \setminus U_1$, then [15, p. 31] or [16, p. 14], shows that $e \in A$, and it is clear that $A \to C_A(C)$ maps $e \to a$.

Now suppose that $r > 1$. We will show that this leads to a contradiction. First note that every $C_1 = G_i \cap C$ is a constructible interval in $C$. Between these sets there is no inclusion possible (since $r$ is minimal). We enumerate the $C_i$ such that $C_1 \subset \overline{C_2} \subset \overline{C_3} \subset \cdots \subset \overline{C_r} = C$. Now we define $V = U_1 \cap U_2$, $b_1 = a_1|V$, $b_2 = a_2|V$, $c = b_2 - b_1$. Then $C \cap V = C \cap U_2$. By constructibility of $V, C \cap V$ has a closed point $y, C \setminus V$ has generic point $z$. Since $c|C \cap V$ can be extended to a compatible family $\bar{c} \in \prod_{\alpha \in C} \rho(\alpha)$ with $\bar{c}(\alpha) = 0$ for $\alpha \in C \setminus V$, [15, p. 31] or [16, p. 14] shows that $\bar{c} \in C_A(C)$. Let $P \subset C_A(C)$ be the convex prime ideal generated by $\bar{c}$. Then

$$\operatorname{supp}(y) \subset P \subset \operatorname{supp}(z).$$

If $\pi\colon A \to C_A(C)$ is canonical then

$$\operatorname{supp}(y) = \pi^{-1}(\operatorname{supp}(y)) \subset \pi^{-1}(P) \subset \pi^{-1}(\operatorname{supp}(z)) = \operatorname{supp}(z)$$

are convex prime ideals of $A$. Hence $\operatorname{supp}(z)$ contains some $0 \leq v \in A$ with

$$|c(y)| = |\bar{c}(y)| < v(y).$$

If $\bar{v} = v|K \setminus V \in C_A(K \setminus V)$, then $S = \{\alpha \in K \setminus V | \bar{v}(\alpha) = 0\}$ is closed constructible in $K \setminus V$ and contains $\overline{\{z\}} = C \setminus V$. The sets $G(K \setminus V \cup S)$ and $G(C \setminus V)$ of generalizations of $K \setminus V \cup S$ and $C \setminus V$ in $K$ are both pro-constructible and have empty intersection. Therefore there is some open constructible $W \subset K$ with

$$W \supset G(C \setminus V), \quad W \cap G(K \setminus V \cup S) = \varnothing.$$

Now $K \setminus W$ is closed constructible and $C \cap (K \setminus W) = \varnothing$. By [15, Corollary II 4.16] or [16, Corollary I 3.28] there is some $w \in A$, $0 \leq w \leq 1$ with $w|C = 1$, $w|K \setminus W = 0$. Then we also have $vw \in A$, $vw|C = v|C$ and, for $\alpha \notin V$, $(vw)(\alpha) = 0$.

So far we have shown that there exists some $0 \leq v \in A$ with

$$V = \{\alpha \in K | v(\alpha) > 0\}$$

and with

$$v(y) > |\bar{c}(y)| = |c(y)|.$$

(Replace $v$ by the $vw$ in the above computation.) The set

$$F = \{\alpha \in V | v(\alpha) \leq |c(\alpha)|\}$$

is closed constructible in $V$ and $F \cap (V \cap C) = \varnothing$. Again there is some $w \in C_A(V)$ with $0 \leq w \leq 1$, $w|F = 0$, $w|V \cap C = 1$ [15, Corollary II 4.16], [16, Corollary I 3.28]. We have $cw|V \cap C = c|V \cap C$ and $(cw)(\alpha) \leq v(\alpha)$ for all $\alpha \in V$. By [15, p. 31] or [16, p. 14], there is some $d \in A$ such that $d|V = c$ and $d|K \setminus V = 0$. We let

$$e = d|U_1 \in C_A(U_1), \quad f = e + a_1.$$

If $\alpha \in C \setminus U_2$ then

$$f(\alpha) = e(\alpha) + a_1(\alpha) = a_1(\alpha) = a(\alpha).$$

If $\alpha \in G_2 \cap C$ then

$$f(\alpha) = e(\alpha) + a_1(\alpha) = c(\alpha) + b_1(\alpha) = b_2(\alpha) = a(\alpha).$$

Now we set $U_0 = U_1$, $H_0 = H_2$, $G_0 = U_0 \cap H_0$ and $a_0 = f \in C_A(U_0)$. Then

$$C \cap (G_1 \cup G_2) = C \cap G_0$$

and

$$a_0|G_0 \cap C = a|G_0 \cap C,$$

and the cover $C \subset G_0 \cup \bigcup_{i=3}^r G_i$ has the same properties as the original cover $C \subset \bigcup_{i=1}^r G_i$, but is shorter. This contradicts the minimality of $r$. □

*Proof of Theorem 3.6.* $X$ has an open affine cover $X = X_1 \cup \cdots \cup X_n$. If we prove the claim for $n = 2$ then it follows for arbitrary $n$ by induction. So let $X = X_1 \cup X_2$. We consider the following rings of sections:

$$A = \Gamma(X), \quad A_1 = \Gamma(X_1), \quad A_2 = \Gamma(X_2), \quad A_{12} = \Gamma(X_1 \cap X_2)$$

with restriction maps $\rho_1$: $A \to A_i$, $\sigma_i$: $A_i \to A_{12}$. These are all real closed rings [16, Theorem I 3.25, Theorem I, 4.12]. $A$ is the fibre product $A_1 \times_{A_{12}} A_2$. Since $\sigma_1$ and $\sigma_2$ are surjective (Lemma 3.7) the same is true for $\rho_1$ and $\rho_2$. This gives us isomorphisms

$$i_1\colon X_1 \subset \operatorname{Sper}^*(A_1) \longrightarrow \operatorname{Sper}^*(A), \quad i_2\colon X_2 \subset \operatorname{Sper}^*(A_2) \longrightarrow \operatorname{Sper}^*(A)$$

onto pro-constructible subspaces $Y_1, Y_2 \subset \operatorname{Sper}^*(A)$. We want to glue $i_1$ and $i_2$ together on $X_1 \cap X_2$. First note that $X$ is a spectral space (by Proposition 3.5). The map

$$i\colon X \longrightarrow \operatorname{Sper}^*(A)\colon x \longrightarrow (A \to O_{X,x} \subset \rho(x))$$

is a morphism of spectral spaces: If $C \subset \operatorname{Sper}^*(A)$ is open constructible then there is some $a \in A$ such that $C = \{\alpha \in \operatorname{Sper}^*(A) | a(\alpha) = 0\}$ [15, Proposition II 4.13], [16, Proposition I 3.17], and this implies

$$X_1 \cap i^{-1}(C) = \{x \in X_1 | \rho_1(a)(x) = 0\},$$
$$X_2 \cap i^{-1}(C) = \{x \in X_2 | \rho_2(a)(x) = 0\},$$

i.e., $i^{-1}(C)$ is open and constructible. It is clear that $i|X_1 = i_1$, $i|X_2 = i_2$. By [16, Proposition I 3.27] there are $a_i' \in A_i$ such that

$$X_1 \cap X_2 = \{x \in X_i | a_i'(x) = 0\}.$$

The sheaf property shows that there are $a_1, a_2 \in A$ with $\rho_1(a_1) = a_1'$, $\rho_1(a_2) = 0$, $\rho_2(a_1) = 0$, $\rho_2(a_2) = a_2'$. If $Y = i(X)$ this shows that $i$: $X \to Y$ is injective and

$$i(X_1) = \{\alpha \in Y | a_2(\alpha) = 0\}, \quad i(X_2) = \{\alpha \in Y | a_1(\alpha) = 0\}$$

are open constructible subsets of $Y$. Thus $i$: $X \to Y$ is an isomorphism of spectral spaces. From $\sigma_1\rho_1 = \sigma_2\rho_2$ it follows immediately that

$$i_1|X_1 \cap X_2 = i_2|X_1 \cap X_2;$$

i.e., $i_1$ and $i_2$ glue together to the morphism $i$: $X \to Y$ of inverse real closed spaces, and $i$ is an isomorphism. Since $Y$ is affine we conclude that $X$ is affine. □

COROLLARY 3.8. *Quasi-compact quasi-separated morphisms of inverse real closed spaces are affine*; *i.e., inverse images of open affine subspaces are open affine.*

These results correspond to [15, Theorem V 4.8; Corollary V 4.9] or [16, Theorem II 5.8 and Corollary II 5.10. However, in the references the additional hypothesis of regularity was required. Regularity of a real closed space means that the specializations of a point form a chain. Translating this into the language of inverse real closed spaces it says that the generalizations of a point form a chain. But this is true in every inverse real closed space.

The results of this section give us some additional information about diagonal morphisms (hence also about graph morphisms): By Proposition 3.2, diagonal morphisms are quasi-separated. Corollary 3.8 shows that the diagonal morphism of a quasi-separated morphism is affine. In this case the image of the diagonal (Example 2.23) is a locally pro-constructible subspace.

## 4. Connections between real closed spaces and inverse real closed spaces

We saw in §1 that the categories of affine real closed spaces and of affine inverse real closed spaces are equivalent via two functors $F, G$. This equivalence is extended in this section.

Recall that real closed spaces are a generalization of locally semi-algebraic spaces [5], [15], [16]. Similarly weakly semi-algebraic spaces [16] are generalized by the inverse real closed spaces (see §7). In [10, Chapter V §3] it is shown that a locally semi-algebraic space is weakly semi-algebraic if and only if it is paracompact. Here we will prove a related result. We start with a definition from [14].

DEFINITION 4.1. Let $X$ be a real closed space or an inverse real closed space. $X$ is *taut* if it is quasi-separated and every $x \in X$ belongs to some closed constructible subspace.

THEOREM 4.2. *The equivalence between the category of affine real closed spaces and the category of affine inverse real closed spaces can be extended to* *an equivalence between the category of taut regular real closed spaces and the category of taut inverse real closed spaces.*

*Proof.* First let $X$ be a taut inverse real closed space. Extending the notation of §1 we let $\overline{\mathcal{C}}(X)$ be the set of closed constructible subspaces of $X$. Since $X$ is taut, $\overline{\mathcal{C}}(X)$ is a cover of $X$. Every $C \in \overline{\mathcal{C}}(X)$ is contained in some quasi-compact open subspace $U_C \subset X$. By Theorem 3.6, $U_C$ is affine, and $G(U_C)$ has already been defined in §1. Since $C \subset U_C$ is a closed constructible subspace, $G(C) \subset G(U_C)$ is an open constructible subspace. We want to glue the $G(C)$ together to obtain a real closed space. If $D \in \overline{\mathcal{C}}(X)$ and $D \subset C$ then $G(D)$ can be canonically identified with an open constructible subspace of $G(C)$. This shows that the $G(C)$, $C \in \overline{\mathcal{C}}(X)$ can be glued together to give a real closed space $G(X)$. By construction, $G(X) = \bigcup_{C \in \overline{\mathcal{C}}(X)} G(C)$ is an open affine cover and $G(C_1) \cap G(C_2) = G(C_1 \cap C_2)$. By [16, Theorem II 4.16], $G(X)$ is quasi-separated.

For every $C \in \overline{\mathcal{C}}(X)$ we have a bijection $\tau_C$: $C \to G(C)$ which is a homeomorphism with respect to the constructible topology. These $\tau_C$ fit together to give a bijection $\tau$: $X \to G(X)$. If $X' \subset X$ is open (closed) constructible then $\tau(X') \subset G(X)$ is closed (open) constructible. To prove regularity of $G(X)$ we pick $\alpha, \beta, \gamma \in G(X)$, $\beta, \gamma \in \overline{\{\alpha\}}$. Then $\tau^{-1}(\beta)$, $\tau^{-1}(\gamma)$ are generalizations of $\tau^{-1}(\alpha)$ in $X$. But in $X$ the generalizations of a point form a chain; i.e.,

$$\tau^{-1}(\beta) \in \overline{\{\alpha^{-1}(\gamma)\}} \quad \text{or} \quad \tau^{-1}(\gamma) \in \overline{\{\tau^{-1}(\beta)\}}.$$

This implies $\gamma \in \overline{\{\beta\}}$ or $\beta \in \overline{\{\gamma\}}$, proving that $G(X)$ is regular. Finally, $G(X)$ is also taut: Pick $\alpha \in G(X)$. There is some $C \in \overline{\mathcal{C}}(X)$ with $\alpha \in G(C)$; i.e., $\tau^{-1}(\alpha) \in C \subset U_C$. Therefore $\alpha \in \tau(U_C)$, and $\tau(U_C)$ is a closed constructible subspace of $G(X)$.

So far we have defined the functor $G$ on the objects. Now let $f$: $X \to Y$ be a morphism of taut inverse real closed spaces. If $C \in \overline{\mathcal{C}}(X)$ then $f(C)$ is quasi-compact and this implies that there is some open constructible subspace $U \subset Y$ containing $f(C)$. By tautness, we can find some $D \in \overline{\mathcal{C}}(Y)$ with $U \subset D$. So, altogether this gives a morphism

$$G(f|C): G(C) \longrightarrow G(D) \subset G(Y).$$

If $C' \in \overline{\mathcal{C}}(X)$, $C' \subset C$ then it is clear that $G(f|C') = G(f|C)|G(C')$. So the $G(f|C)$ glue together to give a morphism $G(f)$: $G(X) \to G(Y)$. This completes the definition of the functor $G$.

Exactly in the same way the functor $F$ from the category of taut regular real closed spaces to the category of taut inverse real closed spaces is defined. From §1 it follows that $G$ and $F$ are quasi-inverse to each other. □

In a quasi-separated space (real closed or inverse real closed) we can define the constructible topology just as in the affine case (cf. [7, I 7.2.2]). From the proof of Theorem 4.2 we see that, starting with a taut inverse real closed space $X$, we may identify $X$ and $G(X)$ with the constructible topologies via the map $\tau$. So, one may say that the taut spaces are exactly those spaces which can be covered equally well by weakly open constructible subsets and by inversely open constructible subsets.

## 5. Complete quasi-separatedness

In [10, p. 4], weakly semi-algebraic spaces are defined by use of special covers. In §7 we will characterize those inverse real closed spaces which correspond to weakly semi-algebraic spaces. To be able to do this we must consider spaces having the same kind of special covers. This causes us to make the following definition:

DEFINITION 5.1. Let $X$ be an inverse real closed space, $\mathscr{L}$ a lattice of subsets covering $X$. $\mathscr{L}$ is *locally constructible* if every $C \in \mathscr{L}$ is locally constructible. If $\mathscr{L}$ is locally constructible and a complete Boolean lattice then $\mathscr{L}$ is called *completely locally constructible*. A cover $X = \bigcup_{i \in I} X_i$ is called *completely locally constructible* if the complete Boolean lattice generated by $\{X_i | i \in I\}$ is completely locally constructible.

We saw in §3 that $X$ is quasi-separated if and only if there is a locally constructible lattice of open affine subspaces (Proposition 3.3, Theorem 3.6).

DEFINITION 5.2. An inverse real closed space is *completely quasi-separated* if it has a completely locally constructible open affine cover. A morphism $f$: $X \to Y$ between inverse real closed spaces is *completely quasi-separated* if $f^{-1}(Y')$ is completely quasi-separated for every open affine subspace $Y' \subset Y$.

It is clear from the definition that every completely quasi-separated space is quasi-separated. By the results of §3, the same is true for morphisms.

Before giving examples we prove the following characterization of completely locally constructible covers.

THEOREM 5.3. *Let $X = \cup X_i$ be a cover of the inverse real closed space $X$. The following statements are equivalent*:

(a) *The cover is completely locally constructible.*

(b) *For every open affine subspace $X' \subset X$ the set $\{X_i \cap X' | i \in I\}$ is finite and $X_i \cap X'$ is constructible.*

*Proof.* (b) $\Rightarrow$ (a). Let $B$ belong to the complete Boolean algebra generated by $\{X_i | i \in I\}$. Then $B \cap X'$ belongs to the complete Boolean algebra of subsets of $X'$ generated by $\{X_i \cap X' | i \in I\}$. Since this Boolean algebra is finite and every $X_i \cap X'$ is constructible in $X'$, $B \cap X'$ is a constructible subset of $X'$.

(a) $\Rightarrow$ (b). We may assume that $\{X_i | i \in I\}$ is a complete Boolean algebra. By definition, $X_i \cap X'$ is constructible in $X'$ for every $i \in I$. Since the constructible topology is compact, the Boolean algebra $\{X_i \cap X' | i \in I\}$ has both the ascending and the descending chain conditions. Therefore it contains finitely many atoms and every element is a finite union of these atoms. Therefore $\{X_i \cap X' | i \in I\}$ is finite. □

An immediate consequence is:

COROLLARY 5.4. *Every finite union of completely locally constructible covers is a completely locally constructible cover.*

*Example* 5.5. Every locally constructible partion $X = \cup X_i$ is a completely locally constructible cover.

*Example* 5.6. Every affine space is completely quasi-separated.

*Example* 5.7. Let $X$ be quasi-separated and paracompact [4, p. 18]. Then $X$ is completely quasi-separated. For, every locally finite open affine cover $X = \cup X_i$ is completely locally constructible.

*Example* 5.8. Let $X$ be a completely quasi-separated space, $Y \subset X$ a locally pro-constructible subspace. Then $Y$ is completely quasi-separated.

To emphasize the analogy with weakly semi-algebraic space we note:

PROPOSITION 5.9. *Let $X$ be completely quasi-separated. Then there is an open affine cover $X = \bigcup_{i \in I} X_i$ which is a lattice and has the properties* E1–E6 *of* [10, *p*. 4].

*Proof.* Let $X = \bigcup_{i \in I} X_i$ be a completely locally constructible open affine cover. The lattice generated by this cover is another completely locally constructible open affine cover. Therefore we assume that $\{X_i | i \in I\}$ is a lattice. If we set $i \leq j$ if $X_i \subseteq X_j$, then $I$ is a lattice.

(E1) $X$ is covered by $(X_i)_{i \in I}$.

(E2) $i \leq j$ implies $X_i \subseteq X_j$ by definition.

(E3) For $i \in I$ the set $\{j \in I | j < i\}$ is finite by Theorem 5.3.

(E4), (E5) For $i, j \in I$ there are $k, l \in I$ with $k \leq i, j \leq l$. Set $k = \inf(i, j)$, $l = \sup(i, j)$.

(E6) $X$ is the direct limit of the subspaces $(X_i)_{i \in I}$. □

Now we need a few basic properties of completely quasi-separated morphisms.

PROPOSITION 5.10. *For a morphism $f\colon X \to Y$ the following properties are equivalent*:

(a) *$f$ is completely quasi-separated.*

(b) *There is an open affine cover $Y = \cup Y_i$ such that each $f^{-1}(Y_i)$ is completely quasi-separated.*

*Proof.* We must only prove (b) $\Rightarrow$ (a). Let $Y = \cup Y_i$ be a cover as in (b), let $Y' \subset Y$ be open affine. We set $Y_i' = Y' \cap Y_i$ and choose an open affine cover

$$Y_i' = \bigcup_{j \in J_i} Y_{ij}'.$$

Then $Y_{ij}' \subset Y_i$ is an open affine subspace, $f^{-1}(Y_{ij}') \subset f^{-1}(Y_i)$ is a locally constructible subspace. Thus $f^{-1}(Y_{ij}')$ is completely quasi-separated (Example 5.8). Since $Y' = \bigcup_{i,j} Y_{ij}'$ is an open affine cover there is a finite subcover

$$Y' = \bigcup_{k=1}^{r} Y_k''.$$

We have

$$f^{-1}(Y') = \bigcup_{k=1}^{r} f^{-1}(Y_k''),$$

and each $f^{-1}(Y_k'')$ has a completely locally constructible open affine cover $(Z_{kl})_{l \in L_k}$. Since $f^{-1}(Y')$ is quasi-separated (Proposition 3.2), every $Z_{kl}$ is locally constructible in $f^{-1}(Y')$ (remark after Proposition 3.5). It remains to show that, for any open affine subspace $Z \subset f^{-1}(Y')$, the set $\{Z \cap Z_{kl} | k, l\}$ is finite (Theorem 5.3): For every $k = 1, \dots, r$, $f^{-1}(Y_k'')$ is open and locally constructible in $f^{-1}(Y')$. Then $Z \cap f^{-1}(Y_k'') \subset Z$ is open and constructible, i.e., $Z \cap f^{-1}(Y_k'')$ is affine. Since $f^{-1}(Y_k'') = \bigcup_l Z_{kl}$ is a completely locally constructible open affine cover, the set

$$\{Z \cap Z_{kl} | l \in L_k\} = \{Z \cap f^{-1}(Y_k'') \cap Z_{kl} | l \in L_k\}$$

is finite. This proves the claim. □

PROPOSITION 5.11. (a) *Every affine morphism is completely quasi-separated. In particular, if $X \subset Y$ is a locally pro-constructible subspace then the inclusion is completely quasi-separated. If $f\colon X \to Y$ is a quasi-separated morphism then $\Delta_f$ is completely quasi-separated.*

(b) *Completely quasi-separated morphisms have properties* (A), (B), (C) (*see* §3).

(c) *If* $f\colon X \to Y$, $g\colon Y \to Z$ *are morphisms*, $gf$ *is completely quasi-separated and* $g$ *is quasi-separated, then* $f$ *is completely quasi-separated.*

For the proof we need the following extension property for completely locally constructible open affine covers:

LEMMA 5.12. *Suppose that* $X$ *is a completely quasi-separated space,* $X' \subset X$ *is an open locally constructible (hence completely quasi-separated—see Example 5.8) subspace. Let* $X' = \bigcup_{i \in I} X'_i$ *be a completely locally constructible open affine cover,* $\{X'_i | i \in I\}$ *a lattice. Then there is a completely locally constructible open affine cover* $X = \bigcup_{j \in J} X_j$, $\{X_j | j \in J\}$ *a lattice with the following properties*:

$$\{X'_i | i \in I\} \subset \{X_j | j \in J\} \quad \text{for every } j \in J,\ X_j \cap X' \in \{X'_i | i \in I\}.$$

*Proof.* Let $X = \bigcup_{k \in K} Y_k$ be a completely locally constructible open affine cover by a lattice. For every $k \in K$, $Y_k \cap X'$ is open and affine. So there is a finite subset $I(k) \subset I$ with

$$Y_k \cap X' \subset \bigcup_{i \in I(k)} X'_i.$$

If we set $Z_k = Y_k \cup \bigcup_{i \in I(k)} X'_i$ then $X = \bigcup_{k \in K} Z_k$ is a completely locally constructible open affine cover (Theorem 5.3). If $\mathscr{L}$ is the lattice generated by

$$\{X'_i | i \in I\} \cup \{Z_k | k \in K\}$$

then Theorem 5.3 implies that $\mathscr{L}$ is a completely locally constructible open affine cover. Clearly, $\{X'_i | i \in I\} \subset \mathscr{L}$ and $\mathscr{L}$ is a lattice. Since $\mathscr{L} \cap X'$ is the lattice generated by

$$\{X'_i | i \in I\} \cup \{Z_k \cap X' | k \in K\} = \{X'_i | i \in I\}$$

and $\{X'_i | i \in I\}$ is a lattice by hypothesis, we see that $\mathscr{L}$ is the desired cover. □

*Proof of Proposition 5.11.* Part (a) is trivial. To prove (b) (A) let $f\colon X \to Y$, $g\colon Y \to Z$ be completely quasi-separated. If $Z' \subset Z$ is open affine it must be shown that $(gf)^{-1}(Z')$ is completely quasi-separated. Since $g$ is completely quasi-separated, $g^{-1}(Z')$ is completely quasi-separated. We must prove: If $f$ is completely quasi-separated and $Y$ is completely quasi-separated then so is $X$. Let $Y = \bigcup_{i\in I} Y_i$ be a completely quasi-separated open affine cover and a lattice. For every $i \in I$ we define

$$n(i) = \left|\left\{Y_j | Y_j \neq \varnothing, \quad j \leq i\right\}\right|.$$

Proposition 5.9 shows that this is a nonnegative integer. For every $i \in I$ we set $X_i = f^{-1}(Y_i)$, a completely quasi-separated space. By induction we will define on every $X_i$ a completely locally constructible open affine cover

$$X_i = \bigcup_{j\in J_i} X_{ij}$$

which is a lattice and such that the following holds:

If $X_k \subset X_i$ then $\{X_{kj} | j \in J_k\} \subset \{X_{ij} | j \in J_i\}$ and for every $j \in J_i$ there is some $l \in J_k$ with $X_k \cap X_{ij} = X_{kl}$.

If $n(i) = 1$ then we choose an arbitrary completely locally constructible open affine lattice cover. Now suppose the construction has been done for all $i$ with $n(i) \leq n$. Now pick $i \in I$ with $n(i) = n + 1$. We must distinguish two cases:

*Case* 1. If $Y_i = \cup\{Y_k | Y_k \subsetneqq Y_i\}$ and if $X_k = \bigcup_{j\in J_k} X_{kj}$ are the covers already defined then the lattice $\mathscr{L}$ generated by $\{X_{kj} | k, j\}$ is a completely locally constructible open affine lattice cover of $X_i$. If $X_m \subsetneqq X_i$ then $\{X_{mj} | j \in J_m\} \subset \mathscr{L}$ holds by definition. Moreover, $X_m \cap \mathscr{L}$ is the lattice generated by $\{X_m \cap X_{kj} | Y_k \subsetneqq Y_i\}$. By definition we know that every $X_m \cap X_{kj}$ belongs to $\{X_{ml} | l \in J_m\}$. Since this is a lattice we are done.

*Case* 2. If $Y_i \supsetneqq U\{Y_k | Y_k \subsetneqq Y_i\}$ we set $Y_l = \cup\{Y_k \subsetneqq Y_i\}$. Then $n(l) = n$ and there is a cover $X_l = \bigcup_{m\in J_l} X_{lm}$. Applying Lemma 5.12 one obtains the desired cover of $X_i$.

This finishes the induction. Now we consider the open affine cover

$$X = \bigcup_{i\in I} \bigcup_{j\in J_i} X_{ij}.$$

Since every $X_{ij}$ is clearly locally constructible it remains to show that $\{Z \cap X_{ij} | i, j\}$ is finite for every open affine subspace $Z \subset X$ (cf. Theorem 5.3). First of all, quasi-compactness of $Z$ implies that there is some $i \in I$ with $Z \subset X_i$. Then $\{Z \cap X_{ij} | j \in J_i\}$ is finite. Now let $k \in I$, $l \in J_k$ be arbitrary and set $m = \inf\{i, k\}$. Then

$$Z \cap X_{kl} = Z \cap X_i \cap X_{kl}$$
$$= Z \cap X_m \cap X_{kl} \in \{Z \cap X_{mn} | n \in J_m\} \subset \{Z \cap X_{ij} | j \in J_i\}.$$

This proves property (A).

To prove (B) we let $f$: $X \to Y$ be completely quasi-separated and $g$: $Z \to Y$ arbitrary, $f'$: $X \times_Y Z \to Z$ the projection. To prove that $f'$ is completely quasi-separated we may assume that $Y$ and $Z$ are both affine (Proposition 5.10). If $X = \bigcup_{i \in I} X_i$ is a completely locally constructible open affine cover, then so is $X \times_Y Z = \bigcup_{i \in I} X_i \times_Y Z$. To finish the proof of (b) we note that property (C) follows from (A) and (B) [7, 0 1.3.9]. Finally, (c) follows from [7, I 5.17]. □

Obviously a space $X$ is completely quasi-separated if and only if the morphism $X \to \mathrm{Sper}^*(R_0)$ is completely quasi-separated. By Proposition 5.11 every morphism of completely quasi-separated spaces is completely quasi-separated. So the completely quasi-separated spaces with the completely quasi-separated morphisms are a full subcategory of the category of inverse real closed spaces.

In the homotopy theory of weakly semi-algebraic spaces [10, Chapter V] patch decompositions play an important role as substitutes for triangulations. We conclude this section by showing that the completely quasi-separated spaces are exactly those inverse real closed spaces having patch decompositions.

Definition 5.13 (cf. [10, Chapter V, §1, Definition 1]). A *constructible decomposition* of the inverse real closed space $S$ is a partition $X = \bigcup_{i \in I} X_i$ with every $X_i$ constructible.

By Example 5.5 every constructible decomposition is a completely locally constructible cover.

If $X = \cup X_i$ is a constructible decomposition we call $X_j$ a *face* of $X_i$ if $X_j \cap \overline{X_i} \neq \emptyset$. We write $X_j < X_i$ in this case. We define the *depth* of $X_i$ to be the number

$$d(X_i) = \sup\{n \in \mathbf{N}_0 | \exists j_0, \ldots, j_n = i : X_{j_0} \gneqq X_{j_1} \gneqq \cdots \gneqq X_{j_n} = X_i\}$$
$$\in \mathbf{N}_0 \cup \{\infty\}$$

(cf. [10, Chapter V, §1]).

Definition 5.14 (cf. loc. cit.). A constructible decomposition $X = \bigcup_{i \in I} X_i$ is called a *patch decomposition* if $d(X_i) \in \mathbf{N}_0$ for every $i \in I$.

THEOREM 5.15. *Let $X$ be a quasi-separated inverse real closed space. The following statements are equivalent*:

(a) *$X$ is completely quasi-separated.*

(b) *$X$ has a patch decomposition.*

*Proof.* (a) $\Rightarrow$ (b). If $X = \bigcup_{i \in I} X_i$ is a completely locally constructible open affine lattice cover then we set

$$C_i = X_i \setminus \bigcup \{ X_j | X_j \subsetneqq X_i \}$$

for every $i \in I$. Obviously, $X = \bigcup_{i \in I} C_i$ is a constructible decomposition. We must show that every $d(C_i)$ is finite. But we clearly have

$$d(C_i) \leq \left| \{ X_j | X_j \subseteqq X_i \} \right|,$$

which is finite.

(b) $\Rightarrow$ (a). Let $X = \bigcup_{i \in I} C_i$ be a patch decomposition. Fix some $C_j$. Then $C_j$ is contained in some open affine subspace $U_j \subset X$. On the other hand, by compactness of the constructible topology, there is a finite subset $J \subset I$ with $U_j \subset \bigcup_{i \in J} C_i$. If $k \in I$ is such that $C_j \lneqq C_k$ then $k$ must belong to $J$ and $d(C_k) < d(C_j)$. By induction we see that

$$K(j) = \{ i \in I | \exists i = j_0, \ldots, j_s = j : C_i = C_{j_0} \gneqq \cdots \gneqq C_{j_s} = C_j \}$$

is finite. So, $\mathrm{st}(C_j) = \bigcup_{i \in K(j)} C_i$, the star of $C_j$ is constructible. To prove that $\mathrm{st}(C_j)$ is also open we must show that it is closed under generalization. So pick $x \in \mathrm{st}(C_j)$, $y \in X$ with $x \in \overline{\{y\}}$. Then there are $k, l \in I$ with $x \in C_k$, $y \in C_l$. By definition $k \in K(j)$. Since $x \in \overline{C}_l$ we see that $C_k = C_l$ or $C_k < C_l$. In either case $l \in K(j)$ and $C_l \subset \mathrm{st}(C_j)$. This shows that $\mathrm{st}(C_j)$ is an open affine subspace of $X$. Of course, $X = \bigcup_{j \in J} \mathrm{st}(C_j)$. This cover is completely locally constructible since it belongs to the lattice generated by the completely locally constructible cover $X = \bigcup C_i$. $\square$

## 6. Finiteness conditions

As in [15, Definition V 6.1], [16, Definition II 7.1] the following notions of finiteness are introduced:

DEFINITION 6.1. A morphism $f: X \to Y$ of inverse real closed spaces is of *finite type at* $x \in X$ if there are open affine neighborhoods $x \in U \subset X$,

$f(x) \in V \subset Y$ and a commutative diagram

$$(*) \qquad \begin{array}{ccc} X & \longrightarrow & Y \\ \cup & & \cup \\ U & \longrightarrow & V \\ {\scriptstyle g}\downarrow\simeq & & \uparrow{\scriptstyle \mathrm{pr}} \\ Z & \longrightarrow & V \times_{R_0} \tilde{R}_0^{n*} \end{array}$$

where $Z$ is a pro-constructible subspace of $V \times_{R_0} \tilde{R}_0^{n*}$. $f$ is *locally of finite type* if it is of finite type at every point of $x$. $f$ is of *finite type* if it is locally of finite type and quasi-compact. $f$ is *finitely presented at* $x \in X$ if it is of finite type at $x$ and the subspace $Z$ in $(*)$ is constructible. $f$ is *locally finitely presented* if it is finitely presented at ever $x \in X$. $f$ is *finitely presented* if it is locally presented, quasi-compact and quasi-separated.

Note that finitely presented morphisms are affine (Corollary 3.9).

*Example* 6.2. For every morphism $f$: $X \to Y$, $\Delta$: $X \to X \times_Y X$ is locally of finite type. To see this let $x \in X$, $z = \Delta(x)$, $y = f(x)$. Let $x \in U \subset X$ and $y \in V \subset Y$ be open affine neighborhoods with $f(U) \subset V$. Then $U \times_V U \subset X \times_Y X$ is an open affine neighborhood of $z$. Since $\Delta$: $U \to U \times_V U$ is a morphism of affine spaces, it has a pro-constructible image. By Example 2.23, $\Delta$ is an isomorphism onto $\Delta(U)$. So, we have the commutative diagram

$$\begin{array}{ccc} X & \stackrel{\Delta}{\longrightarrow} & X \times_Y X \\ \cup & & \cup \\ U & \longrightarrow & U \times_V U \\ \downarrow\simeq & & \uparrow = \\ \Delta(U) & \longrightarrow & U \times_V U. \end{array}$$

We collect a few basic properties of morphisms locally of finite type. The proof is omitted since no special properties of inverse real closed spaces are needed.

PROPOSITION 6.3. (a) *If $X \subset Y$ is a locally pro-constructible subspace then the inclusion is locally of finite type.*

(b) *Morphisms locally of finite type have properties* (A), (B), (C) (*see* §3).

(c) *Let $f$: $X \to Y$, $g$: $Y \to Z$ be morphisms such that $gf$ is locally of finite type. Then so is $f$.*

Now we turn to locally finitely presented morphisms.

*Example* 6.4. If $f\colon X \to Y$ is locally of finite type then $\Delta$ is locally finitely presented.

PROPOSITION 6.5. (a) *If $X \subset Y$ is locally constructible subspace then the inclusion is locally finitely presented.*

(b) *Locally finitely presented morphisms have properties* (A), (B), (C).

(c) *If $f\colon X \to Y$, $g\colon Y \to Z$ are morphisms, $g$ is locally of finite type and $gf$ is locally finitely presented, then $f$ is locally finitely presented.*

Combining these properties with the corresponding properties for quasi-compact and quasi-separated morphisms we get similar lists for morphisms of finite type and for finitely presented morphisms.

The same notions of finiteness were defined for real closed spaces in [15, Definition VI 6.1] and [16, Definition II 7.1]. In view of the connections between real closed spaces and inverse real closed spaces established in §1 and §4 we claim:

PROPOSITION 6.6. *Let $f\colon X \to Y$ be a morphism of affine real closed spaces. $f$ is finitely presented if and only if $F(f)\colon F(X) \to F(Y)$ is finitely presented.*

*Proof.* We choose a finite presentation

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \cup & & \cup \\ U & \longrightarrow & V \\ {\scriptstyle g}\Big\downarrow{\scriptstyle \simeq} & & \Big\uparrow{\scriptstyle \mathrm{pr}} \\ C & \longrightarrow & V \times_R \tilde{R}_0. \end{array}$$

of $f$ in the neighborhood of $x \in X$. Since $Y$ is affine we may assume that $V = Y$. If $y \in \overline{\{x\}}$ is the closed point then we may choose $U$ such that $y \in U$. By an easy compactness argument there is a closed constructible set $K$ with $x \in K \subset U$. Now we apply $F$ to the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \cup & & \Big\uparrow{\scriptstyle =} \\ K & \longrightarrow & Y \\ {\scriptstyle g}\Big\downarrow{\scriptstyle \simeq} & & \Big\uparrow{\scriptstyle \mathrm{pr}} \\ g(K) \subset C & \overset{\subseteq}{\longrightarrow} & Y \times_{R_0} \tilde{R}_0^n. \end{array}$$

to obtain a finite presentation of $F(f)$ in the neighborhood $F(K) \subset F(X)$ of $x$. The other implication of the equivalence is proved similarly. □

As a consequence of Proposition 6.6 we note that a finitely presented morphism of affine inverse real closed spaces has a global finite presentation [15, Proposition V 6.6], [16, Proposition II 7.6].

## 7. Weakly semi-algebraic spaces

We fix a real closed field $R$ and consider a weakly semi-algebraic space $M$ with a lattice exhaustion $M = \bigcup_{i \in I} M_i$ [10, Chapter IV, §1]. For every $i \in I$, there is an affine real closed space $\tilde{M}_i$ over Sper($R$) corresponding to $M_i$ [15, Chapter IV], [16, Chapter III, §1]. Applying the functor $F$ of Section 1, we obtain an affine inverse real closed space $F(\tilde{M}_i)$ over Sper*($R$). Under this functor the closed constructible subsets of $\tilde{M}_i$ correspond to the open constructible subsets of $F(\tilde{M}_i)$. Thus, if $M_j \subset M_i$ then the corresponding morphism $F(\tilde{M}_j) \to F(\tilde{M}_i)$ of inverse real closed spaces is an isomorphism onto an open affine subspace. This implies that the $F(\tilde{M}_i)$, $i \in I$ can be glued together to give an inverse real closed space $F(\tilde{M})$. The construction shows that $F(\tilde{M}) = \bigcup_{i \in I} F(\tilde{M}_i)$ is a completely locally constructible open affine cover, and $F(\tilde{M})$ is completely quasi-separated. It is clear that the structural morphisms $F(\tilde{M}_i) \to$ Sper*($R$) glue together to a morphism $F(\tilde{M}) \to$ Sper*($R$). Since $\tilde{M}_i \to$ Sper($R$) is finitely presented for every $i \in I$ [15, Theorem VI 1.1; Theorem III 1.2], Proposition 6.6 implies that $F(\tilde{M}_i) \to$ Sper*($R$) is finitely presented for every $i$. So $F(\tilde{M})$ is locally finitely presented over Sper*($R$).

To make a functor out of $F(\tilde{\ })$ we must define it on the morphisms: If $M$, $N$ are weakly semi-algebraic spaces with lattice exhaustion $M = \bigcup_{i \in I} M_i$, $N = \bigcup_{j \in J} N_j$ and $f\colon M \to N$ is a morphism, then for every $i \in I$ there is some $j \in J$ with $f(M_i) \subset N_j$, and the restriction $f_{ji}\colon M_i \to N_j$ of $f$ is semi-algebraic [10, Theorem IV 2.3]. So, from [15, Chapter VI §1] or [16, Chapter III, §1] and the functor $F$ in §1 we obtain morphisms

$$F(\tilde{f}_i)\colon F(\tilde{M}_i) \xrightarrow{F(\tilde{f}_{ij})} F(\tilde{N}_j) \subset F(\tilde{N}).$$

If $M_k \subset M_i$, then $f_{ji}$ extends $f_{jk}$ and therefore $F(\tilde{f}_i)$ extends $F(\tilde{f}_k)$. This shows that the $F(\tilde{f}_i)$ can be glued together to give a morphism $F(\tilde{f})$: $F(\tilde{M}) \to F(\tilde{N})$. From the construction it is clear now that $F(\tilde{\ })$ is a functor.

Conversely, let $X$ be a completely quasi-separated inverse real closed space over Sper*($R$) which is locally finitely presented over Sper*($R$). There is a completely locally constructible open affine cover $X = \bigcup_{i \in I} X_i$ which is a lattice. Then every $X_i$ is finitely presented over Sper*($R$), and hence $G(X_i)$ is finitely presented over Sper($R$) (Proposition 6.6). By [15, Theorem VI 1.1] or [16, Theorem III 1.2], the space $G(X_i)(R)$ of $R$-rational points of $G(X_i)$ is an affine semi-algebraic space over $R$. We want to glue the $G(X_i)(R)$ together to obtain a weakly semi-algebraic space. To do so we let $X(R)$ be the set of $R$-rational points of $X$. The sets $X_i(R) = X(R) \cap X_i$ form a cover of $X(R)$. Identifying the sets $X_i(R)$ and $G(X_i)(R)$ we have the structure of an affine semi-algebraic space on $X_i(R)$. If $X_j \subset X_i$ then the results of §1 show that $G(X_j)(R)$ may be identified with a closed semi-algebraic subspace of $G(X_i)(R)$. We see that $X(R) = \bigcup_{i \in I} G(X_i)(R)$ is a cover fulfilling the hypotheses of [10, Theorem IV 1.6]. So $X(R)$ has the structure of a weakly semi-algebraic space. We denote this space by $G(X)(R)$.

Now let $f$: $X \to Y$ be a morphism between completely quasi-separated spaces over Sper*$(R)$ which are locally finitely presented. We choose completely locally constructible open affine lattice covers $X = \bigcup_{i \in I} X_i$, $Y = \bigcup_{j \in J} Y_j$. For every $i \in I$ there is some $j \in J$ with $f(X_i) \subset Y_j$. Let $f_{ji}$: $X_i \to Y_j$ be the restriction of $f$. Then we have morphism

$$G(f_i)(R): G(X_i)(R) \xrightarrow{G(f_{ji})(R)} G(Y_j)(R) \subset G(Y)(R).$$

These morphisms can be glued together to give the morphism

$$G(f)(R): G(X)(R) \to G(Y)(R).$$

Clearly, $G(\ )(R)$ is a functor from the category of those completely quasi-separated inverse real closed spaces over Sper*$(R)$ which are locally finitely presented over Sper*$(R)$ to the category of weakly semi-algebraic spaces over $R$. From §1 and the results of [15, Chapter III], [16, Chapter III, §1] it is clear that the functors $F(\tilde{\ })$ and $G(\ )(R)$ are quasi-inverse to each other. This proves:

THEOREM 7.1. *The category of weakly semi-algebraic spaces over R is equivalent to the category of those completely quasi-separated inverse real closed spaces over* Sper*$(R)$ *which are locally finitely presented over* Sper*$(R)$.

Because of this equivalence of categories we can consider the category of weakly semi-algebraic spaces over $R$ as a full subcategory of the category of inverse real closed spaces over Sper*$(R)$. This is the analogue of the connections between locally semi-algebraic over $R$ and real closed spaces over Sper$(R)$ exhibited in [15, Chapter VI, §1] and [16, Chapter III, §1].

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