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# ENDOMORPHISMS OF CERTAIN IRRATIONAL ROTATION C*-ALGEBRAS

BY

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## 1. Preliminaries for quadratic irrational numbers

First we will give definitions and well known facts on quadratic irrational numbers. Let $GL(2, \mathbf{Z})$ be the group of all $2 \times 2$-matrices over $\mathbf{Z}$ with determinant $\pm 1$. Let

$$g = \begin{bmatrix} k & l \\ m & n \end{bmatrix} \in GL(2, \mathbf{Z})$$

and $\theta$ be an irrational number. We define

$$g\theta = \frac{m + n\theta}{k + l\theta}$$

and we call $g$ a *fractional transformation*. Furthermore if

$$g \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

then we say that $g$ is *non-trivial*.

Let $\mathbf{Q}$ be the ring of rational numbers. We suppose that $\theta$ is a quadratic irrational number. If $\theta = x + y\sqrt{d}$ where $x, y \in \mathbf{Q}$ and $d \in \mathbf{N}$, then we define $\theta' = x - y\sqrt{d}$ and we call $\theta'$ the *conjugate* of $\theta$. We say that $\theta$ is reduced if $\theta > 1$ and $-1 < \theta' < 0$ where $\theta'$ is the conjugate of $\theta$.

For any quadratic irrational number $\theta$ there are a fractional transformation

$$g = \begin{bmatrix} k & l \\ m & n \end{bmatrix} \in GL(2, \mathbf{Z})$$

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and a reduced quadratic irrational number $\theta_1$ such that

$$\theta = g\theta_1 = \frac{m + n\theta_1}{k + l\theta_1}.$$

And for any reduced quadratic irrational number $\theta_1$ there is a non-trivial fractional transformation $h \in GL(2, \mathbf{Z})$ such that $\theta_1 = h\theta_1$. Hence since $\theta_1 = g^{-1}\theta$, we can see that

$$\theta = g\theta_1 = gh\theta_1 = ghg^{-1}\theta.$$

Since $h$ is non-trivial, neither is $ghg^{-1}$. By the above arguments we see that for any quadratic irrational number $\theta$ there is a non-trivial fractional transformation

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbf{Z})$$

such that

$$\theta = \frac{c + d\theta}{a + b\theta}.$$

Furthermore if $a + b\theta > 1$ or $a + b\theta < 0$, we can choose another non-trivial fractional transformation

$$g_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in GL(2, \mathbf{Z})$$

such that

$$\theta = \frac{c_1 + d_1\theta}{a_1 + b_1\theta}, \quad 0 < a_1 + b_1\theta < 1,$$

by an easy computation. Therefore if $\theta$ is a quadratic irrational number, there is a non-trivial fractional transformation

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbf{Z})$$

such that

$$\theta = \frac{c + d\theta}{a + b\theta}, \quad 0 < a + b\theta < 1.$$

Let $\theta$ be a quadratic irrational number and

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbf{Z})$$

be as above. We will show that $\theta$ has its discriminant $D = 5$ if and only if $\theta$ satisfies that there are integers $s, t$ such that

$$\begin{bmatrix} 1-a & -b \\ s & t \end{bmatrix} \in GL(2, \mathbf{Z}) \quad \text{and} \quad \theta = \frac{s + t\theta}{(1-a) - b\theta}.$$

LEMMA 1. *Let $\theta$ be a quadratic irrational number and*

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbf{Z})$$

*be a non-trivial fractional transformation such that*

$$\theta = \frac{c + d\theta}{a + b\theta}, \quad 0 < a + b\theta < 1.$$

*Then the following conditions are equivalent*:

(i) *There are integers $s, t$ such that*

$$\begin{bmatrix} 1-a & -b \\ s & t \end{bmatrix} \in GL(2, \mathbf{Z}) \quad \textit{and} \quad \theta = \frac{s + t\theta}{(1-a) - b\theta}.$$

(ii) *If $ad - bc = 1$, then $a + d = 1$ or $3$ and if $ad - bc = -1$, then $a + d = \pm 1$.*

*Proof.* Suppose condition (i) holds. Then

$$b\theta^2 - (1 - a - t)\theta + s = 0. \tag{1}$$

Since $\theta = (c + d\theta)/(a + b\theta)$,

$$b\theta^2 + (a - d)\theta - c = 0. \tag{2}$$

By (1), (2) we obtain

$$(1 - t - d)\theta - (s + c) = 0. \tag{3}$$

Since $\theta$ is irrational, by (3) we have $t = 1 - d$ and $s = -c$. Furthermore since

$$\begin{bmatrix} 1-a & -b \\ s & t \end{bmatrix} \in GL(2,\mathbf{Z}),$$

we have

$$(1-a)(1-d) - bc = \pm 1.$$

Thus we see that $a + d = 1 + ad - bc \pm 1$. Therefore we obtain condition (ii).

Next, suppose condition (ii) holds. Then by easy computation we can see that

$$\begin{bmatrix} 1-a & -b \\ -c & 1-d \end{bmatrix} \in GL(2,\mathbf{Z}) \quad \text{and} \quad \frac{-c + (1-d)\theta}{(1-a) - b\theta} = \theta, \quad \text{Q.E.D.}$$

The quadratic equation for $\theta$ can be written in the form

$$k\theta^2 + l\theta + m = 0$$

where $k, l, m$ are relatively prime integers and $k > 0$. Let $D = l^2 - 4km > 0$ be the discriminant of $\theta$. Let

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbf{Z})$$

be a non-trivial fractional transformation such that

$$\theta = \frac{c + d\theta}{a + b\theta}, \quad 0 < a + b\theta < 1.$$

The $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \dfrac{t + ls}{2} & ks \\ -ms & \dfrac{t - ls}{2} \end{bmatrix}$$

where $s, t$ are integers such that

$$t^2 - Ds^2 = 4 \quad \text{if } ad - bc = 1$$

or

$$t^2 - Ds^2 = -4 \quad \text{if } ad - bc = -1.$$

LEMMA 2. *Let $\theta$ be a quadratic irrational number and $k\theta^2 + l\theta + m = 0$ be its quadratic equation where $k, l, m$ are relatively prime integers and $k > 0$. Let*

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbf{Z})$$

*be a non-trivial fractional transformation such that*

$$\theta = \frac{c + d\theta}{a + b\theta}, \quad 0 < a + b\theta < 1.$$

*If $\theta$ and $g$ satisfy condition* (ii) *in Lemma* 1, *then the discriminant $D$ of $\theta$ is equal to* 5.

*Proof.* We have the following fact:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{t + ls}{2} & ks \\ -ms & \frac{t - ls}{2} \end{bmatrix}$$

where $s, t$ are integers such that

$$t^2 - Ds^2 = 4 \quad \text{if } ad - bc = 1$$

or

$$t^2 - Ds^2 = -4 \quad \text{if } ad - bc = -1.$$

We suppose that $ad - bc = 1$ and $a + d = 1$. Then $t = 1$. Hence $Ds^2 = -3$. This is a contradiction since $D > 0$. We suppose that $ad - bc = 1$ and $a + d = 3$. Then $t = 3$. Hence $Ds^2 = 5$. Since $D > 0$ and $s$ is an integer, $s = \pm 1$ and $D = 5$. We suppose that $ad - bc = -1$ and $a + d = \pm 1$. Then $t = \pm 1$. Hence $Ds^2 = 5$. Since $D > 0$ and $s$ is an integer, $s = \pm 1$ and $D = 5$, Q.E.D.

LEMMA 3. *Let $\theta$ be a quadratic irrational number and $k\theta^2 + l\theta + m = 0$ be its quadratic equation where $k, l, m$ are relatively prime integers and $k > 0$. If the discriminant $D$ of $\theta$ is equal to* 5, *then there is a non-trivial fractional transformation*

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbf{Z})$$

*satisfying*

$$\theta = \frac{c + d\theta}{a + b\theta}, \quad 0 < a + b\theta < 1$$

*and condition* (ii) *in Lemma* 1.

*Proof.* Since $k\theta^2 + l\theta + m = 0$ and $D = 5, \theta = (-l \pm \sqrt{5})/2k$. Since $D$ is odd, so is $l$. Let $l = 2l_1 - 1$ where $l_1$ is an integer. Then $l_1^2 - l_1 - km = 1$ since $D = 5$. And

$$\theta = \frac{-2l_1 + 1 \pm \sqrt{5}}{2k}.$$

We suppose that

$$\theta = \frac{-2l_1 + 1 + \sqrt{5}}{2k}.$$

Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} l_1 - 1 & k \\ -m & -l_1 \end{bmatrix}.$$

Then

$$ad - bc = (l_1 - 1)(-l_1) + km = -\left(l_1^2 - l_1 - km\right) = -1$$

since $l_1^2 - l_1 - km = 1$. Hence

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbf{Z}) \quad \text{and} \quad a + d = -1.$$

Since $m = -l\theta - k\theta^2$ and $l = 2l_1 - 1$,

$$\begin{aligned} \frac{c + d\theta}{a + b\theta} &= \frac{-m - l_1\theta}{l_1 - 1 + k\theta} \\ &= \frac{(2l_1 - 1)\theta + k\theta^2 - l_1\theta}{l_1 - 1 + k\theta} \\ &= \frac{((l_1 - 1) + k\theta)\theta}{l_1 - 1 + k\theta} = \theta. \end{aligned}$$

Furthermore

$$a + b\theta = l_1 - 1 + k\frac{-2l_1 + 1 + \sqrt{5}}{2k} = \frac{-1 + \sqrt{5}}{2}.$$

Hence $0 < a + b\theta < 1$. We suppose that $\theta = (-2l_1 + 1 - \sqrt{5})/2k$. Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -l_1 & -k \\ m & l_1 - 1 \end{bmatrix}.$$

Then

$$ad - bc = -l_1(l_1 - 1) + km = -\left(l_1^2 - l_1 - km\right) = -1$$

since $l_1^2 - l_1 - km = 1$. Hence

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbf{Z}) \quad \text{and} \quad a + d = -1.$$

Since $m = -l\theta - k\theta^2$ and $l = 2l_1 - 1$,

$$\frac{c + d\theta}{a + b\theta} = \frac{m + (l_1 - 1)\theta}{-l_1 - k\theta} = \frac{(-l_1 - k\theta)\theta}{-l_1 - k\theta} = \theta.$$

Furthermore

$$a + b\theta = -l_1 - k\frac{-2l_1 + 1 - \sqrt{5}}{2k} = \frac{-1 + \sqrt{5}}{2}.$$

Hence $0 < a + b\theta < 1$. Therefore we obtain the conclusion. Q.E.D.

*Remark* 4. By the above proof we can take

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbf{Z})$$

so that

$$\frac{1}{2} < a + b\theta = \frac{-1 + \sqrt{5}}{2} < \frac{2}{3}, \; a + d = -1 \text{ and } ad - bc = -1.$$

PROPOSITION 5. *Let $\theta$ be a quadratic irrational number and $k\theta^2 + l\theta + m = 0$ be its quadratic equation. Then the discriminant $D$ of $\theta$ is equal to 5 if and only if there is a non-trivial fractional transformation $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbf{Z})$ satisfying*

$$\theta = \frac{c + d\theta}{a + b\theta}, \quad 0 < a + b\theta < 1$$

*and condition* (i) *in Lemma* 1.

*Proof.* This is immediate by Lemmas 1, 2, and 3. Q.E.D.

## 2. Construction of endomorphisms of certain irrational rotation $C^*$-algebras

Let $A$ be a unital $C^*$-algebra and $M_n(A)$ be the algebra of all $n \times n$-matrices over $A$ for any $n \in \mathbf{N}$ and we identify $M_n(A)$ with $A \otimes M_n(\mathbf{C})$. Let $I_n$ be the unit element in $M_n(\mathbf{C})$. For any unitary element $x \in M_n(A)$ we denote by $[x]$ the corresponding class in $K_1(A)$.

Let $\theta$ be an irrational number and $A_\theta$ be the corresponding irrational rotation $C^*$-algebra. Let $u$ and $v$ be unitary elements in $A_\theta$ with $uv = e^{2\pi i\theta}vu$ which generate $A_\theta$. Then it is well known that $K_1(A_\theta) = \mathbf{Z}[u] \oplus \mathbf{Z}[v]$. Let $\tau$ be the unique tracial state on $A_\theta$. We extend $\tau$ to the unnormalized finite trace on $M_n(A_\theta)$. We also denote it by $\tau$. Let $m$ and $l$ be integers which generate $\mathbf{Z}$ with $m + l\theta \neq 0$. We also assume $l \neq 0$. Let $V_\theta(m, l : k)$ be the standard module defined in Rieffel [7] where $k \in \mathbf{N}$. Since $V_\theta(m, l : k)$ is a finitely generated projective right $A_\theta$-module, it corresponds to a projection in some $M_n(A_\theta)$. We also denote it by $V_\theta(m, l : k)$. Moreover throughout this paper we assume that endomorphisms of $A_\theta$ are unital.

LEMMA 6. *With the above notations let $f$ be a projection in $M_n(A_\theta)$ where $n$ is a positive integer. Then $\tau(V_\theta(m, l : k)) = \tau(f)$ if and only if $V_\theta(m, l : k)$ is isomorphic to $fA_\theta^n$ as a module.*

*Proof.* It is clear that $\tau(V_\theta(m, l : k)) = \tau(f)$ if $V_\theta(m, l : k)$ is isomorphic to $fA_\theta^n$. Suppose that $\tau(V_\theta(m, l : k)) = \tau(f)$. Then by Rieffel [7, Corollary 2.5], $V_\theta(m, l : k)$ is isomorphic to $fA_\theta^n$, Q.E.D.

From now on we suppose that $\theta$ is a quadratic irrational number with its discriminant $D = 5$. Then by Proposition 5 there is a non-trivial fractional transformation

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbf{Z})$$

such that

$$\theta = \frac{c + d\theta}{a + b\theta}, \quad 0 < a + b\theta < 1$$

and there are integers $s, t$ such that

$$\begin{bmatrix} 1 - a & -b \\ s & t \end{bmatrix} \in GL(2, \mathbf{Z}) \quad \text{and} \quad \theta = \frac{s + t\theta}{(1 - a) - b\theta}.$$

Moreover by Rieffel [6, Theorem 1] there is a projection $q \in A_\theta$ such that $\tau(q) = a + b\theta$.

LEMMA 7. *With the above notations, $qA_\theta q$ is isomorphic to $A_\theta$.*

*Proof.* Since $qA_\theta q$ is a full corner of $A_\theta$, it is strongly Morita equivalent to $A_\theta$ and $qA_\theta$ is the $qA_\theta q - A_\theta$-equivalence bimodule. By Rieffel [7, Theorem 1.4],

$$\tau(V_\theta(a, b : 1)) = a + b\theta.$$

On the other hand by the assumption $\tau(q) = a + b\theta$. Hence by Lemma 6, $qA_\theta$ is isomorphic to $V_\theta(a, b : 1)$ as a module. Thus by Rieffel [7, Theorem 1.1 and Corollary 2.6], $qA_\theta q$ is isomorphic to $A_\eta$ where $\eta = (c + d\theta/a + b\theta)$. However by the assumptions, $\theta = (c + d\theta)/(a + b\theta)$. Therefore $qA_\theta q$ is isomorphic to $A_\theta$, Q.E.D.

LEMMA 8. *With the above notations $q^\perp A_\theta q^\perp$ is isomorphic to $A_\theta$ where $q^\perp = 1 - q$.*

*Proof.* In the same way as in the above lemma, $q^\perp A_\theta q^\perp$ is isomorphic to $A_\eta$ where

$$\eta = \frac{s + t\theta}{(1 - a) - b\theta}.$$

However by the assumptions,

$$\theta = \frac{s + t\theta}{(1 - a) - b\theta}.$$

Therefore $q^\perp A_\theta q^\perp$ is isomorphic to $A_\theta$, Q.E.D.

We denote by $\phi_1$ an isomorphism of $A_\theta$ onto $qA_\theta q$ and by $\phi_2$ an isomorphism of $A_\theta$ onto $q^\perp A_\theta q^\perp$. Let $\phi$ be an endomorphism defined by $\phi(x) = \phi_1(x) + \phi_2(x)$ for any $x \in A_\theta$. We consider an endomorphism $\phi^3 = \phi \circ \phi \circ \phi$ of $A_\theta$. We denote it by $\Phi$. Then by an easy computation we can see that there are an orthogonal family $\{p_j\}_{j=1}^8$ of projections in $A_\theta$ with $\Sigma_{j=1}^8 p_j = 1$ and isomorphisms $\chi_j$ $(j = 1, 2, \ldots, 8)$ of $A_\theta$ onto $p_j A_\theta p_j$ such that $\Phi = \Sigma_{j=1}^8 \chi_j$.

For $j = 1, 2, \ldots, 8$ let $\psi_j$ be the isomorphism of $K_1(p_j A_\theta p_j)$ onto $K_1(A_\theta)$ defined by

$$\psi_j([x]) = \left[x + (1 - p_j) \otimes I_n\right]$$

for any unitary element $x \in M_n(p_j A_\theta p_j)$.

LEMMA 9. *With the above notations,* $\Phi_* = \Sigma_{j=1}^8 \psi_j \circ \chi_{j*}$ *on* $K_1(A_\theta)$.

*Proof.*

$$\begin{aligned}[\Phi(u)] &= \left[\sum_{j=1}^{8} \chi_j(u)\right] = \left[\prod_{j=1}^{8} \left(\chi_j(u) + (1-p_j)\right)\right] \\ &= \sum_{j=1}^{8} \left[\chi_j(u) + (1-p_j)\right] \\ &= \sum_{j=1}^{8} \psi_j\left(\left[\chi_j(u)\right]\right) = \sum_{j=1}^{8} (\psi_j \circ \chi_{j*})([u]).\end{aligned}$$

Similarly

$$[\Phi(v)] = \sum_{j=1}^{8} (\phi_j \circ \chi_{j*})([v]).$$

Therefore we obtain the conclusion. Q.E.D.

Let $SL(2,\mathbf{Z})$ be the group of $2 \times 2$-matrices over $\mathbf{Z}$ with determinant 1. For any

$$h = \begin{bmatrix} k & l \\ m & n \end{bmatrix} \in SL(2,\mathbf{Z})$$

let $\alpha_h$ be the automorphism of $A_\theta$ defined by

$$\alpha_h(u) = u^k v^m, \quad \alpha_h(v) = u^l v^n.$$

Furthermore for any $h \in GL(2,\mathbf{Z})$ let $\det(h)$ be its determinant.

THEOREM 10. *With the above notations there is an endomorphism* $\Phi_0$ *of* $A_\theta$ *with* $\Phi_{0*}$ *an arbitrary endomorphism of* $K_1(A_\theta)$.

*Proof.* By Lemma 9 there is an endomorphism $\Phi = \Sigma_{j=1}^8 \chi_j$ of $A_\theta$ such that

$$\Phi_* = \sum_{j=1}^{8} \psi_j \circ \chi_{j*} \quad \text{on } K_1(A_\theta).$$

Since $\psi_j \circ \chi_{j*}$ is an automorphism of $K_1(A_\theta)$ for $j = 1, 2, \ldots, 8$, there is an element

$$h_j = \begin{bmatrix} k_j & l_j \\ m_j & n_j \end{bmatrix} \in GL(2,\mathbf{Z})$$

such that

$$(\psi_j \circ \chi_{j*})([u]) = k_j[u] + m_j[v],$$
$$(\psi_j \circ \chi_{j*})([v]) = l_j[u] + n_j[v].$$

For $g_1, g_2, \dots, g_8 \in SL(2, \mathbf{Z})$ let $\Phi_{g_1, g_2, \dots, g_8}$ be an endomorphism of $A_\theta$ defined by

$$\Phi_{g_1, g_2, \dots, g_8} = \sum_{j=1}^{8} \chi_j \circ \alpha_{g_j}.$$

Then, as in Lemma 9,

$$\Phi_{g_1, g_2, \dots, g_8 *} = \sum_{j=1}^{8} \psi_j \circ \chi_{j *} \circ \alpha_{g_j *}$$

on $K_1(A_\theta)$. Since $K_1(A_\theta) \cong \mathbf{Z}^2$, we can consider $\alpha_{g_j *}$ and $\psi_j \circ \chi_{j*}$ ($j = 1, 2, \dots, 8$) as elements in $GL(2, \mathbf{Z})$. Then since $\alpha_{g_j *} = g_j \in SL(2, \mathbf{Z})$ and

$$\psi_j \circ \chi_{j *} = h_j \in GL(2, \mathbf{Z}) \ (j = 1, 2, \dots, 8),$$

we can easily see that

$$\Phi_{g_1, g_2, \dots, g_8 *} = \sum_{j=1}^{8} h_j g_j.$$

For any $r_1$, $r_2$, $r_3$ and $r_4 \in \mathbf{Z}$ we can find elements $g_1, g_2, \dots, g_8 \in SL(2, \mathbf{Z})$ satisfying

$$(1)\ h_1 g_1 + h_2 g_2 = \begin{cases} \begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } \det(h_1 h_2) = 1 \\ \begin{bmatrix} r_1 & 2 \\ 0 & 0 \end{bmatrix} & \text{if } \det(h_1 h_2) = -1, \end{cases}$$

$$(2)\ h_3 g_3 + h_4 g_4 = \begin{cases} \begin{bmatrix} 0 & r_2 \\ 0 & 0 \end{bmatrix} & \text{if } \det(h_3 h_4) = 1 \\ \begin{bmatrix} 0 & r_2 \\ 2 & 0 \end{bmatrix} & \text{if } \det(h_3 h_4) = -1, \end{cases}$$

$$(3)\ h_5 g_5 + h_6 g_6 = \begin{cases} \begin{bmatrix} 0 & 0 \\ r_3 & 0 \end{bmatrix} & \text{if } \det(h_5 h_6) = 1 \\ \begin{bmatrix} 0 & 2 \\ r_3 & 0 \end{bmatrix} & \text{if } \det(h_5 h_6) = -1, \end{cases}$$

$$(4)\ h_7 g_7 + h_8 g_8 = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & r_4 \end{bmatrix} & \text{if } \det(h_7 h_8) = 1 \\ \begin{bmatrix} 0 & 0 \\ 2 & r_4 \end{bmatrix} & \text{if } \det(h_7 h_8) = -1. \end{cases}$$

Hence we can see that for any $z \in M_2(\mathbf{Z})$ there are $g_{1,0}, g_{2,0}, \ldots, g_{8,0} \in SL(2, \mathbf{Z})$ such that $(\Phi_{g_{1,0}, g_{2,0}, \ldots, g_{8,0}})_* = z$ on $K_1(A_\theta)$ by (1), (2), (3) and (4) where $M_2(\mathbf{Z})$ is the set of all $2 \times 2$-matrices over $\mathbf{Z}$. Let $\Phi_0 = \Phi_{g_{1,0}, g_{2,0}, \ldots, g_{8,0}}$. Then we obtain the conclusion. Q.E.D.

## 3. The minimizing index for a C*-subalgebra

Let $\theta$ and $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbf{Z})$ be as in Section 2. Then by Remark 4 we can assume that

$$\frac{1}{2} < a + b\theta = \frac{-1 + \sqrt{5}}{2} < \frac{2}{3},$$

$a + d = -1$ and $ad - bc = -1$. Let $q$ be a projection in $A_\theta$ with $\tau(q) = a + b\theta$ and let $\phi_1$ (resp. $\phi_2$) be the isomorphism of $A_\theta$ onto $qA_\theta q$ (resp. $q^\perp A_\theta q^\perp$) as in Section 2. Let $\phi$ be the endomorphism of $A_\theta$ defined by $\phi(x) = \phi_1(x) + \phi_2(x)$ for any $x \in A_\theta$.

Let $E_1$ be the linear map of $A_\theta$ onto $\phi(A_\theta)$ defined by

$$E_1(x) = qxq + \phi_2(\phi_1^{-1}(qxq))$$

for any $x \in A_\theta$ and let $E_2$ be the linear map of $A_\theta$ onto $\phi(A_\theta)$ defined by

$$E_2(x) = \phi_1(\phi_2^{-1}(q^\perp xq^\perp)) + q^\perp xq^\perp$$

for any $x \in A_\theta$. By an easy computation we can see that $E_1$ and $E_2$ are conditional expectations of $A_\theta$ onto $\phi(A_\theta)$. Furthermore let $E = \frac{1}{2}(E_1 + E_2)$. Then by an easy computation we see that $E$ is a faithful conditional expectation of $A_\theta$ onto $\phi(A_\theta)$.

LEMMA 11. *There is a unitary element $w \in A_\theta$ such that $q \geq w^* q^\perp w$.*

*Proof.* Let $\tau_1$ be the unique tracial state on $qA_\theta q$. Then by Rieffel [6, Theorem 1],

$$\tau_1(\operatorname{Proj}(qA_\theta q)) = \mathbf{Z} + \mathbf{Z}\theta \cap [0, 1]$$

since $qA_\theta q$ is isomorphic to $A_\theta$ where $\operatorname{Proj}(qA_\theta q)$ is the set of all projections in $qA_\theta q$. On the other hand since $qA_\theta q$ is a $C^*$-subalgebra of $A_\theta$, by the uniqueness of the tracial state,

$$\tau_1 = \tau(q)^{-1}\tau = \frac{1}{a + b\theta}\tau.$$

Hence

$$\frac{1}{a+b\theta}\tau(\mathrm{Proj}(qA_\theta q)) = \mathbf{Z} + \mathbf{Z}\theta \cap [0,1].$$

We claim that there is a projection $\tilde{q} \in qA_\theta q$ such that $\tau(\tilde{q}) = \tau(q^\perp) = 1 - a - b\theta$. In fact it is sufficient to show that there are $m, n \in \mathbf{Z}$ such that

$$(a+b\theta)(m+n\theta) = 1 - a - b\theta, \quad 0 < m + n\theta < 1.$$

By the assumption on $\theta$, $\theta = (c + d\theta)/(a + b\theta)$. Thus

$$b\theta^2 = (d-a)\theta + c.$$

Let $m = -(d+1)$ and $n = b$. Then by the above equation

$$\begin{aligned}(a+b\theta)(m+n\theta) &= (a+b\theta)(-(d+1)+b\theta)\\ &= -ad - a + (ab - bd - b)\theta + b^2\theta^2\\ &= bc - ad - a - b\theta.\end{aligned}$$

Since $ad - bc = -1$, $(a + b\theta)(m + n\theta) = 1 - a - b\theta$. Moreover

$$m + n\theta = -(d+1) + b\theta = a + b\theta$$

since $a + d = -1$. Hence $0 < m + n\theta < 1$. Thus there is a projection $\tilde{q} \in qA_\theta q$ such that

$$\tau(\tilde{q}) = 1 - a - b\theta.$$

By Rieffel [7, Corollary 2.5] there is a unitary element $w \in A_\theta$ such that $q^\perp = w\tilde{q}w^*$. Hence $w^*q^\perp w = \tilde{q} \le q$. Q.E.D.

LEMMA 12. *There are a projection $\bar{q} \in A_\theta$ and a unitary element $z \in A_\theta$ such that $z\bar{q}z^* = q - w^*q^\perp w$ and $\bar{q} \le q^\perp$.*

*Proof.* Let $\mathrm{Proj}(q^\perp A_\theta q^\perp)$ be the set of all projections in $q^\perp A_\theta q^\perp$. We will find a projection $\bar{q} \in q^\perp A_\theta q^\perp$ such that $\tau(\bar{q}) = \tau(q - w^*q^\perp w)$. In the same way as in Lemma 11, we can see that

$$\frac{1}{1-a-b\theta}\tau(\mathrm{Proj}(q^\perp A_\theta q^\perp)) = \mathbf{Z} + \mathbf{Z}\theta \cap [0,1].$$

Hence it is sufficient to show that there are $m, n \in \mathbf{Z}$ such that

$$\begin{aligned}(1-a-b\theta)(m+n\theta) &= \tau(q - w^*q^\perp w)\\ &= (2a-1) + 2b\theta, \quad 0 < m + n\theta < 1.\end{aligned}$$

By the assumption on $\theta, \theta = (c + d\theta)/(a + b\theta)$. Thus

$$b\theta^2 = (d - a)\theta + c.$$

Let $m = -(d + 1)$ and $n = b$. Then by the above equation

$$\begin{aligned}(1 - a - b\theta)(m + n\theta) &= (1 - a - b\theta)(-(d + 1) + b\theta)\\ &= -(1 - a)(d + 1) + (d - a + 2)b\theta - b^2\theta^2\\ &= ad - bc + a - d - 1 + 2b\theta.\end{aligned}$$

Since $ad - bc = -1$ and $a + d = -1$, $(1 - a - b\theta)(m + n\theta) = 2a - 1 + 2b\theta$. Moreover

$$m + n\theta = -(d + 1) + b\theta = a + b\theta$$

since $a + d = -1$. Hence $0 < m + n\theta < 1$. Thus there is a projection $\bar{q} \in q^{\perp}A_\theta q^{\perp}$ such that $\tau(\bar{q}) = (2a - 1) + 2b\theta$. By Rieffel [7, Corollary 2.5] there is a unitary element $z \in A_\theta$ such that $z\bar{q}z^* = q - w^*q^{\perp}w$. Therefore we obtain the conclusion. Q.E.D.

THEOREM 13. *Let $q, q^{\perp}, \bar{q}$ and $w, z$ be as in Lemmas* 11 *and* 12. *A family*

$$\{(2q, q), (2q^{\perp}, q^{\perp}), (2q^{\perp}wq, qw^*), (2w^*q^{\perp}, q^{\perp}w), (2z\bar{q}, q^{\perp}z^*)\}$$

*is a quasi-basis for $E$ where $E$ is the conditional expectation defined in the beginning of this section.*

*Proof.* We will show that for any $x \in A_\theta$,

$$\begin{aligned}x &= 2qE(qx) + 2q^{\perp}E(q^{\perp}x) + 2q^{\perp}wqE(qw^*x)\\ &\quad + 2w^*q^{\perp}E(q^{\perp}wx) + 2z\bar{q}E(q^{\perp}z^*x)\\ &= 2E(xq)q + 2E(xq^{\perp})q^{\perp} + 2E(xq^{\perp}wq)qw^*\\ &\quad + 2E(xw^*q^{\perp})q^{\perp}w + 2E(xz\bar{q})q^{\perp}z^*.\end{aligned}$$

For any $x \in A_\theta$ we see by an easy computation that

$$qE(qx) = \frac{1}{2}qxq,$$

$$q^{\perp}E(q^{\perp}x) = \frac{1}{2}q^{\perp}xq^{\perp},$$

$$q^{\perp}wqE(qw^*x) = \frac{1}{2}q^{\perp}wqw^*xq = \frac{1}{2}q^{\perp}xq$$

since $q \geq w^*q^{\perp}w$ by Lemma 11. And we see that

$$w^*q^{\perp}E(q^{\perp}wx) = \frac{1}{2}w^*q^{\perp}wxq^{\perp},$$

$$z\bar{q}E(q^{\perp}z^*x) = \frac{1}{2}\left(z\bar{q}\phi_1\left(\phi_2^{-1}(q^{\perp}z^*xq^{\perp})\right) + z\bar{q}q^{\perp}z^*xq^{\perp}\right).$$

Since $\bar{q} \leq q^{\perp}$ by Lemma 12, $\bar{q}q^{\perp} = \bar{q}$ and $\bar{q}q = 0$. Thus we obtain

$$z\bar{q}E(q^{\perp}z^*x) = \frac{1}{2}z\bar{q}z^*xq^{\perp} = \frac{1}{2}(q - w^*q^{\perp}w)xq^{\perp}$$

since $z\bar{q}z^* = q - w^*q^{\perp}w$ by Lemma 12. Therefore

$$qE(qx) + q^{\perp}E(q^{\perp}x) + q^{\perp}wqE(qw^*x) + w^*q^{\perp}E(q^{\perp}wx) + z\bar{q}E(q^{\perp}z^*x) = \frac{1}{2}x.$$

Next for any $x \in A_\theta$,

$$E(xq)q = \frac{1}{2}qxq,$$

$$E(xq^{\perp})q^{\perp} = \frac{1}{2}q^{\perp}xq^{\perp},$$

$$E(xq^{\perp}wq)qw^* = \frac{1}{2}qxq^{\perp}wqw^* = \frac{1}{2}qxq^{\perp}$$

since $q \geq w^*q^{\perp}w$ by Lemma 11. And we see that

$$E(xw^*q^{\perp})q^{\perp}w = \frac{1}{2}q^{\perp}xw^*q^{\perp}w,$$

$$E(xz\bar{q})q^{\perp}z^* = \frac{1}{2}q^{\perp}xz\bar{q}q^{\perp}z^* = \frac{1}{2}q^{\perp}xz\bar{q}z^*$$

since $\bar{q} \leq q^{\perp}$ by Lemma 12. Thus since $z\bar{q}z^* = q - w^*q^{\perp}w$ by Lemma 12,

$$E(xz\bar{q})q^{\perp}z^* = \frac{1}{2}q^{\perp}x(q - w^*q^{\perp}w).$$

Therefore

$$E(xq)q + E(xq^{\perp})q^{\perp} + E(xq^{\perp}wq)qw^* + E(xw^*q^{\perp})q^{\perp}w + E(xz\bar{q})q^{\perp}z^* = \frac{1}{2}x,$$

Q.E.D.

COROLLARY 14. *Let $E$ be the conditional expectation in the beginning of this section. Then* Index $E = 4$.

*Proof.*

$$\begin{aligned}\text{Index } E &= 2q + 2q^{\perp} + 2q^{\perp}wqw^* + 2w^*q^{\perp}w + 2z\bar{q}q^{\perp}z^* \\ &= 2 + 2q^{\perp} + 2w^*q^{\perp}w + 2z\bar{q}z^* \\ &= 2 + 2q^{\perp} + 2w^*q^{\perp}w + 2(q - w^*q^{\perp}w) \\ &= 4\end{aligned}$$

by Lemmas 11 and 12, Q.E.D.

Let $\varepsilon_0(A_\theta, \phi(A_\theta))$ be the set of all expectations of $A_\theta$ onto $\phi(A_\theta)$ of index-finite type. We define the minimizing index $[A_\theta : \phi(A_\theta)]_0$ by

$$[A_\theta : \phi(A_\theta)]_0 = \min\{\text{Index } F | F \in \varepsilon_0(A_\theta, \phi(A_\theta))\}.$$

COROLLARY 15. *With the above notations*, $[A_\theta : \phi(A_\theta)]_0 = 4$.

*Proof.* This is immediate by Corollary 14 and Watatani [8, Theorem 2, 12.3].

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