ILLINOIS JOURNAL OF MATHEMATICS
Volume 36, Number 3, Fall 1992

# BETTI NUMBERS, CHARACTERISTIC CLASSES AND SUB-RIEMANNIAN GEOMETRY

BY

ZHONG GE

## Introduction

In this paper we will develop generalized characteristic classes and (a part of) the Hodge theory in the context of degenerate metrics (called sub-Riemannian metrics). As an application, we study topological obstructions to putting a connection on a fiber bundle over a Riemmanian manifold with prescribed curvature. The novelty in the application is that we make no assumption on the geometry of the fiber.

Roughly speaking, a sub-Riemannian metric on a manifold $M$ is a fiber-wise metric on a subbundle $H \subset TM$ satisfying Hörmander's condition. Associated with this metric is the distance between any two points, called Carnot-Carathéodory distance, defined to be the minimum of the length functional over the space of absolutely continuous curves tangent almost everywhere to $H$ and connecting the two points. This metric and the corresponding distance have appeared in a number of different contexts (cf. [2], [3], [7], [8], [9], [11], [13], [18], [20], [21], [22], [25], [27], [29]).

In §1 we first study the geometry of sub-Riemannian metrics. In particular, we generalize the Gauss-Bonnet-Chern type formulas to sub-Riemannian metrics, showing that certain global invariants of the underlying distribution (certain "horizontal cohomology classes") can be given by the data of the sub-Riemannian metrics, in a slightly less canonical way in general. This construction is canonical if $H$ is contact.

One of the difficulties in the study of sub-Riemannian geometry is that so far no intrinsic connection has been defined (cf. [27]) in general. However, if we choose a complementary subbundle to $H$, we can develop an analogue of the Levi-Civita connection, which enables us to parallel translate horizontal tangent vectors along horizontal paths. This connection was encountered in the study of collapsing of Riemannian metrics to sub-Riemannian metrics [9]. Similar connections in the context of principal bundles have been introduced by Kamber and Tondeur (cf. [15], p. 14). However, unlike in the Riemannian case, the curvature is not a tensor in the ordinary sense. In this paper we

Received September 18, 1990.
1991 Mathematics Subject Classification. Primary 53C05; Secondary 58G05.

show that the curvature, modulo a differential ideal, is a tensor, and gives rise, via the Chern-Weil homomorphism, to global characteristic classes which are horizontal cohomology classes.

The global invariants of $H$ here will be cohomology classes of a differential complex associated with $H$. This differential complex is constructed as follows. If $H \subset TM$, say, is locally defined by $k$ 1-forms $\omega_1 = \cdots = \omega_k = 0$, then the differential ideal $\bigwedge_N(M) \subset \bigwedge(M)$ is locally generated by $k$ 1-forms $\omega_1 = \cdots = \omega_k = 0$, then the differential ideal $\bigwedge_N(M) \subset \bigwedge(M)$ is locally generated by $\omega_1, \dots, \omega_k, d\omega_1, \dots, d\omega_k$. Then the complex is the quotient $\bigwedge_H(M) = \bigwedge(M)/\bigwedge_N(M)$, with the induced exterior differentiation $d_H$, and the cohomology groups (to be called horizontal cohomology) is that of the differential complex $\bigwedge_H(M)$. Though this cohomology group is easy to define, until recently it has not been used much in geometry (see Rumin [25]). Recently Ginzburg observed that if $H$ is a contact distribution, then the lower dimensional cohomology groups of $\bigwedge_H(M)$ are isomorphic to the de Rham cohomology groups (interestingly enough, a similar result on the homology level was in Thom [28]). In §1.2. we generalize his result to certain 2-step generating distributions (i.e., $H + [H, H] = TM$).

Having developed the geometry of sub-Riemannian metrics, in §2 we will develop a part of the Hodge theorem for sub-Riemannian metrics. To do this, we assume that a volume form $dv$ on $M$ is given, in addition to the sub-Riemannian metric. If $H$ is contact, we can choose a canonical volume form. Our main result in §2 is the proof of the hypoellipticity of $\Delta^1_H = d_H\delta_H + \delta_H d_H$ acting on $\bigwedge^1_H(M)$ under certain explicit condition on the tangent cone. Here some identities obtained in §1 play a fundamental role. Our results are inspired by a result of Rumin [25] for the case where $M$ is pseudo-hermitian. Also recall that if $H$ is integrable, then there is a harmonic integration theory due to Kamber-Tondeur [16], [17], Reinhart [23], and Kacimi-Hector [14]. So the results in this paper can be considered as generalizations of a part of their results.

The generalization of the Hodge theorem to degenerate metrics seems particularly suitable for the study of the problem of putting a connection on a fiber bundle $M$ over an Riemannian manifold with a prescribed curvature, since the sub-Laplacian $\Delta^1_H$ has a relatively simple form in this case. As an application of Theorem 2.1, in §3 we study the case where $M$ is the total space of a fiber bundle over a Lie group

$$W \to M \to G$$

with a given connection which has an "almost left invariant" curvature, showing that if the curvature satisfies certain complicated but explicit inequalities, then the first Betti number of $M$ must be zero (cf. Theorem 3.2 and the remarks following).

It is our pleasure to thank Prof. Philippe Tondeur for suggesting improvements in this paper.

## 1. Geometry of sub-Riemannian metrics and generalized characteristics

### 1.1. Geometry of sub-Riemannian metrics

In this subsection we will recall some basic properties of sub-Riremannian metrics and introduce a local invariant of the underlying distribution.

Let $M$ be a connected, compact manifold, and $H \subset TM$ a smooth subbundle of $TM$. A *sub-Riemannian metric* on $M$ is a symmetric positive bilinear form $(\cdot, \cdot)$ on $H, (\cdot, \cdot)$: $H \times H \to R$. If $H$ satisfies Hörmander's condition, there is a Carnot-Carathéodory distance between $x, y \in M$, defined to be

$$d(x, y) = \min_{\gamma \in \Omega_H M(x, y)} \left( \int (\dot{\gamma}(t), \dot{\gamma}(t)) \, dt \right)^{1/2}.$$

Here $\Omega_H(x, y)$ is the space of horizontal paths connecting $x, y$.

An important class of sub-Riemannian metrics are constructed as follows: suppose that $M$ is the total space of a fiber bundle $W \to M \to B$ over a Riemannian manifold, and $H$ comes from a given connection, i.e., $TM = H \oplus K$ where $K$ is tangent to the fibers. Then define a sub-Riemannian metric on $M$ by horizontally lifting the Riemannian metric on $B$ to $H$.

Now we introduce a local invariant of $H$ which will play a most important role in later developments. We will use a construction which is very similar to the construction of a tangent cone (cf. [8], [9], [19], [24]). Let $H_1 = H + [H, H]$ be the subbundle of $TM$ consisting of such elements $c$ which locally can be written as $c = b_0 + [b_1, b_2], b_0, b_1, b_2 \in C^\infty(H)$. Then there is an antisymmetric bilinear map $\mu(\cdot, \cdot)_x$: $H \times H \to H_1/H$ defined by

$$\mu(a, b)_x = [a, b] \mod(H). \tag{1.1}$$

It is easy to verify that (1.1) is well defined.

Note that if $M$ is the total space of a principle fiber bundle and $H$ comes from a connection, then $\mu$ is just the curvature of the connection.

Suppose that the vector bundle $H_1/H$ is of rank $k_1$, then $\mu(,)_x$ is a $R^{k_1}$-valued 2-form on $H_x$, thus determines $k_1$ elements of $\wedge^2(H_x)$, which we will denote by $\theta^1, \dots, \theta^{k_1}$. Thus we can write $\mu_x = (\theta^1, \dots, \theta^{k_1})$ in a non-canonical way. Let $I_x(\theta^1, \dots, \theta^{k_1})$ be the exterior algebraic ideal in $\wedge(H_x)$ generated by $\theta^1, \dots, \theta^{k_1}$. Sometimes we will write $I_x(\theta^1, \dots, \theta^{k_1})$ simply as $I_x$.

We say that $H$ is of non-degeneracy $r$ if $\tau$ is the biggest number such that for $(r-1)$-forms $a_1, \dots, a_{k_1}$ on $H_x$,

$$a_1 \wedge \theta^1 + \cdots + a_{k_1} \wedge \theta^{k_1} \neq 0$$

unless $a_1 = \cdots = a_{k_1} = 0$. Note that if $H$ has non-degeneracy $r > 0$, then the distribution $H$ must be two-step bracket generating, i.e., $H_1 = H + [H, H] = TM$.

We will prove that $H$ has non-degeneracy $r > 0$ if $H$ is strongly bracket generating (cf. [27]), i.e. for any $v_1 \in H_x$, $v_1 \neq 0$, the induced map $H_x \to TM_x/H_x$, $v_2 \to \mu(v_1, v_2)$ is a submersion. If $M$ is the total space of a fiber bundle and $H$ comes from a connection, then $H$ is strongly bracket generating iff $M$ is a fat bundle (Weinstein [30]).

LEMMA 1.1. *If $M$ is strongly bracket generating and $(M, H)$ is not a 3-dimensional contact manifold, then $H$ has non-degeneracy $r > 0$.*

*Proof.* Assume otherwise, i.e., there are 1-forms $a_1, \dots, a_{k_1}$, which are not all zero, such that

$$a_1 \wedge \theta^1 + \cdots + a_{k_1} \wedge \theta^{k_1} = 0. \tag{1.2}$$

Without loss of generality we assume that $a_1, \dots, a_k$ are linearly independent at $x \in M$. Choose a coordinate system $\{x_i\}$ such that $a_1 = dx_1, \dots, a_{k_1} = dx_{k_1}$ at $x$. Write

$$\theta^i = \sum_{lk} \theta^i_{lk}\, dx_l \wedge dx_k,$$

then from (1.2) at $x$ we have

$$\sum_{l \geq k_1+1,\, k \geq k_1+a} \theta^i_{lk}\, dx_l \wedge dx_k = 0,$$

which is in contradiction with the fact that $H$ is strongly bracket generating.

*Remark.* There are subbundles $H$ which have non-degeneracy $> 0$ and yet are not strongly bracket generating. For example, take $(M, H)$ where $M = R^{2n+2}$, $H$ is defined by two 1-forms

$$dz_1 - x_1\, dy_1 - \cdots - x_{n_1}\, dy_{n1} = dz_2 - x_{n_1+1}\, dy_{n_1+1} - \cdots - x_n\, dy_n = 0.$$

Here $(x_1, y_1, \dots, x_n, y_n, z_1, z_2)$ is a coordinate system on $R^{2n+2}$, $2 \leq n_1 \leq n-2$. Then it is easy to see that $H$ is not strongly bracket generating but yet has non-degeneracy $> 0$.

We recall the definition of partial connections, which is a generalization of the Levi Civita connection to sub-Riemannian metrics (cf. [8], [9]). To define such a partial connection, we need to choose a subbundle $K$ in $TM$ complementary to $H$, $TM = H \oplus K$, and denote $\pi\colon TM \to H$ the projection. Then a bilinear map

$$(a, b) \in H_x \times C^\infty(H) \to D_a b \in H_x,$$

depending smoothly on $x$, is a partial connection if

$$(1) \quad D_a(fb) = < df, a > b + fD_a b, \quad a, b \in C^\infty(H), f \in C^\infty(M)$$

where $\langle \ , \ \rangle$ is the dual bracket between $T^*M$ and $TM$.

$$(2) \qquad D_a b - D_b a = \pi[a, b], \quad a, b \in C^\infty(H),$$

$$(3) \qquad a(b, c) = (D_a b, c) + (b, D_a c).$$

As an example, suppose that $M$ is the total space of a fiber bundle $W \to M \to B$ over a Riemannian manifold and $H$ comes from a connection on the fiber bundle, then horizontally lifting the Levi-Civita connection on $B$ to $H$, we obtain a partial connection.

In [9] it is proved that for given $H$, $K$, and $(\cdot, \cdot)$ on $H$, there exists a unique partial connection.

An orthonormal frame $e_i$ for $H$ is *normal at a given point* $x_0 \in M$ if $D_{e_j} e_i(x_0) = 0$. In [9] it is proved that such a normal frame always exists. Note that if $e_i$ is normal at $x_0$, $\pi[e_i, e_j](x_0) = 0$.

The partial curvature of the partial connection is a trilinear map

$$R\colon C^\infty(H) \times C^\infty(H) \times C^\infty(H) \to C^\infty(H)$$

defined by

$$R(a, b)c = D_a D_b c - D_b D_a c - D_{\pi[a, b]} c.$$

As the following result shows, unlike the curvature of the Levi-Civita connection, $R(a, b)$ is not a tensor in the "usual" sense.

LEMMA 1.2. *Let $a, b, c$ be smooth horizontal vector fields on $M$ and $f$ a smooth function. Then*

$$R(fa, b)c = fR(a, b)c, \quad R(a, b)fc = (\mu(a, b)f)c + fR(a, b)c.$$

For a proof see [8].

In general there is *no* partial connection and volume form canonically associated with the sub-Riemannian metric. However, if $H$ is a contact distribution, then there is a natural volume form $dv$ and a complementary bundle $K$ to $H$ defined as follows: let $\alpha$ be the 1-form such that $\alpha = 0$ defines $H$ and

$$(x, y) = d\alpha(x, Jy), x, y \in H, \tag{1.3}$$

where $J$ is an endmorphism of $H$ such that $\det J = 1$. It is easy to see that such a 1-form exists uniquely. Having determined $\alpha$, then we define

$$K = \{x, d\alpha(x, \cdot) = 0\} \tag{1.4}$$

and $dv = \alpha \wedge (d\alpha)^n$. In this case the induced partial connection $D$ will be called the canonical partial connection of the sub-Riemannian metric.

### 1.2. Horizontal cohomology

In this subsection we will define global invariants of $H$, the cohomology groups of $H$ (also called horizontal cohomology groups), and study their properties.

Let $\wedge(M) = \oplus \wedge^k(M)$ be the sheaf of smooth differential forms on $M$, and $\wedge_N(M)$ be the subsheaf consisting of $\omega$ such that if $H$ is locally defined by $k$ 1-forms $\omega_1 = \cdots = \omega_k = 0$, then

$$\omega = \sum (f_i \wedge \omega_i + g_i \wedge d\omega_i),$$

where $f_i$, $g_i$ are smooth differential forms.

There is a natural filtration $\wedge_N(M) = \oplus \wedge_N^k(M)$, and $d(\wedge_N^k(M)) \subset \wedge_N^{k+1}(M)$. $\wedge_N(M)$ is both an algebraic and a differential ideal of $\wedge(M)$. The $k$-th *vertical cohomology* is defined by

$$H_N^k(M) = \frac{\ker d_N^k}{\operatorname{Im} d_N^{k-1}}$$

where $d_N^k$: $\wedge_N^k(M) \to \wedge_N^{k+1}(M)$ is the restriction of the exterior differentiation.

Let $\wedge_H(M)$ be the quotient sheaf $\wedge(M)/\wedge_N(M)$, defined by the exact sequence

$$0 \to \wedge_N(M) \to \wedge(M) \to \wedge_H(M) \to 0. \tag{1.5}$$

$\wedge_H(M)$ has a natural filtration $\wedge_H(M) = \oplus \wedge_H^k(M)$, and a natural operator

$$d_H = d_H^k\colon \wedge_H^k(M) \to \wedge_H^{k+1}(M)$$

defined in the following way: if $p_H\colon \wedge(M) \to \wedge_H(M)$ is the projection,

$$d_H p_H(\omega) = p_H(d\omega).$$

DEFINITION 1.1. *The k-th cohomology of H is*

$$H^k(H) = \frac{\ker d_H^k}{\operatorname{Im} d_H^{k-1}}.$$

Later on we will need the following technical condition: we say that $\wedge_H^k(M)$ satisfies condition (L) if $\omega \in \wedge_H^k(M)$ satisfies $\omega(x) = 0$ for every $x \in M$ (as a cross-section of $\wedge^k(TM)$) then $\omega = 0$.

LEMMA 1.3. *Suppose that $H$ satisfies the following condition: there are 1-forms $\omega_1, \dots, \omega_k$, such that $H$ is defined by $\omega_1 = \cdots = \omega_k = 0$ locally, and $d\omega_{k_1+1}, \dots, d\omega_k$ can be uniquely written as*

$$d\omega_{k_1+i} = \sum_{j=1}^{k} f_j^i \wedge \omega_j + \sum_{j=1}^{k_1} g_j^i \, d\omega_j, \quad i = 1, \dots, k - k_1,$$

*where $f_j^i$, $g_j^i$ are smooth forms, then $\wedge_H^2(M)$ satisfies condition $(L)$.*

COROLLARY 1.4. *If $H$ is two-step generating, then $\wedge_H^2(M)$ satisfies condition $(L)$.*

Next we will determine the stalk of $\wedge_H^k(M)$ over $x \in M$, $\wedge_H^k T_x M$ explicitly. Obviously if $k = 1$ then $\wedge_H^1 T_x M = H_x$. However, for $k \geq 2$, $\wedge_H^k T_x M$ is not freely generated by $H_x$.

LEMMA 1.5. *Suppose that the vector bundle $H_1/H$ is of rank $k_1$, $\mu_x = (\theta^1, \dots, \theta^{k_1})$. Then the stalk of $\wedge_H^2(M)$ over $x \in M$ is*

$$\wedge_H^2 T_x(M) = \wedge^2(H_x) \big/ \operatorname{span}(\theta^1, \dots, \theta^{k_1}).$$

*Proof.* Select a subbundle $V_1$ in $TM$ which is complementary to $H$. Suppose that $H_x$ is spanned by $e_1, \dots, e_m$, $V_1$ spanned by $b_1, \dots, b_k$, and

$$[e_i, e_j](x) = \sum c_{ij}^l(x) b_l(x) \mod(e_1, \dots, e_m), \quad c_{ij}^l = -c_{ji}^l.$$

Then one can choose a local coordinate neighborhood $\{x_1, \dots, x_m, y_1, \dots, y_k\}$

such that $H$ is defined by $\omega_1 = \cdots = \omega_k = 0$, where

$$\omega_l = \begin{cases} dy_l - \sum c_{ij}^l x_i\, dx_j + O(y^2 + x^2), & 1 = 1, \dots, k_1; \\ O(y^2 + x^2), & 1 = k_1 + 1, \dots, k. \end{cases}$$

Here $O(x^2 + y^2)$ denotes a 1-form $\Sigma f_i\, dx_i + \Sigma g_j\, dy_j$, where $f_i = O(x^2 + y^2)$, $f_j = O(x^2 + y^2)$. So

$$d\omega_l = \begin{cases} -\sum c_{ij}^l\, dx_i \wedge dx_j + O(|y| + |x|), & 1 = 1, \dots, k_1; \\ O(|y| + |x|), & 1 = k_1 + 1, \dots, k \end{cases}$$

then it is easy to see that the lemma follows.

The above result can be easily generalized to $k > 2$,

LEMMA 1.6. *The stalk of* $\wedge_H(M)$ *over* $x \in M$ *is*

$$\wedge_H T_x(M) = \wedge(H_x)/I_x(\theta^1, \dots, \theta^{k_1});$$

*i.e., we have the exact sequence*

$$0 \to I_x \to \wedge(H_x) \to \wedge_H T_x M \to 0.$$

Following an idea of Ginzburg, consider the short exact sequence (1.5), from which follows the long exact sequence

$$0 \to H_N^1(M) \to H^1(M) \to H_H^1(M) \\ \to H_n^2(M) \to H^2(M) \to H_H^2(M) \to \cdots . \quad (1.6)$$

Ginzburg observed in certain important cases that $H_N^i(M) = 0$, e.g., $(M, H)$ is a contact manifold of dimension $2r + 1$; then $H_H^k(M)$ is isomorphic to $H^k(M)$ for $k = 1, \dots, r - 1$ (see also Rumin [25]). We will generalize his result to certain 2-step generating subbundle $H$ (cf. [27]). We first begin with:

LEMMA 1.7. *If every* $x \in M$ *admits a neighborhood* $U$ *such that* $H_N^k(U) = 0$, $i = 0, 1, \dots, r + 1 < n$, *then* $H^k(M)$ *is isomorphic to* $H_H^k(M)$, $i = 1, \dots, r$.7

*Proof.* We have the commutative exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & \wedge(U \cup V) & \longrightarrow & \wedge(U) \oplus \wedge(V) & \longrightarrow & \wedge(U \cap V) & \longrightarrow 0 \\ & \downarrow{\scriptstyle p_H} & & \downarrow{\scriptstyle p_H} & & \downarrow{\scriptstyle p_H} & \\ 0 \longrightarrow & \wedge_H(U \cup V) & \longrightarrow & \wedge_H(U) \oplus \wedge_H(V) & \longrightarrow & \wedge_H(U \cap V) & \longrightarrow 0 \end{array}$$

so

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^1(U\cup V) & \longrightarrow & H^1(U)\oplus H^1(V) & \longrightarrow & H^1(U\cap V) & \longrightarrow & H^2(U\cup V) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle p_H} & & \downarrow{\scriptstyle p_H} & & \downarrow{\scriptstyle p_H} & & \downarrow{\scriptstyle p_H} & \\
0 \longrightarrow & H^1_H(U\cup V) & \longrightarrow & H^1_H(U)\oplus H^1_H(V) & \longrightarrow & H^1_H(U\cap V) & \longrightarrow & H^2_H(U\cup V) & \longrightarrow \cdots
\end{array}$$

and by a standard argument (see Bott et al. [1]) we can prove the lemma.

LEMMA 1.8. *If at every $x \in M$, $H_x$ has non-degeneracy $r$, then $H^1_N(M) = \cdots = H^r_N(M) = 0$.*

*Proof.* Fix a point $p \in M$, then there is a coordinate system $(x_i, y_j)$ and $k$ 1-forms $\omega_1, \dots, \omega_k$ such that $H$ is defined by $\omega_1 = \cdots = \omega_k = 0$, where

$$\omega_j = dy_j - \sum a^j_{il} x_i \, dx_l + O\big(|x|^2 + |y|^2\big), \quad j = 1, \dots, k,$$

and

$$d\omega_j = \theta^j + O\big(|x|^1 + |y|^1\big), \quad j = 1, \dots, k.$$

Now let $\alpha_s$ be a closed $s$-form ($s \le r$) of the form $\Sigma f_i \wedge \omega_i + \Sigma g_i \wedge d\omega_i$. Then

$$d\alpha_r = \sum df_i \wedge \omega_i + \sum\big((-1)^{s-1} f_i + dg_i\big) \wedge d\omega,$$

hence by the assumption we have $(-1)^{s-1} f_i + dg_i = 0 \bmod\{\omega\}$, where $\{\omega\}$ is the algebraic ideal generated by $\omega_1, \dots, \omega_k$. Now we need only to prove that for an $s$-form $\alpha = \Sigma_{i_1 < \cdots < i_k} f_J \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{ik}$, $d\alpha = 0$ iff $\alpha = 0$. Here $J = (i_1, \dots, i_k)$, $f_J$ is an $(s-k)$-form $f_J = \Sigma h_{(j_1, j_2, \dots, j_{s-k})} \, dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_{s-k}}$. Now

$$\begin{aligned} d\alpha = {} & \sum df_J \wedge \omega^J + \sum (-1)^{s-i} f_I \wedge d\omega_{i_1} \wedge \cdots \wedge \omega_{i_k} \\ & + \cdots + \sum (-1)^{s-u-1} f_I \wedge \omega_{i_1} \wedge \cdots \wedge d\omega_{i_u} \wedge \omega_{i_{u+1}} \wedge \cdots \omega_{i_{s-k}}, \end{aligned}$$

from which follows $\Sigma_{j \ge k} f_{(1,2,\dots,k-1,j)} \wedge d\omega_j = 0$. Again by the assumption that $H$ has non-degeneracy $r \ge s$, we have $f_{(1,2,\dots,k-1,j)} = 0$. Similarly $f_J = 0$ for any $J$. So the lemma is proved.

COROLLARY 1.9. *Under the same condition as in Lemma 1.8, $H^i_H(M) = H^i(M)$, $i = 1, \dots, r-1$.*

Before concluding this subsection, we look at the geometric meaning of the cohomology of $H$.

We say that a differentiable map $f\colon N \to M$ is horizontal if the pull back of $H^*$ by $f$, $f^*(H^*)$ is zero. Such maps have appeared in various contexts, such as variations of Hodge structures (cf. Carlson and Toledo [3], Griffiths [10]).

Denote $I^q = [0,1] \times \cdots \times [0,1]$. Let $C_q(M)$ be the free abelian group generated by the $q$-singular cubes $f\colon I^q \to M$, and $C_{q,H}(M)$ be the subgroup generated by horizontal ones, and

$$C(M) = \oplus C_q(M), \quad C_H(M) = \oplus C_{q,H}(M).$$

Define the *k-th horizontal singular homology group* by

$$H_{q,H}(M) = \frac{\ker \delta^q}{\operatorname{Im} \delta^{q-1}}.$$

Here $\delta$ is the restriction of the boundary operators to $C_H(M)$. There is a well defined pairing between $H_H^q(M)$ and $H_{q,H}(M)$. Suppose that $f$ represents a $k$-th horizontal singular homology, and $\omega$ represents a $k$-th horizontal cohomology, then define

$$\langle [f], [\omega] \rangle = \int_f \omega. \tag{1.7}$$

LEMMA 1.10. *The pairing* (1.7) *is well defined.*

*Proof.* Let $\omega'$ (resp. $f'$) represents the same element as $\omega$ (resp. $f$). So there is a horizontal $k$ such that $f' = f + \delta k$. Without loss of generality we assume that $H$ is defined by $k$ 1-forms $e_1 = \cdots = e_k = 0$ within the image of $f, f', k$. Then $\omega'_r = \omega + \Sigma h_i \wedge e_i + g_i \wedge de_i$,

$$\int_{f'} \omega' = \int_{f'} (\omega' - \omega) + \int_{f'} \omega = \int_{f'} (\omega' - \omega) + \int_k d\omega + \int_f \omega. \tag{1.8}$$

Now the first term above is

$$\int_{f'} h_i \wedge e_i + g_i \wedge de_i = \int_{f'} g_i \wedge de_i = (-1)^{\deg(g_i)} \int_{f'} dg_i \wedge e_i = 0.$$

As for the second term in (1.8), note that by definition $d\omega$ can be written as $d\omega = \Sigma h'_i \wedge e_i + g'_i \wedge de_i$, so

$$\int_k d\omega = \int_k g'_i \wedge de_i = \int_k dg'_i \wedge e_i - \left( \int_{f'} - \int_f \right) g'_i \wedge e_i = 0.$$

Hence $\int_i \omega = \int_{f'} \omega'$.

Now by the result of Thom [28]: if $(M, H)$ is a contact $(2r+1)$-manifold, $H_{q,H}(M)$ is isomorphic to $H_q(M)$, $r = 0, 1, \ldots, r-1$, we know that the pairing (1.7) is nondegenerate modulo torsion elements.

### 1.3. Characteristic classes for horizontal connections

Let $V$ be a vector bundle over $M$, and $H^* \subset T^*M$ the subbundle dual to $H$. In this subsection we will study the geometric properties of a "horizontal connection" in which the connection is only defined for horizontal vector fields. In particular, partial connections associated with sub-Riemannian metrics are examples of horizontal connections. Our main goal here is to generalize the classical theory of connections (cf. Chern [4]) to horizontal connections.

DEFINITION 1.2. *A horizontal connection is a linear smooth map*

$$D\colon C^\infty(V) \to C^\infty(H^* \otimes V)$$

*which satisfies*

$$D(fs) = d_H f \otimes s + fDs, \quad f \in C^\infty(M),\, s \in C^\infty(H).$$

*Example* 1. Let $TM = H \oplus K$ be a splitting, where $K$ is a vector bundle over $M$, and let $p_K\colon TM \to K$ be the projection onto $K$. Define $D\colon C^\infty(V) \to C^\infty(H^* \otimes V)$ by

$$Ds = \sum_i p_K[s, e_i] \otimes e^i, \quad s \in C^\infty(K),$$

where $e_i$ is a local frame for $K$. It is easy to see that $D$ is a horizontal connection.

*Example* 2. If $M$ is the total space of a fiber bundle $W \to M \to B$ and $H$ comes from a connection, and $D_B$ is the Levi-Civita connection on $B$, and $\overline{D}$ the horizontal lift of $D_B$, then define

$$Ds = \sum \left(\overline{D}_{e_i} s\right) \otimes e^i,\, s \in C^\infty(H),$$

where $e_i$ is an orthonormal frame for $H$. It is easy to verify that $D$ is a horizontal connection.

*Example* 3. Let $D_a b$, $a \in H$, $b \in C^\infty(H)$, be a partial connection for the sub-Riemannian metric. Obviously the partial connection is an example of horizontal connection. If an orthogonal frame $e_i$ spans $H$, define a horizontal connection $D\colon C^\infty(H) \to C^\infty(H^* \otimes H)$ by

$$Ds = \sum e^i \otimes D_{e_i}(s), \tag{1.9}$$

where $e^i$ are the dual frame of $e_i$. It is easy to check that (1.9) is well defined.

Now let $D$ be a horizontal connection. Let $(s_1, \dots, s_k)$ be a local frame for $V$. Write $s = \Sigma f_i s_i$, then

$$Ds = d_H f_i \otimes s_i + f_i \omega_{ij} \otimes s_j$$

where $Ds_i = \omega_{ij} \otimes s_j$, and $\omega_{ij} \in \Lambda^1_H(M)$. The connection 1-form relative to the local frame $s_i$ is the matrix valued horizontal 1-form $\omega = (\omega_{ij})$.

We choose another $s'$ frame for $V$, $s'_i = h_{ij} s_j$. Let $h^{-1} = (h_{ij})^{-1}$ represent the inverse matrix, then we compute:

$$\omega' = d_H h \cdot h^{-1} + h\omega h^{-1}.$$

We extend $D$ to be a derivation mapping

$$C^\infty\big(\Lambda^p_H(M) \otimes V\big) \to C^\infty\big(\Lambda^{p+1}_H(M) \otimes V\big)$$

by

$$D\big(\theta_p \otimes s\big) = d_H \theta \otimes s + (-1)^p \theta_p \wedge Ds.$$

Then

$$\begin{aligned} D^2(fs) &= D(d_H f \otimes s + fDs) \\ &= d_H^2 f \otimes s - d_H f \wedge Ds + d_H f \wedge Ds + fD^2 s = fD^2 s. \end{aligned}$$

Let $D^2(s)(x_0) = \Omega(x_0)s(x_0)$. $\Omega$ will be called the curvature for the horizontal connection $D$. In terms of a local frame $s_i$,

$$\Omega = d_H \omega - \omega \wedge \omega.$$

If we change to another local frame, $s'_j = h_{ij} s_j$, then $\Omega' = h\Omega h^{-1}$.

We say $P$: $\mathrm{End}(C^k) \to C$ is an invariant polynomial mapping, if $P(hAh^{-1}) = P(A)$ for any $h \in GL(C^k)$. Define $P(D) = P(\Omega)$.

THEOREM 1.11. *Let $P$ be an invariant polynomial mapping.*

(a) $d_H P(D) = 0$.

(b) *Given two connection $D_0$ and $D_1$, we can define a differential form $TP(D_0, D_1)$ so that*

$$P(D_1) - P(D_0) = d_H\{TP(D_1, D_0)\}.$$

*Proof.* Without loss of generality we assume that $P$ is homogeneous of order $k$. Let $P(A_1, \ldots, A_k)$ denote the complete polarization of $P$, so $dP(A) = P(dA, A, \ldots, A)$. Note that this implies $d_H P(A) = P(d_H A, A, \ldots, A)$.

Let $D'_i$: $C^\infty(V) \to C^\infty(T^*M \times V)$, $i = 0, 1$, be two connections such that $p_H(D'_i s) = D_i s$, $i = 0, 1$, for $s \in C^\infty(V)$. Such connections exist at least locally. In fact, take a local frame $s_j$ for $C^\infty(V)$, and let $\omega_i$ be the connection 1-form for $D_i$, $i = 0, 1$. Now $\omega_i$ can also be considered as matrix-valued 1-forms on $M$. Then let $D'_i$ be the connections whose connection 1-forms are $\omega_i$ respectively.

Now let $\Omega'_i$ be the curvature of $D'_i$. Then

$$\begin{aligned} p_H(\Omega'_i) &= p_H(d\omega_i - \omega_i \wedge \omega_i) \\ &= d_H p_H(\omega_i) - p_H(\omega_i) \wedge p_H(\omega_i) = \Omega_i, \quad i = 0, 1. \end{aligned}$$

Next let $D'_t = tD'_1 + (1 - t)D'_0$ with the connection 1-form $\omega'_t = \omega'_0 + t\theta'$ where $\theta' = \omega'_1 - \omega'_0$.

Define $TP(D'_1, D'_0) = k\int_0^1 P(\theta', \Omega'_t, \ldots, \Omega'_t)\, dt$. Then, as is well known (cf. [4]),

$$\begin{aligned} dP(D'_i) &= 0, \quad i = 0, 1; \\ P(D'_1) - P(D'_0) &= dP(\theta', \Omega'_t, \ldots, \Omega'_t). \end{aligned}$$

Now

$$d_H P(D_i) = p_H(dP(D'_i)) = 0,$$

and

$$\begin{aligned} P(D_1) - P(D_0) &= p_H(P(D'_1) - P(D'_0)) = p_H(dTP(D'_1, D'_0)) \\ &= d_H(p_H(\{TP(D'_1, D'_0)\})). \end{aligned}$$

On the other hand,

$$\begin{aligned} p_H(TP(D'_1, D'_0)) &= p_H\left(\int_0^1 P(\theta', \Omega'_t, \ldots, \Omega'_t)\, dt\right) \\ &= \int_0^1 P(p_H(\theta'), p_H(\Omega'_t), \ldots, p_H(\Omega'_t))\, dt \\ &= \int_0^1 P(\theta, \Omega_t, \ldots, \Omega_t)\, dt = TP(D_1, D_0). \end{aligned}$$

From the above proof we have:

LEMMA 1.12. *If $D'$: $C^\infty(M) \to C^\infty(T^*M \otimes V)$ is a connection such that $p_H(D's) = Ds$, $s \in C^\infty(V)$, and $P$ is an invariant polynomial, then $p_H(P(D')) = P(D)$.*

If $V$ is a complex vector bundle, then as in standard vector bundle theory [4], we define the total horizontal Chern class

$$c(D) = \det\left(I + \frac{i}{2\pi}\Omega\right) = c_1(D) + c_2(D) + \cdots$$

where $c_k(D)$ is the $2k$-form, called the $k$-th *horizontal Chern class*. Similarly, we define the *total horizontal Chern character*

$$ch(D) = \mathrm{Tr}(\exp(i\Omega/2\pi)).$$

If $V$ is a real vector bundle with a fiberwise metric $\langle \cdot , \cdot \rangle_V$, then we say a horizontal connection $D$ is *sub-Riemannian* if

$$d\langle s_1, s_2\rangle_V = \langle Ds_1, s_2\rangle_V + \langle s_1, Ds_2\rangle_V, \quad s_1, s_2 \in C^\infty(V).$$

If $D$ is a sub-Riemannian connection, we define the total horizontal Pontragin class as

$$p(D) = \det\left(I + \frac{1}{2\pi}\Omega\right) = p_1(D) + p_2(D) + \cdots,$$

where $p_k(D)$ is the $4k$-form, called the $k$-th horizontal Pontragin class. Moreover, if the vector bundle $V$ has even rank $2r$, then one can define the Euler class ($\Omega = (\theta_{ij})$)

$$e(D) = \frac{(-1)^r}{2^q\pi^r r!}\sum \varepsilon_{i_1\ldots i2r}\theta_{i_1i_2} \wedge \cdots \wedge \theta_{i_{2r-1}i_{2r}}.$$

Similarly one can define secondary invariants.

In the following we will let $P$ be an invariant homogeneous polynomial of degree $4k$.

LEMMA 1.13. *Let $D_\tau$ be a family of horizontal connections on $V$, let $\phi = \partial D_\tau/\partial\tau$ and*

$$V(\tau) = \int_0^1 t^{k-1}P(\phi, \Omega(\tau), \Omega(\tau), \ldots, \Omega(\tau))\, dt.$$

*Then*

$$\frac{\partial}{\partial\tau}TP(D_\tau, D_0) = k(k-1)\,dV(\tau) + hP(\phi, \Omega(\tau), \ldots, \Omega(\tau)). \quad (1.10)$$

*Proof.* Suppose that $D'_\tau$: $C^\infty(V) \to C^\infty(M \times V)$ is a connection such that $D_\tau(s) = p_H(D'_\tau s)$, $s \in \Gamma(V)$, and

$$V'(\tau) = \int_0^1 t^{k-1} P(\phi', \Omega'(\tau), \Omega'(\tau), \ldots, \Omega'(\tau))\, dt,$$

where $\phi' = \partial D'_\tau/\partial\tau$ and $\Omega'(\tau)$ is the curvature of $D'_\tau$. Then $V(\tau) = p_H(V'(\tau))$, and

$$\begin{aligned}\frac{\partial}{\partial\tau}TP(D_\tau, D_0) &= p_H\left(\frac{\partial}{\partial\tau}TP(D'_\tau, D'_0)\right)\\ &= p_H(k(k-1)\,dV'(\tau)) + kP(\phi, \Omega'(\tau), \ldots, \Omega'(\tau)).\\ &= k(k-1)d_H(p_H(V'(\tau)))\\ &\quad + kP(p_H(\phi), p_H(\Omega')(\tau), \ldots, p_H(\Omega')(\tau)).\end{aligned}$$

Observe that if $\omega'(\tau)$ and $\Omega'(\tau)$ are the connection 1-form and the curvature of $D'_\tau$ respectively, then the connection 1-form for $D_\tau$ is $\omega(\tau) = p_H(\omega'(\tau))$, and

$$\Omega(\tau) = d_H(p_H(\omega'(\tau))) - p_H(\omega'(\tau)) \wedge p_H(\omega'(\tau)) = p_H(\Omega'(\tau));$$

hence (1.10) is proved.

The next theorem follows immediately from the lemma.

THEOREM 1.14. *Let $P$ be an invariant polynomial mapping. Let $D_\tau$ be a family of horizontal connections with curvatures $\Omega(\tau)$, which satisfy*

$$p_H(P(\Omega(\tau), \ldots, \Omega(\tau))) = 0,$$
$$p_H\left(P\left(\frac{\partial D_\tau}{\partial\tau}, \Omega(\tau), \ldots, \Omega(\tau)\right)\right) = 0.$$

*Then the horizontal cohomology class $TP(D_\tau, D_0) \in H_H(M)$ is independent of $\tau$.*

### 1.4. Curvature for sub-Riemannian metrics

In this sub-section we will apply the results in §1.3 to sub-Riemannian metrics.

Let $D$ be the partial connection associated with a splitting $TM = H \oplus K$. We have seen that the partial connection is an example of horizontal connection (see §1.3). Now we compute its curvature.

Let $e_i$ be an orthonormal frame for $H$.

THEOREM 1.15. *Suppose that* $\Lambda^2_H(M)$ *satisfies the condition* $(L)$. *Then the curvature of the horizontal connection* (1.19) *can be expressed in terms of the partial curvature as follows*:

$$\Omega s = \sum_{i<j} p_H\big(e^i \wedge e^j \otimes R(e_i, e_j)s\big). \tag{1.11}$$

*Moreover, if* $p_k$, $P_k$ *are the k-th Pontragian class of* $H \to M$ *and k-th Pontragian polynomial respectively, then*

$$P_k(\Omega) = p_H(p_k).$$

*Proof.* By the condition (L), we only need to prove (1.11) at a point $x_0$. Note that the right hand side of (1.11) is defined independent of a local frame $e_i$. So we need only to prove (1.11) for a local frame $e_i$ normal at $x_0$. Now

$$\Omega s(x_0) = \sum p_H\big(d_H e^i \otimes D_{e_i} s\big)(x_0) + \sum_{i<j} e^i \wedge e^j \otimes R(e_i, e_j)s(x_0)\big).$$

We need to prove $de^i(e_j, e_k)(x_0) = 0$. In fact,

$$de^i(e_j, e_k) = \tfrac{1}{2}\Big(e_j\big(e^i(e_k)\big) - e_k\big(e^i(e_j)\big) - e^i\big([e_j, e_k]\big)\Big)(x_0) = 0.$$

So $(d_H e^i \otimes D_{e_i} s)(x_0) = 0$.

*Remark.* If $I_x$ is generated by $\theta^1, \dots, \theta^k$ which are orthonormal with respect to the inner product on $\Lambda^2(H)$,

$$\theta^r = \sum_{ij} \theta^r_{ij} e^i \wedge e^j,$$

where $e_i$ is an orthonormal frame for $H$, then (1.11) can be written as

$$\Omega = \sum_{ij}\left(R(e_i, e_j) - \sum_{lkr} R(e_l, e_k)\theta^r_{lk}\theta^r_{ij}\right) \otimes e^i \wedge e^j.$$

So we see that

$$R(e_i, e_j) - \sum_r \sum_{lk} \theta^r_{lk}\theta^r_{ij} R(e_l, e_k) \tag{1.12}$$

is a tensor. However, in view of the importance of (1.12), we will prove that (1.12) is a tensor without condition (L).

LEMMA 1.16. (1.12) *is a tensor*.

*Proof.* In view of Lemma 1.2, we need only to prove that

$$\mu(e_i, e_j) - \sum_r \sum_{lk} \theta^r_{lk}\theta^r_{ij}\mu(e_l, e_k) = 0. \tag{1.13}$$

If $H$ is given by 1-forms $\omega_1 = \cdots = \omega_k = 0$, where

$$d\omega_i = \theta^i \bmod (e^j)$$

then $[e_i, e_j] = 2\Sigma_r \theta^r_{ij} n_r \bmod (e_r)$, where $n_r$ is the dual vector field to $\omega_r$. So

$$\mu(e_i, e_j) = 2\sum_r \theta^r_{ij} n_r;$$

thus

$$\begin{aligned}\mu(e_i, e_j) - \sum_r \sum_{lk} \theta^r_{lk}\theta^r_{ij}\mu(e_l, e_k) &= 2\sum \theta^r_{ij} n_r - 2\sum_r \sum_{lk} \sum_t \theta^r_{lk}\theta^r_{ij}\theta^t_{lk} n_t \\ &= 2\sum \theta^r_{ij} n_r - 2\sum_r \sum_{lk} \sum_t \delta_{rt}\theta^r_{ij} n_t = 0.\end{aligned}$$

Now by the results in §1.3, we can express the horizontal Pontragin classes in terms of the 2-nd jets of the sub-Riemannian metric, moreover, if $H$ is contact, the construction is canonical and the lower horizontal Pontragin classes are in fact the Pontragin classes of $H$ (see Gromov [12], p. 65, for a related problem).

Define a tri-linear map $T$: $H \otimes H \otimes H \to H$ by

$$T(x, y, z) = R(x, y)z - \sum \tfrac{1}{4}(\theta^r, \bar{x} \wedge \bar{y})(\theta^r, e^i \wedge e^j)R(e_i, e_j)z.$$

Here $\bar{x}$ denotes the dual of $x \in H$ in $H^*$.

LEMMA 1.17. *$T$ is a well defined tensor*.

*Proof.* Observe $\theta^r_{ij} = (\theta^r, e^i \wedge e^j)/2$, expand $x = (x, e_1)e_1 + \cdots + (x, e_m)e_m$ and similarly expand $y$, and using Lemma 1.16, we prove the lemma.

## 2. The Hodge theory of $H^1(M)$ for degenerate metrics

The classical Hodge theorem says that on a Riemannian manifold the $k$-th de Rham cohomology group is isomorphic to the kernel of the Laplacian acting on $k$-forms. In this section we will generalize a part of the Hodge theorem to degenerate metrics (sub-Riemannian metrics).

Throughout this section, without loss of generality, we will work in the following setting. Let $Q$ be a Riemannian metric on $M$ which agrees with the sub-Riemannian metric $(\cdot, \cdot)$ on $H$, $K = H^{\perp}$ be the subbundle orthogonal to $H$, and let $D$ be the (unique) partial connection associated with the splitting $TM = H \oplus K$.

$Q$ is called an extension of the sub-Riemannian metric. In general there is no canonical extension, however, if $H$ is contact, there is a canonical way to extend the sub-Riemannian metric to a Riemannian metric on $M$: if $\alpha$ is the canonical 1-form in (1.3), then we take $Q$ such that $\alpha$ has norm 1, i.e.,

$$Q(a+b, a+b) = d\alpha(a, Ja) + (\alpha, b)^2, \quad a \in H, b \in K.$$

### 2.1. Main results

We first introduce some notations.

To begin with, let $D^Q$ be the Levi Civita connection of $(M, Q)$. The relation between the Levi Civita connection of $Q$ and the partial connection of the sub-Riemannian metric is (cf. [9])

$$D_a b = \pi D_a^Q b, \quad a \in H, b \in C^\infty(H), \tag{2.1}$$

where $\pi$: $TM \to H$ is the projection.

If $\omega_1, \omega_2$ are two horizontal forms of the same degree, their inner product is

$$(\omega_1, \omega_2)_0 = \int_M (\omega_1, \omega_2)_x \, dv$$

where $(\cdot, \cdot)_x$ is the inner product induced on $\wedge(H_x)$. Define $\delta_H$ to be the dual of $d_H$ with respect to $(\cdot, \cdot)$, and define

$$\Delta_H = d_H \delta_H + \delta_H d_H.$$

If $\omega \in \wedge_H(M)$, its weighted Sobolev norm (cf. [24]) will be denoted by

$$\|\omega\|_1^2 = (\omega, \omega)_1 = \int \sum_i \left(D_{e_i}\omega, D_{e_i}\omega\right) dv(x)$$

where $e_i$ is an orthonormal frame on $H$. In the following we suppose that $I_x$ is generated by $\theta^1, \dots, \theta^k$, which are orthonormal with respect to the induced inner product on $\wedge^2(H)$.

LEMMA 2.1. *If $e_i$ is an orthonormal frame, $y_1, \dots, y_k$ is an orthonormal frame for $K = H^\perp$, then if $\omega$ is a horizontal* 1*-form or* 2*-form,*

$$d_H\omega = \sum_i e^i \wedge D_{e_i}\omega - \sum\left(\theta^r, \sum_i e^i \wedge D_{e_i}\omega\right)\theta^r, \tag{2.2}$$

$$\delta_H = -\sum_i i(e_i)D_{e_i} - D^0, \tag{2.3}$$

*where $D^0$ is the* 0*-order operator*

$$D^0 = \sum_j p_H\big(i(y_j)D^Q_{yj}\big). \tag{2.4}$$

*Remark.* $D^0$ only depends on $dv$, $Q$, and $K$. In particular, if $H$ is contact, then $D^0$ is a canonically defined tensor, thus is another invariant of the sub-Riemannian metric.

*Proof.* Let $p_1$: $\wedge(M) \to \wedge(H)$ and $p_2$: $\wedge(H) \to \wedge_H(M)$ be the orthogonal projections respectively, then $p_H = p_2 \circ p_1$ and define $\bar{d} = p_1 d$. Then, using (2.1), we can rewrite $\bar{d}$ as

$$\bar{d} = \sum_i e^i \wedge D_{e_i}, \tag{2.5}$$

thus when acting on horizontal 1-forms or 2-forms,

$$d_H\omega = p_2\,\bar{d}\omega = \bar{d}\omega - \sum_r(\theta^r, \bar{d}\omega)\theta^r.$$

So (2.2) is proved. Now we compute $\delta_H$. Let $\delta^Q$ be the adjoint of $d$ with respect to $Q$,

$$\begin{aligned}\delta_H\omega &= p_1\delta^Q\omega \\ &= p_1\Big(\sum i(e_i)D^Q_{ei}\omega + i(y_i)D^Q_{y_i}\omega\Big) \\ &= \sum i(e_i)D_{e_i}\omega + p_1\big(i(y_i)D^Q_{y_i}\omega\big).\end{aligned}$$

LEMMA 2.2. *If for any $x, y \in C^\infty(H^\perp)$, $D^g_x y \in C^\infty(H^\perp)$, then $D^0 = 0$.*

*Remark.* If $H^{\perp}$ is an integrable distribution (e.g., $H$ is contact), then $D^0 = 0$ if every leaf of $H^{\perp}$ is totally geodesic with respect to $Q$.

LEMMA 2.3. *If $\omega$ is a horizontal 1-form,*

$$-\Delta^1_H\omega = \sum_i D_{e_i}D_{e_i}\omega - D_{D_{e_i}e_i}\omega + \sum_{ij} e^i \wedge i(e_j)R(D_{e_i}, D_{e_i})\omega + D_0\sum_i e^i \wedge D_{e_i}$$
$$+ \sum_i e^i \wedge D_{e_i}D_0\omega - \sum_{rj} e_j\Big(\theta^r, \sum_i e^i \wedge D_{e_i}\omega\Big)i(e_j)\theta^r$$
$$- \sum_r \Big(\theta^r, \sum_i e^i \wedge D_{e_i}\omega\Big)i(e_j)D_{e_j}\theta^r. \tag{2.6}$$

*Proof.* Select an orthonormal frame $e_i$ which is normal at $x_0 \in M$. Using (2.5),

$$\Delta^1_H\omega = (\bar{d}\delta + \delta\bar{d})\omega - \delta\Big(\sum(\bar{\omega}, \theta^r)\theta^r\Big).$$

The last term above is the last two terms in (2.6), while the first term above is easily seen to be equal to (cf. Wu [31])

$$D_{e_i}D_{e_i}\omega - D_{D_{e_i}e_i}\omega + \sum_{ij} e^i \wedge i(e_j)R(D_{e_i}, D_{e_i})\omega + D_0\sum_i e^i \wedge D_{e_i}.$$

If $M$ is the total space of a fiber bundle, then $\Delta^1_H$ takes a much simpler form

COROLLARY 2.4. *If $M$ is the total space of a fiber bundle $W \to M \to B$ over a Riemannian manifold with totally geodesic fibers, and the sub-Riemannian metric is the horizontal lifting of the Riemannian metric on $B$, then*

$$-\Delta^1_H\omega = \sum_i D_{e_i}D_{e_i}\omega - D_{D_{e_i}e_i}\omega + \sum_{ij} e^i \wedge i(e_j)R(D_{e_i}, D_{e_i})\omega$$
$$- \sum_{rj} e_j\Big(\theta^r, \sum_i e^i \wedge D_{e_i}\omega\Big)i(e_j)\theta^r - \sum_r \Big(\theta^r, \sum_i e^i \wedge D_{e_i}\omega\Big)i(e_j)D_{e_j}\theta^r, \tag{2.7}$$

*where $D$ is the horizontal lift of the Levi Civita connection on $B$.*

To state our main result, we need to define some quantities associated with $H$. To begin with, suppose that $I_x$ is generated by $\theta^1, \ldots, \theta^k$

$$\theta^r = \sum_{ij} \theta^r_{ij} e^i \wedge e^j, \quad \theta^r_{ij} = -\theta^r_{ji}.$$

Without loss of generality we assume that they are orthonormal:

$$\sum_{ij} \theta^s_{ij}\theta^t_{ij} = \delta_{st}.$$

Define

$$\lambda_1(x) = \max\left|\sum_r \frac{2\Sigma_{ijst}\theta^r_{ij}\theta^r_{st}(u_{si}, u_{tj}) - \Sigma_{ijst}\theta^r_{ij}\theta^r_{st}(u_{st}, u_{ij})}{\Sigma_{si}|u_{si}|^2}\right| \tag{2.8}$$

$$\lambda_2(x) = \max\left|\sum_{ru} \frac{\Sigma_{ijlkst}\theta^r_{ij}\theta^r_{lk}\theta^u_{st}\theta^u_{il}(u_{sj}, u_{tk})}{\Sigma_{si}|u_{si}|^2}\right| \tag{2.9}$$

LEMMA 2.5. *$\lambda_1(x), \lambda_2(x)$ only depend on $I_x$ (and not on the choice of $\theta^1, \ldots, \theta^r, e_1, \ldots, e_n$).*

*Proof.* We first prove that $\lambda_1, \lambda_2$ are independent of $e_i$. Suppose that $\bar{e}_i$ is another orthonormal frame, $e_i = \Sigma_{i_1} a_{ii_1}\bar{e}_{i_1}$; then we have

$$\theta^r_{ij} = \sum_{i_1 j_1} \bar{\theta}^r_{i_1 j_1} a_{i_1 i} a_{j_1 j}.$$

Now define a transformation $u_{ij} \to \bar{u}_{ij}$ by $u_{ij} = \Sigma_{i_1 j_1}\bar{u}_{i_1 j_1} a_{i_1 i} a_{j_1 j}$, which is orthogonal with respect to $\Sigma_{si}(u_{si}, u_{si})$.

Now we compute the various terms in (2.8). The first term in (2.8) is

$$\begin{aligned}\sum \theta^r_{ij}\theta^r_{st}(u_{si}, u_{tj}) &= \sum \bar{\theta}^r_{i_1 j_1} a_{i_1 i} a_{j_1 j}\bar{\theta}^r_{s_1 t_1}\left(\bar{u}_{s_2 i_2}, \bar{u}_{t_2 j_2}\right) a_{s_2 s} a_{i_2 i} a_{t_2 t} a_{j_2 j} \\ &= \sum \theta^r_{ij}\bar{\theta}^r_{st}\left(\bar{u}_{si}, \bar{u}_{tj}\right).\end{aligned}$$

Similarly, we can prove that other terms are invariant under the transformations $e^i \to \bar{e}^i$, $u_{ij} \to \bar{u}_{ij}$. Hence (2.9) is independent of the choice of $e_i$. Next we prove that $\lambda_1, \lambda_2$ are independent of the choice of $\theta^i$. If $\theta^i = \Sigma_j b_{ir}\bar{\theta}^r$ where $\bar{\theta}^r$ is another orthogonal frame, then

$$\sum \theta^r_{ij}\theta^r_{st} = \sum b_{rr_1} b_{rr_2}\bar{\theta}^{r_1}_{ij}\bar{\theta}^{r_2}_{st} = \sum \delta_{r_1 r_2}\bar{\theta}^{r_1}_{ij}\bar{\theta}^{r_2}_{st} = \sum \bar{\theta}^r_{ij}\bar{\theta}^r_{st},$$

hence (2.8) is independent of the choice of $\theta^r$. Similarly we can prove that (2.9) is independent of $e^i, \theta^r$.

THEOREM 2.6. *If at every point $x \in M$, $1 - \lambda_1(x) - 2\lambda_2(x) > 0$, then $\Delta^1_H$ is hypoelliptic.*

COROLLARY 2.7. *If $H$ has non-degeneracy $> 0$, and $1 - \lambda_1(x) - 2\lambda_2(x) > 0$, then*

$$H^1(M) = \{\omega \in \Lambda^1_H(M),\, d_H\omega = \delta_H\omega = 0.\}$$

Now let us look at the case where $H$ is a contact manifold, and we assume $\theta = \sum_i e^i \wedge e^{n+i}/n^{1/2}$. This sub-Riemannian metric is usually called an almost Heisenberg metric. Then we compute $\lambda_1 < 3/2n$, $\lambda_2 \le 1/2n^2$. Thus if $n > 1$, $1 - \lambda_1(x) - 2\lambda_2(x) > 0$, so (compare [25]).

COROLLARY 2.8. *If $M$ is a $(2n + 1)$-dimensional almost Heisenberg manifold, $n > 1$, then $\Delta^1_H$ is hypoelliptic.*

**2.2. The proof of Theorem 2.6**

By definition, we need to prove that there is a positive $\delta_0 > 0$ such that

$$\left(\Delta^1_H\omega, \omega\right)_0 \ge \delta_0(\omega, \omega)_1 - N(\omega, \omega)_0. \tag{2.10}$$

Now

$$\begin{aligned}\left(\Delta^1_H\omega, \omega\right) &= (d_H\omega, d_H\omega)_0 + (\delta_H\omega, \delta_H\omega)_0 \\ &= (\bar{d}\omega, \bar{d}\omega)_0 - \sum(\bar{d}\omega, \theta^r)_0^2 + (\delta_H\omega, \delta_H\omega)_0 \\ &= \left(\left(\bar{d}\delta_H + \delta\bar{d}\right)\omega, \omega\right)_0 - \sum_r(\bar{d}\omega, \theta^r)_0^2.\end{aligned}$$

Modulo a 0-order operators, $\delta_H = \sum_i i(e_i)D_{e_i}$, hence modulo first order operators,

$$\bar{d}\delta + \delta\bar{d} = \sum_i D_{e_i}D_{e_i} + \sum_{ij} e^i \wedge i(e_i)R\left(D_{e_i}, D_{e_j}\right)$$

Let $\omega = \sum_i u_i e^i$. In what follows we will use $O_1$ to denote a sum of terms of the form $(D_i u_j, u_k)_0$, which is bounded (for any positive $\varepsilon > 0$) by

$$|O_1(\omega)|_0 \le \varepsilon\|\omega\|_1^2 + N_\varepsilon\|\omega\|_0^2.$$

Now we have

$$(\theta^r, \bar{d}\omega)^2 = \left(\sum_{ij}\theta^r_{ij}D_{e_i}u_j\right)^2 + O_1.$$

Thus

$$(\Delta_H \omega, \omega)_0 = \sum_{ij} (D_{e_i} u_i, D_{e_i} u_i) + \sum_{ij} \big(R(D_{e_i}, D_{e_j}) u_i, u_j\big)_0$$
$$- \sum_r \Big(\sum_{ij} \theta^r_{ij} D_{e_i} u_j\Big)^2_0 + O_1. \tag{2.11}$$

By integration by parts the second term above is

$$\sum_{ij} \big(R(D_{e_i}, D_{e_j}) u_i, u_j\big) = \sum_{ij} \sum_{lk} \big(\theta^r_{lk} \theta^r_{ij} R(D_{e_l}, D_{e_k}) u_i, u_j\big)_0 + O_1$$
$$= 2 \sum \theta^r_{lk} \theta^r_{ij} (D_{e_l} u_i, D_{e_k} u_j)_0 + O_1. \tag{2.12}$$

Here we have made use of the fact (cf. Lemma 1.16) that modulo 0-order operators,

$$R(D_{e_i}, D_{e_j}) = \sum_r \sum_{lk} \theta^r_{lk} \theta^r_{ij} R(D_{e_l}, D_{e_k}). \tag{2.13}$$

Now, using integration by parts repeatedly, the third term in (2.11) is

$$\sum_r \Big(\sum_{ij} D_{e_i} u_j\Big)^2_0 = \sum_{ijlkr} \theta^r_{ij} \theta^r_{lk} (D_{e_i} u_j, D_{e_l} u_k)_0$$
$$= \sum_{rf} \sum_{ijlkr} \theta^r_{ij} \theta^r_{lk} (D_{e_l} u_j, D_{e_i} u_k)_0$$
$$- \sum_{ijlkr} \theta^r_{ij} \theta^r_{lk} \big(R(D_{e_i}, D_{e_l}) u_j, u_k\big)_0 + O_1$$
$$= \sum_r \sum_{ijlkr} \theta^r_{ij} \theta^r_{lk} (D_{e_l} u_j, D_{e_i} u_k)_0$$
$$- \sum_{ijlkru} \theta^r_{ij} \theta^r_{lk} \theta^u_{il} \theta^u_{st} \big(R(D_{e_s}, D_{e_t}) u_j, u_k\big)_0 + O_1$$
$$= \sum_r \sum_{ijlkr} \theta^r_{ij} \theta^r_{lk} (D_{e_l} u_j, D_{e_i} u_k)_0$$
$$- \sum_{ijlkru} \theta^r_{ij} \theta^r_{lk} \theta^u_{il} \theta^u_{st} (D_{e_s} u_j, D_{e_t} u_k)_0$$
$$+ \sum_{ijlkru} \theta^r_{ij} \theta^r_{lk} \theta^u_{il} \theta^u_{st} (D_{e_t} u_j, D_{e_s} u_k)_0 + O_1. \tag{2.14}$$

Here we have used (2.13) again. Inserting (2.12) and (2.14) into (2.11), we obtain

$$
\begin{aligned}
(\Delta_H \omega, \omega)_0 \geq & \sum_{ij} (D_{e_i} u_i, D_{e_i} u_i)_0 - 2 \sum \theta^r_{lk} \theta^r_{ij} (D_{e_l} u_i, D_{e_k} u_j)_0 \\
& + \sum_{r} \sum_{ijlkr} \theta^r_{ij} \theta^r_{lk} (D_{e_l} u_j, D_{e_i} u_k)_0 \\
& - \sum_{ijlkru} \theta^r_{ij} \theta^r_{lk} \theta^u_{il} \theta^u_{st} (D_{e_s} u_j, D_{e_t} u_k)_0 \\
& + \sum_{ijlkru} \theta^r_{ij} \theta^r_{lk} \theta^u_{il} \theta^u_{st} (D_{e_t} u_j, D_{e_s} u_k)_0 + O_1 \\
\geq & (1 - \lambda_1 - 2\lambda_2) \sum_{ij} (D_{e_i} u_j, D_{e_i} u_j)_0 + O_1.
\end{aligned}
$$

Hence we have proved (2.10).

## 3. Application: A vanishing theorem

In this section we will apply Theorem 2.6 to the case where $M$ is the total space of a fiber bundle over a Lie group with a connection whose curvature is "almost left-invariant", showing that if the curvature satisfies certain inequalities, then the 1-st Betti number of the total space must be zero. The novelty here is that no assumption on the fiber is made.

For the problem of finding a connection with prescribed curvature, in general very little is known. Weinstein [30] proved that a fat bundle is not flat. For the special case where $M$ is the total space of a 3-sphere bundle over the 4-sphere, Derdzinski and Rigas [6], using the theory of self-dual connections, showed that if $M$ is a fat bundle, then the fiber bundle must be the Hopf-fibration $S^3 \to S^7 \to S^4$.

### 3.1. Vanishing theorem for a connection on $M$ with prescribed curvature

In this subsection we will first state our main result of this section.

Let $W \to M \to G$ be a Riemannian submersion, where $G$ is a Lie group with a left-invariant metric and fibers are totally geodesic. The horizontal bundle is obtained as follows: if $K$ is the subbundle of $M$ tangent to the fibers, then $H = K^{\perp}$ is the orthogonal complement. Let $e_i$ be a left invariant orthonormal frame for $TG$, then $e_i$ can be lifted to be an orthonormal frame for $H$, which we still denote by $e_i$. Such an orthonormal frame on $M$ will be called a lifted left invariant orthonormal frame.

DEFINITION 3.1. *We say that $H$ has left invariant curvature if $\mu(\cdot,\cdot)$: $H \times H \to TM/H$ can be generated by $\theta^1,\dots,\theta^r$, such that with respect to a lifted left invariant orthonormal frame $e_i$ for $H$,*

$$\theta^r = \sum_{ij} \theta^r_{ij} e^i \wedge e^j \tag{3.1}$$

*where $e_k(\theta^r_{ij}) = 0$.*

*Remark.* If $H$ is Hörmander, then $\theta^r_{ij}$ is constant.

Without loss of generality we will assume that $\theta^r$ are orthonormal.

We first define some quantities associated with $H$ and the left invariant metric on the Lie group $G$.

Let

$$D_{e_i} e_j = \sum_{ij} \Gamma^r_{ij} e_r. \tag{3.2}$$

In the following formula $v, v_l$ will denote elements of $g$, the Lie algebra of $G$, $\omega = \sum_i u_i e^i$. Define

$$\begin{aligned} R_H(\omega) = {} & \sum_{ijlk} \Big( e^i \wedge i(e_j)\big( R\big(D_{e_i}, D_{e_j}\big) - \theta^r_{ij}\theta^r_{lk} R\big(D_{e_l}, D_{e_k}\big)\big)\omega, \omega\Big) \\ & + \sum_{ijlmst} \theta^r_{ij}\theta^r_{st}\big(u_l \Gamma^j_{il}\Gamma^t_{sm} u_m - u_l \Gamma^t_{il}\Gamma^j_{sm} u_m\big) \\ & + \sum_{ijlkrstu} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}\big(\Gamma^m_{lk} - \Gamma^m_{kl}\big)\Gamma^j_{mv}(u_v, u_t) \\ & + \sum_{ijlrst} \theta^r_{ij}\theta^r_{st}\big(\Gamma^l_{is} - \Gamma^l_{si}\big)\big(\Gamma^j_{lm} u_m, u_t\big) - 2 \sum_{ijklmrstuv} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}\Gamma^j_{lm} u_m \Gamma^t_{kv} u_v, \end{aligned} \tag{3.3}$$

and

$$\gamma_1 = \max_{\omega \neq 0} \frac{R_H(\omega)}{(\omega, \omega)}, \tag{3.4}$$

$$\beta_1(\phi) = \max \Big| 2 \sum_{ijlkr} \left( \theta^r_{ij} \theta^r_{lk} D_{e_l}\!\left(e^i \wedge i(e_j)\right) v_k, v \right) + \sum_{ijlkr} \left( \theta^r_{ij} \theta^r_{lk} (\Gamma^m_{lk} - \Gamma^m_{kl}) e^i \wedge i(e_j) v_m, v \right) \Big| \Big/ \left\{ (1-\phi)(v,v) + \phi \sum_l (v_l, v_l)^2 \right\}, \tag{3.5}$$

$$\begin{aligned} \beta_2(\phi) = \max \Big| & \sum_{ijklmrstu} \theta^r_{ij} \theta^r_{st} \theta^u_{is} \theta^u_{lk} \left\{ 2u_{lj} \Gamma^t_{kv} u_v + 2u_{kt} \Gamma^j_{lm} u_m \right\} \\ & - \sum_{ijlkrstuv} \theta^r_{ij} \theta^r_{st} \theta^u_{is} \theta^u_{lk} (\Gamma^m_{lk} - \Gamma^m_{kl})(u_{mj}, u_t) \\ & - \sum_{ijlrst} \theta^r_{ij} \theta^r_{st} \left( \Gamma^l_{is} - \Gamma^l_{si} \right) u_{lj}, u_t \Big) \\ & - \sum_{ijlstr} \theta^r_{ij} \theta^r_{st} \left\{ \Gamma^j_{il} u_t u_{st} + \Gamma^t_{sl} u_l u_{ij} - \Gamma^t_{sl} u_l u_{sj} - \Gamma^j_{sl} u_l u_{it} \right\} \Big| \\ & \Big/ \left\{ (1-\phi) \sum_i u^2_m + \phi \sum_{ij} u^2_{ij} \right\}. \end{aligned} \tag{3.6}$$

Here $\phi$ is a fixed number, $0 < \phi < 1$.

LEMMA 3.1. *$\beta_1(\phi), \beta_2(\phi), \gamma_1$, $(0 < \phi < 1)$ are independent of the choice of the left invariant frame $e_i, \theta^j$.*

*Proof.* The proof is similar to that of Lemma 2.5.

THEOREM 3.2. *If at every point $H$ has non-degeneracy $r > 0$, $1 - \lambda_1 - 2\lambda_2 > 0$ and the following inequalities are satisfied for some $0 < \phi < 1$,*

$$1 - \lambda_1 - 2\lambda_2 - \phi(\beta_1(\phi) + \beta_2(\phi)) \geq 0, \tag{3.7}$$

$$\gamma_1 - (1-\phi)(\beta_1(\phi) + \beta_2(\phi)) \geq 0, \tag{3.8}$$

*then* $\dim H^1(M) \leq m$. *Moreover, if the inequality* (3.8) *is strict, then* $H^1(M) = 0$.

*Remark* 1. Note that no assumption on the fiber is made.

*Remark* 2. $\Delta^1_H$ acts on $\Lambda^1_H(M)$, the space of smooth cross-sections of $H^*$. Observe that $\Lambda^1_H(M)$ has a description independent of $H$: if $p_1\colon M \to B$

is the projection of the fiber bundle, then $H^*$ can be identified with the pulled back bundle $p_1^*T^*B$, so $\wedge_H^1(M)$ is the space of smooth cross-sections of $p_1^*T^*B$.

*Remark* 3. If $H'$ is a connection whose curvature is not left invariant but sufficiently close to the curvature of a left invariant connection $H$ satisfying the conditions in Theorem 3.2, in particular, satisfying the strict inequality (3.8), then $H'(M) = 0$ (cf. Corollary 3.7).

Now we look at the simplest case.

COROLLARY 3.3. *Let $M$ be the total space a fiber bundle $W \to M \to T^m$ over a flat $m$-dimensional tori, $H$ a connection on $M$ with left invariant curvature satisfying* $1 - \lambda_1 - 2\lambda_2 > 0$, *then* $\dim H^1(M) \le m$.

*Proof.* In this case $\beta_1 = \beta_2 = 0$, so the conclusion follows from Theorem 3.2.

### 3.2. Proof of Theorem 3.2

To prove the theorem, we need to compute $(\Delta_H^1\omega, \omega)$, which is quite involved.

LEMMA 3.4. *If $\omega$ is a horizontal 1-form,*

$$\begin{aligned}
\sum_{ijlkr} &\left(\theta_{ij}^r\theta_{lk}^r e^i \wedge i(e_j)R\left(D_{e_l}, D_{e_k}\right)\omega, \omega\right) \\
&= -2\sum_{ijlkr}\left(\theta_{ij}^r\theta_{lk}^r e^i \wedge i(e_j)D_{e_k}\omega, D_{e_l}\omega\right) \\
&\quad - 2\sum_{ijlkr}\left(\theta_{ij}^r\theta_{lk}^r D_{e_l}\left(e^i \wedge i(e_j)\right)D_{e_k}\omega, \omega\right) \\
&\quad - \sum_{ijlmkr}\left(\theta_{ij}^r\theta_{lk}^r(\Gamma_{lk}^m - \Gamma_{kl}^m)e^i \wedge i(e_j)D_{e_m}\omega, \omega\right).
\end{aligned} \tag{3.9}$$

*Proof.* Using the integration by parts, we have

$$\begin{aligned}
\sum_{ijlkr} &\left(\theta_{ij}^r\theta_{lk}^r e^i \wedge i(e_j)R\left(D_{e_l}, D_{e_k}\right)\omega, \omega\right) \\
&= -2\sum_{ijlkr}\left(\theta_{ij}^r\theta_{lk}^r e^i \wedge i(e_j)D_{e_k}\omega, D_{e_l}\omega\right) \\
&\quad - 2\sum_{ijlkr}\left(\theta_{ij}^r\theta_{lk}^r D_{e_l}\left(e^i \wedge i(e_j)\right)D_{e_k}\omega, \omega\right) \\
&\quad - \sum_{ijlmkr}\left(\theta_{ij}^r\theta_{lk}^r e^i \wedge i(e_j)(\Gamma_{lk}^m - \Gamma_{kl}^m)D_{e_m}\omega, \omega\right).
\end{aligned}$$

In the following let $\omega = \Sigma_i u_i e^i$, $u_{ij} = e_i(u_j) + \Sigma_l \Gamma_{il}^j u_l$, so $D_{e_i}\omega = \Sigma u_{ij}e^j$.

LEMMA 3.5.

$$\begin{aligned}
\sum_r (\theta^r, \bar{d}\omega)^2 &= \int \sum_{ijstr} \theta^r_{ij}\theta^r_{st} u_{it} u_{sj} \\
&\quad + \int \sum_{ijklmrstu} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}\Big\{u_{lj}u_{kt} - 2u_{l_j}\Gamma^t_{kv}u_v - 2u_{kt}\Gamma^j_{lm}u_m \\
&\qquad + 2\Gamma^j_{lm}u_m\Gamma^t_{kv}u_v\Big\} \\
&\quad + \int \sum_{ijlkrstu} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}(\Gamma^m_{lk} - \Gamma^m_{kl})\big\{(u_{mj}, u_t) - \big(\Gamma^j_{mv}u_v, u_t\big)\big\} \\
&\quad + \int \sum_{ijlstr} \theta^r_{ij}\theta^r_{st}\big\{\Gamma^j_{il}u_t u_{st} + \Gamma^t_{sl}u_l u_{ij} - \Gamma^t_{sl}u_l u_{sj} - \Gamma^j_{sl}u_l u_{it}\big\} \\
&\quad - \int \sum_{ijlmst} \theta^r_{ij}\theta^r_{st}\big(u_l\Gamma^j_{il}\Gamma^t_{sm}u_m - u_l\Gamma^t_{il}\Gamma^j_{sm}u_m\big).
\end{aligned}$$

*Proof.* By definition,

$$\begin{aligned}
\sum_r (\theta^r, \bar{d}\omega)^2 &= \int \theta^r_{ij}u_{ij}\theta^r_{st}u_{st} \\
&= \int \sum_{rijst} \Big\{\theta^r_{ij}u_{it}\theta^r_{st}u_{sj} - \sum_{ijlmst} \theta^r_{ij}\theta^r_{st}\big(u_l\Gamma^j_{il}\Gamma^t_{sm}u_m - u_l\Gamma^t_{il}\Gamma^j_{sm}u_m\big)\Big\} \\
&\quad + \int \sum_{ijlrst} \theta^r_{ij}\theta^r_{st}\big(u_l e_s(u_t)\Gamma^j_{il} - u_l\Gamma^t_{il}D_{e_s}u_j \\
&\qquad + u_l\Gamma^t_{sl}e_i(u_j) - u_l\Gamma^j_{sl}e_i(u_t)\big) \\
&\quad + \int \sum_{ijrst} \theta^r_{ij}\theta^r_{st}\big(e_i(u_j)e_s(u_t) - e_i(u_t)e_s(u_j)\big) \\
&= I_1 + I_2 + I_3. \qquad (3.10)
\end{aligned}$$

The second term in (3.10) is

$$I_2 = \int \sum_{ijrst} \theta^r_{ij}\theta^r_{st}\big(\mu(e_i, e_s)u_j, u_t\big) + \int \sum_{ijrst} \theta^r_{ij}\theta^r_{st}\big(\pi[e_i, e_s]u_j, u_t\big) = I_{21} + I_{22}.$$

Using the formula

$$\pi[e_i, e_s] = D_{e_i}e_s - D_{e_s}e_i = \sum_l \big(\Gamma^l_{is} - \Gamma^l_{si}\big)e_l,$$

we have

$$I_{22} = \int \sum_{ijlrst} \theta^r_{ij}\theta^r_{st}(\Gamma^l_{is} - \Gamma^l_{si})(u_{lj}, u_t) - \int \sum_{ijlrst} \theta^r_{ij}\theta^r_{st}(\Gamma^l_{is} - \Gamma^l_{si})(\Gamma^j_{lm}u_m, u_t). \tag{3.11}$$

On the other hand, using (1.13),

$$\begin{aligned}
I_{21} &= 2\int \sum_{ijlkrstu} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}\big(e_l(u_j), e_k(u_t)\big) \\
&\quad - \int \sum_{ijlkrstu} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}\big(\pi[e_l, e_k]u_j, u_t\big) \\
&= \int \sum_{ijlkrstu} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}(\Gamma^m_{lk} - \Gamma^m_{kl})\Big(e_m(u_j) + \sum_v \Gamma^j_{mv}u_v, u_t\Big) \\
&\quad - \int \sum_{ijlkrstuv} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}(\Gamma^m_{lk} - \Gamma^m_{kl})(\Gamma^j_{mv}u_v, u_t) \\
&\quad + 2\int \sum_{ijlkrstumv} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}(u_{lj}, u_{kt}) - 2\int \sum_{ijlmkrstuv} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}\big(u_{lj}, \Gamma^t_{kv}u_v\big) \\
&\quad - 2\int \sum_{ijlmkrstuv} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}\big(\Gamma^j_{lm}u_m, u_{kt}\big) \\
&\quad + 2\int \sum_{ijlmkrstuv} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}\big(\Gamma^j_{lm}\Gamma^t_{kv}u_m, u_v\big).
\end{aligned} \tag{3.12}$$

Now we compute the third term in (3.10):

$$\begin{aligned}
I_3 &= \int \sum_{ijrst} \theta^r_{ij}\theta^r_{st}\big\{u_l e_s(u_t)\Gamma^j_{il} - u_l\Gamma^t_{il}D_{e_s}u_j + u_l\Gamma^j_{sl}e_i(u_j) - u_l\Gamma^j_{sl}e_i(u_t)\big\} \\
&= \int \sum_{ijrst} \theta^r_{ij}\theta^r_{st}\big\{(u_l, u_{st})\Gamma^j_{il} - \big(u_l\Gamma^t_{il}, u_{sj}\big) + \big(u_l\Gamma^t_{sl}, u_{ij}\big) - \big(u_l\Gamma^j_{sl}, u_{it}\big)\big\} \\
&\quad - \int \sum_{ijmrst} \theta^r_{ij}\theta^r_{st}\big\{\big(\Gamma^j_{il}u_l, \Gamma^t_{sm}u_m\big) - \big(\Gamma^t_{il}u_l, \Gamma^j_{sm}u_m\big) + \big(\Gamma^t_{sl}u_l, \Gamma^j_{im}u_m\big) \\
&\qquad - \big(\Gamma^j_{sl}u_l, \Gamma^t_{im}u_m\big)\big\}
\end{aligned} \tag{3.13}$$

Insert (3.11), (3.12), (3.13) into (3.10), we prove the lemma.

Now we can write $(\Delta^1_H(\omega), \omega)$ explicitly.

COROLLARY 3.6. *If* $\omega = \Sigma u_i e^i$,

$$(\Delta_H \omega, \omega) = (\omega, \omega)_1 + \sum_{ijlk} \Big(e^i \wedge i(e_j)\Big(\Big(R\big(D_{e_i}, D_{e_j}\big)\Big)$$

$$- \theta^r_{ij}\theta^r_{lk} R\big(D_{e_l}, D_{e_k}\big)\Big)\omega, \omega\Big)$$

$$- 2\sum_{ijlkr} \Big(\theta^r_{ij}\theta^r_{lk} e^i \wedge i(e_j) D_{e_k}\omega, D_{e_l}\omega\Big)$$

$$- 2\sum_{ijlkr} \Big(\theta^r_{ij}\theta^r_{lk} D_{e_l}\big(e^i \wedge i(e_j)\big) D_{e_k}\omega, \omega\Big)$$

$$- \sum_{ijlmkr} \Big(\theta^r_{ij}\theta^r_{lk}(\Gamma^m_{lk} - \Gamma^m_{kl}) e^i \wedge i(e_j) D_{e_m}\omega, \omega\Big) - \int \sum_{ijstr} \theta^r_{ij}\theta^r_{st} u_{it} u_{sj}$$

$$- \int \sum_{ijklmrstu} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}\big\{u_{lj}u_{kt} - 2u_{lj}\Gamma^t_{kv}u_v$$

$$- 2u_{kt}\Gamma^j_{lm}u_m + 2\Gamma^j_{lm}u_m\Gamma^t_{kv}u_v\big\}$$

$$- \int \sum_{ijlrst} \theta^r_{ij}\theta^r_{st}\big(\Gamma^l_{is} - \Gamma^l_{si}\big)\big\{(u_{lj}, u_t) - \big(\Gamma^j_{lm}u_m, u_t\big)\big\}$$

$$- \int \sum_{ijlrstuv} \theta^r_{ij}\theta^r_{st}\theta^u_{is}\theta^u_{lk}(\Gamma^m_{lk} - \Gamma^m_{kl})\big\{(u_{mj}, u_t) - \Gamma^j_{mv}(u_v, u_t)\big\}$$

$$- \int \sum_{ijlstr} \theta^r_{ij}\theta^r_{st}\big\{\Gamma^j_{il}u_l u_{st} + \Gamma^t_{sl}u_l u_{ij} - \Gamma^t_{sl}u_l u_{sj} - \Gamma^j_{sl}u_l u_{it}\big\}$$

$$+ \int \sum_{ijlmst} \theta^r_{ij}\theta^r_{st}\big(u_l\Gamma^j_{il}\Gamma^t_{sm}u_m - u_l\Gamma^t_{il}\Gamma^j_{sm}u_m\big).$$

The following inequality, which is an easy consequence of Corollary 3.6, will complete the proof of Theorem 3.2.

COROLLARY 3.7. *We have the following inequality*:

$$(\Delta_H \omega, \omega) \geq \big(1 - \lambda_1 - 2\lambda_2 - \phi\big(\beta_1(\phi) + \beta_2(\phi)\big)\big)(\omega, \omega)_1$$

$$+ \big(\gamma_1 - (1 - \phi)\big(\beta_1(\phi) + \beta_2(\phi)\big)\big)(\omega, \omega)_0.$$

## References


1. R. Bott and L.W. Tu, *Differential forms in algebraic topology*, Springer-Verlag, New York, 1982.
2. R.W. Brockett, "Control theory and singular Riemannian geometry" in *New Direction in Applied Mathematics* (P.J. Hilton and G.S. Young eds.), Springer, New York, 1981, pp. 11–27.
3. J.A. Carlson and D. Toledo, *Variations of Hodge structure, Legendre submanifolds and accessibility*, Trans. Amer. Math. Soc., vol. 31 (1989), pp. 391–411.
4. S.S. Chern, *Geometry of characteristic classes*, Proceeding of the Thirteenth Biennial Seminar of the Canadian Mathematical Congress., Canad. Math. Congress; Montreal, 1972, pp. 1–40.
5. A.G. Chernjakov, "Obstruction to the existence of the non-holonomic structure on a smooth manifold" in *Topology and Geometry-Rohlin Seminar*, Lecture Notes in Math., vol. 1346, Springer-Verlag, New York, 1988.
6. A. Derdzinski and A. Rigas, *Unflat connection in 3-sphere bundles over $S^4$*, Trans. Amer. Math. Soc., vol. 265 (1981), pp. 485–493.
7. B. Gaveau, *Principe de moindre action, propagation de la chaleur et estimées sous-elliptiques sur certains groupes nilpotent*, Acta Math., vol. 139 (1977), pp. 85–153.
8. Zhong Ge, *On horizontal paths spaces and Carnot-Carathéodory metrics*, preprint.
9. ______, *Collapsing Riemannian metrics to Carnot-Carathéodory metrics and Laplacians to sub-Laplacians*, preprint.
10. P.A. Griffiths, *Periods of integrals on algebraic manifolds, III.*, Publ. Math. I. H. E. S., vol. 38 (1970), pp. 125–180.
11. M. Gromov, *Structures métriques pour les variétés Riemanniennes*, Cedic, Paris, 1981.
12. ______, *Partial differential relations*, Springer-Verglag, New York, 1986.
13. U. Hamenstädt, *On the geometry of Carnot-Carathéodory metrics*, preprint.
14. A. El Kacimi et G. Hector, *Décomposition de Hodge basique pour un feuilletage riemannien pour un feuilletage riemannien*, Ann. Inst. Fourier, vol. 36 (1986), pp. 207–227.
15. F. Kamber and P. Tondeur, *Foliated bundles and characteristic classes*, Lecture Notes in Math., vol. 493, Springer, New York, 1975.
16. ______, "Foliations and metrics" in *Differential Geometry, Progress in Math.*, vol. 32, R. Brooks, A. Gray, B.L. Reinhart, Birkhäuser, 1983.
17. ______, *De Rham-Hodge Theory for Riemannian Foliations*, Math. Ann., vol. 277 (1987), pp. 415–431.
18. A. Korányi, *Geometric properties of Heisenberg-type groups*, Adv. in Math., vol. 56 (1985), pp. 28–38.
19. J. Mitchell, *On Carnot-Carathéodory metrics*, J. Differential Geom., vol. 21 (1985), pp. 35–45.
20. R. Montgomery, *Shortest loops with fixed holonomy*, MSRI, preprint.
21. P. Pansu, Croissance des boules et des géodésique fermeés dans les nilvariétés, Ergodic Theory Dynamics Systems, vol. 3 (1983), pp. 441–443.
22. P. Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math., vol. 129 (1989), pp. 1–60.
23. B.L. Reinhart, *Harmonic integrals on foliated manifolds*, Amer. J. Math., vol. 81 (1959), pp. 529–536.
24. L. Rothschild and E. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math., vol. 137 (1976), pp. 247–320.
25. M. Rumin, *Un complexe de formes differentielles sur les variétés de contact*, C. R. Acad. Sci. Paris, t.310 (1990), serie I., p. 401–404.
26. D.C. Spencer, "Remark on the perturbation of structure" in *Analytical functions*, Princeton Univ. Press, Princeton, N.J., 1960, pp. 67–87.

27. R. Strichartz, *Sub-Riemannian geometry*, J. Differential Geom., vol. 24 (1986), pp. 221–263.
28. R. Thom, *Remarques sur les problems comportant des inequations différentielles globales*, Bull. Soc. Math. France, vol. 87 (1959), pp. 455–461.
29. T. Taylor, *Some aspects of differential geometry associated with hypoelliptic second order operators*, Pacific J. Math., vol. 136 (1989), pp. 355–378.
30. A. Weinstein, *Fat bundles and symplectic manifolds*, Adv. in Math., vol. 37 (1980), pp. 239–250.
31. H. Wu, *The Bochner technique*, Proc. of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, vol. 2, Science Press, Beijing 1982.

University of Arizona
Tucson, Arizona