A NEW CHARACTERIZATION OF DIRICHLET TYPE SPACES AND APPLICATIONS

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1. Introduction

Let \( D \) be the unit disk of the complex plane \( \mathbb{C} \) and \( dA(z) = 1/\pi \, dx \, dy \) be the normalized Lebesgue measure on \( D \). For \( \alpha < 1 \), let

\[
dA_\alpha(z) = (2 - 2\alpha)(1 - |z|^2)^{1-2\alpha} \, dA(z).
\]

The Sobolev space \( L^{2,\alpha} \) is the Hilbert space of functions \( u: D \to \mathbb{C} \), for which the norm

\[
\|u\| = \left( \int_D u \, dA_\alpha(z) \right)^2 + \int_\Delta \left( |\partial u / \partial z|^2 + |\partial u / \partial \bar{z}|^2 \right) dA_\alpha(z)
\]

is finite. The space \( D_\alpha \) is the subspace of all analytic functions in \( L^{2,\alpha} \). This scale of spaces includes the Dirichlet type spaces (\( \alpha > 0 \)), the Hardy space (\( \alpha = 0 \)) and the Bergman spaces (\( \alpha < 0 \)). (The Hardy and Bergman spaces are usually described differently, however see Lemma 3 of Section 3.) Let

\[
\hat{D}_\alpha = \{ g \in D_\alpha : g(0) = 0 \}
\]

and let

\[
\hat{P} = \{ g \text{ is a polynomial on } D : g(0) = 0 \}.
\]

Clearly \( \hat{P} \) is dense in \( \hat{D}_\alpha \). Let \( P_\alpha \) denote the orthogonal projection from \( L^{2,\alpha} \) onto \( \hat{D}_\alpha \). For a function \( f \in L^{2,\alpha} \), it is possible to define the (small) Hankel operator with symbol \( f \), \( h_f^{(\alpha)} \), on \( \hat{P} \) by (see also [W1])

\[
h_f^{(\alpha)} = P_\alpha(\overline{fg}).
\]

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When we say $h_f^{(\alpha)}$ is bounded, we mean there exists a constant $C > 0$ such that
\[ \|h_f^{(\alpha)}(g)\|_\alpha \leq C\|g\|_\alpha, \quad \forall g \in \hat{D}. \]

If we use the normalized monomials as a basis for $\hat{D}_\alpha$ and their conjugates as a basis for $\hat{D}_\alpha$, then the matrix of $h_f^{(\alpha)}$ is
\[ \left( \frac{j\sqrt{\beta_{k+j,\alpha}}}{(k+j)\sqrt{\beta_{k,\alpha}}\sqrt{\beta_{j,\alpha}}} \right)_{k,j \geq 1} \sim \left( \frac{k^{-\alpha j^{1-\alpha}}}{(k+j)^{1-2\alpha}} \right). \]

Here
\[ \beta_{n,\alpha} = \frac{n^2}{2 - 2\alpha} B(n, 2 - 2\alpha) = n^{2\alpha} \]
($B(\cdot, \cdot)$ is the classical Beta function) and $\{f_n\}$ are the Taylor coefficients of the analytic part of the symbol $f$:
\[ P_\alpha(f)(z) = \sum_{n=0}^{\infty} f_n z^n. \]

For $\alpha < 1$, define the space $W_\alpha$ to be the space of all analytic functions $f$ on $D$ for which
\[ \|f\|_{W_\alpha} = \sup_{\|g\|_{\alpha} \leq 1} \left( \int_D |g(z)|^2 |f'(z)|^2 dA_\alpha(z) \right)^{1/2} < \infty. \]

Clearly $W_\alpha \subseteq D_\alpha$. And it is easy to see that $W_0 = BMO$ and $W_\alpha = D_\alpha$ if $\alpha > 1/2$. (See [W1] and [W3] for more about $W_\alpha$.)

There are many equivalent norm characterizations of $D_\alpha$. The one that we are going to present here can be viewed as a generalization of one of the results in [AFP, Proposition 3.6] (see also [AFJP]).

The question of characterizing the symbol functions on $D$ for which the Hankel operators on the Dirichlet type space $D_\alpha$ are bounded was raised in [W1]. The space $W_\alpha$ is related to the boundedness of the Hankel operators (See [Ax], [P], [RS], [AFP] and [J] for $\alpha \leq 0$; [W2] for $\alpha > 1/2$). Our decomposition theorem for $W_\alpha$ (Theorem 3 below) includes theorems similar to those proved in [R] and [RS] for the Bloch space (= $W_\alpha$, $\alpha < 0$) and the space BMO (= $W_0$).

Throughout this paper, we will use the symbol $C$ to denote a positive constant which may vary at each occurrence, but will not depend on any.
function or measure that we deal with. We also use the symbol \( \approx \) to mean comparable.

Our main results are:

**Theorem 1.** Suppose \( g \) is an analytic function on \( D, \alpha \leq 1/2, \sigma, \tau > -1 \) and \( \min(\sigma, \tau) + 2\alpha > -1 \). Then we have

\[
\int_D \int_D \frac{|g(z) - g(w)|^2}{|1 - \bar{w}w|^{3+\alpha+\tau+2\alpha}} (1 - |z|^2)^\sigma (1 - |w|^2)^\tau \, dA(z) \, dA(w) \\
\approx \int_D |g'(z)|^2 (1 - |z|^2)^{1-2\alpha} \, dA(z).
\]

**Theorem 2.** Assume \( f \) is analytic on \( D \) and \( \alpha \leq 1/2 \), then \( h_f^{(\alpha)} \) is bounded if and only if \( f \in W_\alpha \).

For any fixed \( z \) in \( D \), \( \delta_z \) is the point measure on \( D \) defined by

\[
\delta_z(w) = \begin{cases} 1 & \text{if } w = z; \\ 0 & \text{if } w \neq z. \end{cases}
\]

**Theorem 3.** Let \( \alpha \leq 1/2 \) and \( b > 1/2 \) if \( \alpha = 1/2 \), \( b > 1 \) if \( \alpha < 1/2 \). There exists a \( d_0 > 0 \), so that for \( 0 < d < d_0 \) and any \( d \)-lattice \( \{z_j\}^\infty_{j=0} \) in \( D \), we have:

(a) If \( f \in W_\alpha \) then

\[
f(z) = \sum_{j=0}^{\infty} \lambda_j \left( \frac{1 - |z_j|^2}{1 - \bar{z}_j z} \right)^{b-1/2+\alpha},
\]

with

\[
\left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \right\|_\alpha \leq C |f|_{W_\alpha}^2.
\]

(b) If \( \{\lambda_j\}^\infty_{j=0} \) satisfies

\[
\left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \right\|_\alpha < \infty,
\]

then \( f \), defined by (1.1), converges in \( D_\alpha \) with

\[
\|f\|_{W_\alpha}^2 \leq C \left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \right\|_\alpha.
\]

(The \( d \)-lattice and the norm \( \| \cdot \|_\alpha \) will be defined in Section 2.)
For $\alpha = 1/2$ and $\sigma = \tau$, Theorem 1 is proved in [AFP, Proposition 3.6] (see also [AFJP]) with "...". Notice that (see [AFJP]) we can't prove Theorem 1 by using the identity

$$|f(z) - f(w)|^2 = |f(z)|^2 - f(z)\overline{f(w)} - \overline{f(z)}f(w) + |f(w)|^2$$

and then integrating each term; that will simply give $\infty - \infty - \infty + \infty$. We should be very careful when we use Fubini's theorem. Theorem 2 is also true for $\alpha > 1/2$ (see [W2] or [W4]). Theorem 3 has its root in [CR], [R] and [RS]. Proofs for Theorem 2 (or Theorem 3) for the case of $\alpha \leq 0$ can be found, for example, in [P], [R] and [W4] (or [CR], [R] and [RS]). The difficulties, for the case of $0 < \alpha \leq 1/2$, are that the reproducing kernel of the space $D_\alpha$, unlike the other case, can't give us sufficient information (see for example [RW] and [W4]) and, unlike the 0-Carleson measure, the $\alpha$-Carleson measure can't be characterized by a single box (see [G], [A], [S] and [J]). Our method, however, works for all $\alpha \leq 1/2$.

In Section 2 we will give the background and the preliminaries needed for the rest part of this paper. In Section 3, we will prove Theorem 1. In Section 4, we will apply Theorem 1 to get Theorem 2 and 3. Finally we will end this paper with some questions.

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2. Background and preliminaries

For $\beta > -1$ and $0 < p < \infty$, let

$$d\mu_\beta(z) = (1 + \beta)(1 - |z|^2)^\beta dA(z).$$

The Bergman space $A_{p,\beta}$ is the space of all analytic functions in $L_p(d\mu_\beta)$. $L^2(d\mu_\beta)$ and $A_{2,\beta} (= D_{-(1+\beta)/2})$ are Hilbert spaces. The orthogonal projection from $L^2(d\mu_\beta)$ to $A_{2,\beta}$ is (see [ZL])

$$u \rightarrow \int \frac{u(z)}{d(1 - \bar{w})^{\beta+2}} d\mu_\beta(z).$$

In particular if $u \in A_{2,\beta}$, then

$$u(w) = \int \frac{u(z)}{d(1 - \bar{w})^{\beta+2}} d\mu_\beta(z).$$

This formula is sometimes called the reproducing formula of $A_{2,\beta}$.
Denote by $K_\alpha(z, w)$ the reproducing kernel of the space $\hat{D}_\alpha$. We know $K_\alpha(\cdot, w) \in \hat{D}_\alpha$ and the orthogonal projection $P_\alpha: L^{2, \alpha} \rightarrow \hat{D}_\alpha$ is (see also \[W2\])

\[
P_\alpha(u)(w) = \int_{\mathbb{D}} \frac{\partial u}{\partial z}(z) \overline{\frac{\partial K_\alpha}{\partial z}(z, w)} \, dA_\alpha(z).
\]

It has the property

\[
\frac{\partial}{\partial w}(P_\alpha(u))(w) = \int_{\mathbb{D}} \frac{\partial u}{\partial z}(z) \frac{(1 - \overline{w} z)^{3-2\alpha}}{dA_\alpha(z)}, \quad u \in L^{2, \alpha}.
\]

The Bloch space $B$ and the space $BMO$, on $\mathbb{D}$, are defined respectively to be the functions $f$ which are analytic in $\mathbb{D}$ and satisfy (see \[G\] or \[Z\])

\[
\|f\|_B = \sup_{z \in \mathbb{D}} \left\{ \left| f'(z) \right| \left( 1 - |z|^2 \right) \right\} < \infty;
\]

\[
\|f\|_{BMO} = \sup_{z \in \mathbb{D}} \left\{ \int_{\mathbb{D}} \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{w} z|^2} \left| f'(w) \right|^2 \, dA(w) \right\} < \infty.
\]

Let $w \in \mathbb{D}$, let $\phi_w$ be the function defined by $\phi_w(z) = (w - z)/(1 - \overline{w} z)$. We know $\phi_w: \mathbb{D} \rightarrow \mathbb{D}$ is an analytic, 1-1, and onto map. The hyperbolic distance on $\mathbb{D}$, which is Moebius invariant, is defined by

\[
d(z, w) = \log \frac{1 + |\phi_w(z)|}{1 - |\phi_w(z)|}.
\]

A sequence $\{z_j\}_{j=0}^\infty$ in $\mathbb{D}$ is called a $d$-lattice, (see \[R\]), if every point of $\mathbb{D}$ is within hyperbolic distance $5d$ of some $z_j$ and no two points of this sequence are within hyperbolic distance $d/5$ of each other.

A nonnegative measure $\mu$ on $\mathbb{D}$ is called an $\alpha$-Carleson measure if

\[
\int_{\mathbb{D}} |g(z)|^2 \, d\mu(z) \leq C\|g\|_\alpha^2, \quad \forall g \in D_\alpha.
\]

The best constant $C$, denoted by $\|\mu\|_\alpha$, is said to be the $\alpha$-Carleson measure norm of $\mu$.

0-Carleson measures are just the classical Carleson measures (see \[G\]). There are many equivalent characterizations on $\alpha$-Carleson measure (see \[A\], \[KS\], \[S\] and \[J\]). In this paper, however, we don’t need them. The above definition seems easier to work with in our proofs. The space $W_\alpha$ can also be defined as the space of all analytic functions $f$ on $\mathbb{D}$ for which the measure $|f'(z)|^2 \, dA_\alpha(z)$ is an $\alpha$-Carleson measure.
The following results can be found in [R, Theorems 2.2, 2.10] (see also [CR] and [RS]).

**Theorem A.** Suppose $0 < p < \infty$, $-1 < \beta$ and $b > (1 + \beta)/p + \max(1, 1/p)$. There is a positive number $d_0$ such that for any $0 < d < d_0$ and any $d$-lattice $(z_j)_{0}^{\infty}$, there is a $C = C(\beta, p, b, d)$ so that:

(a) If $f \in A^{p, \beta}$ then

\[
(2.3) \quad f(z) = \sum_{j=0}^{\infty} \lambda_j \left(1 - |z_j|^2\right)^{b-(2+\beta)/p} \left(1 - \overline{z_j}z\right)^b,
\]

with

\[
\sum_{j=0}^{\infty} |\lambda_j|^p \leq C\|f\|_{A^{p, \beta}}^p.
\]

(b) Conversely, if $(\lambda_j)_0^{\infty}$ satisfies $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$, then $f$, defined by (2.3), converges in $A^{p, \beta}$ with

\[
\|f\|_{A^{p, \beta}} \leq C \sum_{j=0}^{\infty} |\lambda_j|^p.
\]

**Theorem B.** Suppose $b > 1$. There is a positive $d_0$ such that for any $d$, $0 < d < d_0$, and any $d$-lattice $(z_j)_{0}^{\infty}$, there is a $C > 0$ so that:

(a) If $f \in B$ (or BMO), then

\[
(2.4) \quad f(z) = \sum_{j=0}^{\infty} \lambda_j \left(1 - |z_j|^2\right)^b \left(1 - \overline{z_j}z\right)^b,
\]

with

\[
\sup_{j \geq 0} |\lambda_j| \leq C\|f\|_B
\]

\[
\left(\text{or} \left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \left(1 - |z_j|^2\right) \delta_{z_j} \right\|_0 \leq C\|f\|_{BMO} \right).
\]

(b) Conversely, if $(\lambda_j)_0^{\infty}$ satisfies

\[
\sup_{j \geq 0} |\lambda_j| < \infty
\]

\[
\left(\text{or} \left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \left(1 - |z_j|^2\right) \delta_{z_j} \right\|_0 < \infty \right),
\]
then \( f \), defined by (2.4), converges in the weak* topology in \( B \) (or \( BMO \)) with
\[
\|f\|_B \leq C \sup_{j \geq 0} \{ |\lambda_j| \}
\]

\[
\text{or} \quad \|f\|_{BMO} \leq C \left( \sum_{j=0}^{\infty} |\lambda_j|^2 \left( 1 - |z_j|^2 \right) \right)^{1/2}.
\]

Remark. The assumption on (b) of Theorem A in [R] is \( b > (2 + \beta)\max(1, 1/p) \). It is easy to check that we can change to the above assumption (for the detail see [W1]). The original form of Theorem B in [R] also contains the results for Besov spaces.

The ideas of the proofs of Theorem A and B in [CR], [R] and [RS], which we also need here, are to start with the reproducing formula
\[
f(w) = (b - 1) \int_D \frac{f(z)}{(1 - \overline{z}w)^b} \left( 1 - |z|^2 \right)^{b-2} dA(z), \quad b > 1,
\]

and then to approximate this integral by a Riemann sum
\[
(Af)(w) = C \sum_{j=0}^{\infty} f(z_j) |D_j| \left( 1 - |z_j|^2 \right)^{b-2} \left( 1 - \frac{|z_j|}{|w|} \right)^b.
\]

Here \( \{D_j\}_{j=0}^{\infty} \) is a proper disjoint cover of \( D \), and \( |D_j| = \int_{D_j} dA(z) \) is the normalized area of \( D_j \).

The key steps using these ideas are summarized as the following lemmas (see [CR, pp. 22–25] or [R] and [RS]):

**Lemma A.**
1. If \( \beta > -1 \) and \( b > 1 + (1 + \beta)/2 \), then the operator
\[
(Tf)(w) = \int_D \frac{f(z)}{|1 - \overline{z}w|^b} \left( 1 - |z|^2 \right)^{b-2} dA(z)
\]
is bounded on \( L^2(d\mu(z)) \).
2. If \( b > 2 \), then the operator \( T \) is bounded on the space
\[
\left\{ u : \|u(z)\|^2 \left( 1 - |z|^2 \right) dA(z) \right\}_0 < \infty
\]

**Lemma B.** Let \( \{z_j\}_{j=0}^{\infty} \) be a \( d \)-lattice in \( D \), then there exists a disjoint decomposition \( \{D_j\}_{j=0}^{\infty} \) of \( D \), i.e., \( \bigcup_{j=0}^{\infty} D_j = D \), such that \( |D_j| \approx (1 - |z_j|^2)^2 \), \( z_j \in D_j \) and
\[
|f(w) - (Af)(w)| \leq Cd(Tf)(w).
\]
3. A characterization of $D_\alpha$

In this section, we will prove Theorem 1 which generalizes a result in [AFP], which says ($\beta > -1$)

$$\int_{D} \int_{D} \frac{|f(z) - f(w)|^2}{|1 - \bar{z}w|^{4 + 2\beta}} d\mu_{\beta}(z) d\mu_{\beta}(w) = \int_{D} |f'(z)|^2 dA(z).$$

Notice that $1/(1 - \bar{z}w)^{2+\beta}$ is the Bergman reproducing kernel of $D$ with respect to the measure $d\mu_{\beta}(z)$. If we consider any "good" plane domain and the corresponding Bergman kernel with respect to a more general nonnegative measure $d\nu(z)$, then a similar formula is still true (see [AFJP]).

We need some lemmas for proving Theorem 1.

**Lemma 1.** For $x, y > 0$, the Gamma and Beta function are defined as

$$\Gamma(x) = \int_{0}^{\infty} t^{x-1}e^{-t} dt, \quad B(x, y) = \int_{1}^{y} r^{x-1}(1 - r)^{y-1} dr.$$

For fixed $x$ and $y$, we have for any natural numbers $j$ and $k$

(3.1) \hspace{1em} \frac{\Gamma(j + x)}{\Gamma(j + y)} \approx (j + 1)^{x-y}, \quad B(k, x) \approx k^{-x};

(3.2) \hspace{1em} B(j + 1 + x, y) - B(j + k + 1 + x, y) \approx (j + 1)^{y} - (k + j + 1)^{y}.

Here "$\approx$" is independent of $j$ and $k$.

**Proof.** (3.1) can be found in [T, section 1.87]. For (3.2), we have

$$B(j + 1 + x, y) - B(j + k + 1 + x, y) = \int_{0}^{1} (r^{j+x} - r^{j+k+x})(1 - r)^{y-1} dr$$

$$= \int_{0}^{1} r^{j+x} \left( \sum_{n=0}^{k-1} r^n \right) (1 - r)^{y-1} dr$$

$$= \sum_{n=0}^{k-1} B(n + j + x + 1, y + 1)$$

$$\approx \sum_{n=0}^{k-1} (n + j + 1)^{-y-1}$$

$$\approx \int_{0}^{k} (t + j + 1)^{-y-1} dt$$

$$\approx (j + 1)^{-y} - (k + j + 1)^{-y}.$$  

The proof is complete. $\square$
LEMMA 2. For $\alpha \leq 1/2$, $\sigma + 2\alpha > -1$, we have
\[
\int_0^{\infty} t^{\sigma+2\alpha}(1 + t)^{\alpha-\sigma-3/2}((1 + t)^{1/2-\alpha} - t^{1/2-\alpha}) \, dt < \infty.
\]
Proof. Obvious. \hfill \Box

LEMMA 3. If $f(z) = \sum_{j=0}^{\infty} a_j z^j \in D_\alpha$, then $\|f\|_\alpha^2 = \sum_{j=0}^{\infty} (j + 1)^{2\alpha}|a_j|^2$
Proof. Obvious. \hfill \Box

Before proving Theorem 1, notice that if $\sigma > \tau$, say, $\sigma \geq \tau$, then by the
fact that $(1 - |z|), (1 - |w|) \leq |1 - \bar{z}w|$, for $z, w \in \mathbb{D}$, we have
\[
\frac{(1 - |z|^2)^\sigma(1 - |w|^2)^\tau}{|1 - \bar{z}w|^{3+2\sigma+2\alpha}} \leq \frac{(1 - |z|^2)^\sigma(1 - |w|^2)^\tau}{|1 - \bar{z}w|^{3+\sigma+\tau+2\alpha}} \leq \frac{1 - |z|^2)^\tau(1 - |w|^2)^\tau}{|1 - \bar{z}w|^{3+2\tau+2\alpha}}.
\]
Hence, in Theorem 1, the case $\sigma \neq \tau$ can be obtained from the case $\sigma = \tau$.

Proof of Theorem 1. We only need to consider the case of $\sigma = \tau$ and $\alpha < 1/2$.

For convenience, let $\beta = 3/2 + \sigma + \alpha$ and
\[
f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad \chi(k) = \begin{cases} 1, & \text{if } k \geq 0; \\ 0, & \text{if } k < 0. \end{cases}
\]
By setting $z = re^{i\theta}$, $w = se^{i\phi}$, $t = \phi - \theta$ and $\zeta = se^{it}$, we can write
\[
\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - \bar{z}w|^{2\beta}} (1 - |z|^2)^\sigma (1 - |w|^2)^\sigma \, dA(z) \, dA(w)
\]
\[
= \frac{1}{\pi^2} \int_0^1 \int_{\partial \mathbb{D}} \int_0^1 \int_{\partial \mathbb{D}} \frac{|f(re^{i\theta}) - f(se^{i(\theta + r)})|^2}{|1 - re^{it}|^{2\beta}}
\]
\[
\times (1 - r^2)^\sigma (1 - s^2)^\sigma \, d\theta \, dr \, dt \, ds \, ds
\]
\[
= \frac{2}{\pi} \int_0^1 \int_{\partial \mathbb{D}} \int_0^1 \sum_{k=1}^{\infty} |a_k|^2 \frac{|r^k - s^k e^{ikt}|^2}{|1 - re^{it}|^{2\beta}} (1 - r^2)^\sigma (1 - s^2)^\sigma \, dr \, dt \, ds \, ds
\]
\[
= 2 \sum_{k=1}^{\infty} |a_k|^2 \int_0^1 \int_{\mathbb{D}} \frac{|r^k - \xi^k|^2}{|1 - r\xi|^{|2\beta}} (1 - r^2)^\sigma (1 - \xi^2)^\sigma \, dA(\xi) \, dr.
\]
Let $g_r(\zeta) = (r^k - \zeta^k)/(1 - r\zeta)^\beta$. By Lemma 3, we only need to show

$$I(k) = \int_0^1 \left| g_r(\zeta) \right|^2 (1 - r^2)^\sigma (1 - |\zeta|^2)^\sigma \, dA(\zeta) \, r \, dr \approx k^{2\alpha}, \quad k \geq 1.$$ 

Notice that for $r \in [0, 1)$, $g_r(\zeta)$ is in the Bergman space $A^{2,\sigma}$. Hence the reproducing formula for $A^{2,\sigma}$ allows us to write

$$g_r(\zeta) = (1 + \sigma) \int_\mathbb{D} \frac{g_r(\eta)}{(1 - \overline{\eta}\zeta)^{2+\sigma}} (1 - |\eta|^2)^\sigma \, dA(\eta)$$

$$= \sum_{j=0}^{\infty} \frac{1}{B(j + 1, \sigma + 1)} \int_\mathbb{D} g_r(\eta) \overline{\eta}^j \zeta^j (1 - |\eta|^2)^\sigma \, dA(\eta).$$

Thus we have

$$\int_\mathbb{D} \left| g_r(\zeta) \right|^2 (1 - |\zeta|^2)^\sigma \, dA(\zeta)$$

$$= \sum_{j=0}^{\infty} \frac{1}{B(j + 1, \sigma + 1)} \left| \int_\mathbb{D} g_r(\eta) \overline{\eta}^j (1 - |\eta|^2)^\sigma \, dA(\eta) \right|^2;$$

i.e.,

$$I(k) = \sum_{j=0}^{\infty} \frac{1}{B(j + 1, \sigma + 1)} \int_0^1 \left| \int_\mathbb{D} g_r(\eta) \overline{\eta}^j (1 - |\eta|^2)^\sigma \, dA(\eta) \right|^2$$

$$\times (1 - r^2)^\sigma r \, dr.$$ 

We now compute the integral above by observing that

$$g_r(\eta) = (r^k - \eta^k) \sum_{n=0}^{\infty} \frac{\Gamma(n + \beta)}{\Gamma(\beta) \Gamma(n + 1)} r^n \eta^n,$$

hence

$$\int_\mathbb{D} g_r(\eta) \overline{\eta}^j (1 - |\eta|^2)^\sigma \, dA(\eta)$$

$$= \Gamma(\beta)^{-1} \left( \frac{\Gamma(j + \beta) B(j + 1, \sigma + 1)}{\Gamma(j + 1)} r^{k+j}$$

$$- \chi(j - k) \frac{\Gamma(j - k + \beta) B(j + 1, \sigma + 1)}{\Gamma(j - k + 1)} r^{j-k} \right).$$
and

\[ \int_0^1 \left| \int_{D} g_r(\eta) \eta^j (1 - |\eta|^2)^\sigma dA(\eta) \right|^2 (1 - r^2)^\sigma r \, dr \]

\[ = \Gamma(\beta)^{-2} B(j + 1, \sigma + 1)^2 \left( \frac{\Gamma(j + \beta)^2 B(k + j + 1, \sigma + 1)}{\Gamma(j + 1)^2} \right. \]

\[ - 2 \chi(j - k) \frac{\Gamma(j + \beta) \Gamma(j - k + \beta)}{\Gamma(j + 1) \Gamma(j - k + 1)} B(j + 1, \sigma + 1) \]

\[ + \frac{\chi(j - k) \Gamma(j - k + \beta)^2 B(j - k + 1, \sigma + 1)}{\Gamma(j - k + 1)^2} \right). \]

So

\[ I(k) = \Gamma(\beta)^{-2} \lim_{m \to \infty} \sum_{j=0}^{m} B(j + 1, \sigma + 1) \left( \frac{\Gamma(j + \beta)^2 B(k + j + 1, \sigma + 1)}{\Gamma(j + 1)^2} \right. \]

\[ - 2 \chi(j - k) \frac{\Gamma(j + \beta) \Gamma(j - k + \beta)}{\Gamma(j + 1) \Gamma(j - k + 1)} B(j + 1, \sigma + 1) \]

\[ + \frac{\chi(j - k) \Gamma(j - k + \beta)^2 B(j - k + 1, \sigma + 1)}{\Gamma(j - k + 1)^2} \right) \]

\[ \approx \lim_{m \to \infty} \left\{ \sum_{j=m-k+1}^{m} \frac{B(j + 1, \sigma + 1) B(j + k + 1, \sigma + 1) \Gamma(j + \beta)^2}{\Gamma(j + 1)^2} \right. \]

\[ + 2 \sum_{j=0}^{m-k} \left( \frac{B(j + 1, \sigma + 1) B(j + k + 1, \sigma + 1) \Gamma(j + \beta)^2}{\Gamma(j + 1)^2} \right. \]

\[ \left. - \frac{B(j + k + 1, \sigma + 1)^2 \Gamma(j + k + \beta) \Gamma(j + \beta)}{\Gamma(j + k + 1) \Gamma(j + 1)} \right\}. \]

For any \(j\), by (3.1) of Lemma 1, we have

\[ \frac{B(j + 1, \sigma + 1) B(j + k + 1, \sigma + 1) \Gamma(j + \beta)^2}{\Gamma(j + 1)^2} \]

\[ \approx j^{-1-\sigma} (k + j)^{-1-\sigma} j^{2\beta-2} \leq j^{2\alpha-1}, \]
hence for $\alpha < 1/2$

$$\lim_{m \to \infty} \sum_{j=-m-k+1}^{m} \frac{B(j+1,\sigma+1)B(j+k+1,\sigma+1)\Gamma(j+\beta)^2}{\Gamma(j+1)^2} = 0;$$

If we let $x = \beta - 1$ and $y = 1/2 - \alpha$, then by Lemma 1, we get

$$\frac{B(j+1,\sigma+1)B(k+j+1,\sigma+1)\Gamma(j+\beta)^2}{\Gamma(j+1)^2} = \frac{B(j+k+1,\sigma+1)\Gamma(j+k+\beta)\Gamma(j+\beta)}{\Gamma(j+k+1)\Gamma(j+1)} = \frac{B(k+j+1,\sigma+1)\Gamma(j+\beta)\Gamma(\sigma+1)}{\Gamma(j+1)\Gamma(1/2 - \alpha)} \times (B(j+k+1/2 - \alpha) - B(j+k+\beta,1/2 - \alpha)).$$

$$\approx (j+1)^{\beta-1}(k+j+1)^{-\sigma-1}((j+1)^{\alpha-1/2} - (j+k+1)^{\alpha-1/2}).$$

Combine these computations to get (using Lemma 2)

$$I(k) \approx \lim_{m \to \infty} \sum_{j=0}^{m-k} (j+1)^{\beta-1}(k+j+1)^{-\sigma-1}$$

$$\times \left((j+1)^{\alpha-1/2} - (j+k+1)^{\alpha-1/2}\right)$$

$$\approx \int_{0}^{\infty} x^{\beta-1} (k+x)^{-\sigma-1} \left(x^{\alpha-1/2} - (k+x)^{\alpha-1/2}\right) dx$$

$$= k^{2\alpha} \int_{0}^{\infty} t^{\alpha+2\alpha}(1+t)^{-\sigma-3/2}((1+t)^{1/2-\alpha} - t^{1/2-\alpha}) dt$$

$$\approx k^{2\alpha}.$$
4. Applications

In this section we prove Theorem 2 and 3. We first need the following lemmas.

For $\gamma > -1$ and $u \in L^2(dA)$, define the operator

$$\tilde{h}_{u,\gamma}(g)(w) = \int_{D} \frac{u(z)g(z)}{(1 - \bar{w}z)^{2+\gamma}} d\mu_{\gamma}(z), \quad \forall g \in \hat{H}.$$

**Lemma 4.** Suppose $a < 1$, $\gamma > -\alpha$, $u \in A^{2,1-2\alpha}$ and $\tilde{h}_{u,\gamma}$ is bounded from $\hat{D}$ to $L^2(dA)$, then $\sup_{z \in D}(|u(z)|(1 - |z|^2)) < \infty$.

**Proof.** (cf. [W2, Theorem 1]). Let $[\alpha]$ be the greatest integer in $\alpha$ and set $n = \lceil \alpha \rceil$. We consider the functions

$$f_a(z) = (1 - |a|^2)^{1/2 + \alpha + n} \frac{z^{n+1}}{(1 - \bar{a}z)^{n+1}},$$

$$e_a(z) = (1 - |a|^2)^{3/2 - \alpha} \frac{(1 - |z|^2)^{\gamma - 1 + 2\alpha}}{(1 - \bar{a}z)^{2+\gamma}}.$$

Clearly for any $a \in D$, $f_a$ is in $\hat{D}$, with $\|f_a\|_\alpha \approx 1$ and $e_a$ is in $L^2(dA)$ with $\|e_a\|_{L^2(dA)} \approx 1$. It is easy to check that

$$\int_{D} \tilde{h}_{u,\gamma}(f_a)(w) e_a(w) dA_\alpha(w)$$

$$= (1 - |a|^2)^{2+n} \int_{D} \int_{D} \frac{u(z)\bar{z}^{n+1}}{(1 - \bar{w}z)^{2+\gamma}(1 - a\bar{z})^{n+1}}$$

$$\times \frac{(1 - |w|^2)^{\gamma - 1 + 2\alpha}}{(1 - a\bar{w})^{2+\gamma}} d\mu_{\gamma}(z) dA_\alpha(w)$$

$$= \frac{2 - 2\alpha}{\gamma + 1} (1 - |a|^2)^{n+2} \int_{D} \frac{u(z)\bar{z}^{n+1}}{(1 - \bar{a}z)^{3+\gamma+n}} d\mu_{\gamma}(z)$$

$$= \frac{(2 - 2\alpha)}{(1 + \gamma)(2 + \gamma) \cdots (n+2 + \gamma)} (1 - |a|^2)^{n+2} u^{(n+1)}(a).$$

This implies

$$\sup_{a \in D} \left\{(1 - |a|^2)^{n+2} |u^{(n+1)}(a)|\right\} \leq C \|\tilde{h}_{u,\gamma}\| \|f_a\|_\alpha \|e_a\|_{L^2(dA)}.$$
Recall
\[ \sup_{a \in \mathbb{D}} \left\{ |u(a)| (1 - |a|^2) \right\} \approx \sum_{j=0}^{n} |u^{(j)}(0)| + \sup_{a \in \mathbb{D}} \left\{ (1 - |a|^2)^{n+2} |u^{(n+1)}(a)| \right\}. \]
Hence the proof is complete.

**Lemma 5.** Let \( \alpha \leq 1/2 \) and \( \varepsilon > 0 \). If \( \mu \) is an \( \alpha \)-Carleson measure, then for any \( w \in \mathbb{D} \),
\[ \int_{\mathbb{D}} \frac{(1 - |w|^2)^\varepsilon}{|1 - \bar{w}z|^{1+\varepsilon-2\alpha}} \, d\mu(z) \leq C\|\mu\|_\alpha. \]

**Remark.** For \( \alpha = 0 \) and \( \varepsilon = 1 \), this condition is also sufficient (see [G, p. 239]).

**Proof.** For fixed \( w \in \mathbb{D} \), a straightforward computation shows that
\[ g(z) = (1 - |w|^2)^{\varepsilon/2} (1 - \bar{w}z)^{\alpha-1/2-\varepsilon/2} \]
is in \( D_\alpha \) and \( \|g\|_\alpha \leq C \) independently of \( w \). Hence
\[ \int_{\mathbb{D}} \frac{(1 - |w|^2)^\varepsilon}{|1 - \bar{w}z|^{1+\varepsilon-2\alpha}} \, d\mu(z) = \int_{\mathbb{D}} |g(z)|^2 \, d\mu(z) \leq \|\mu\|_\alpha \|g\|_\alpha^2 \leq C\|\mu\|_\alpha. \]
The proof is now complete.

For \( b > 1 \), consider the operator
\[ (Tf)(w) = \int_{\mathbb{D}} \frac{f(z)}{|1 - \bar{w}z|^b} (1 - |z|^2)^{b-2} \, dA(z). \]

**Lemma 6.** Let \( \alpha \leq 1/2 \), \( \beta > -1 \), \( \beta + 2\alpha > -1 \) and
\[ b > \max\left\{ \frac{\beta + 3}{2}, \frac{\beta + 3}{2} - \alpha \right\}. \]
Suppose \( v(z) \) is a function in \( L^2(d\mu_\beta) \). If the measure \( |v(z)|^2 \, d\mu_\beta(z) \) is an \( \alpha \)-Carleson measure, then the measure \( |T(v)(z)|^2 \, d\mu_\beta(z) \) is also an \( \alpha \)-Carleson measure.

**Remark.** For the case of \( \alpha = 0 \) and \( \beta = 1 \) (which is part 2) of Lemma A), Lemma 6 is proved in [RS]. The method we are going to use here is quite
different from theirs (which is based on the fact that the 0-Carleson measure can be characterized by a single box). Also it seems very hard (at least for us) to prove this lemma by using the results in [A], [S], [J] and [KS], because the corresponding conditions in there are hard to verify.

**Proof of Lemma 6.** Notice that \( |w(z)|^2 \, d\mu_\beta(z) \) is an \( \alpha \)-Carleson measure if and only if the multiplier \( M_w : D_\alpha \to L^2(d\mu_\beta) \) is bounded. We only need to prove that the multiplier \( M_{T(v)} \) is bounded from \( D_\alpha \) to \( L^2(d\mu_\beta) \). Because \( T \) is bounded on \( L^2(d\mu_\beta) \), by Lemma A, we have \( T \) is bounded from \( D_\alpha \) to \( L^2(d\mu_\beta) \), hence we only need to show the difference \( M_{T(v)} - T \) is bounded from \( D_\alpha \) to \( L^2(d\mu_\beta) \).

In fact, \( \forall g \in D_\alpha \), we have

\[
\left\| (M_{T(v)} - TM_v)(g)(w) \right\|^2 = \left| \int_{D} v(z) \frac{g(w) - g(z)}{|1 - \bar{z}w|^b} (1 - |z|^2)^{b-2} \, dA(z) \right|^2.
\]

If \( \alpha = 1/2 \), then

\[
\left| \int_{D} v(z) \frac{g(w) - g(z)}{|1 - \bar{z}w|^b} (1 - |z|^2)^{b-2} \, dA(z) \right|^2 
\leq C \|v\|_{L^2(d\mu_\beta)}^2 \int_{D} \frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2b}} (1 - |z|^2)^{2b-4-\beta} \, dA(z);
\]

hence, by Theorem 1 (\( \sigma = 2b - 4 - \beta, \tau = \beta \)),

\[
\left\| (M_{T(v)} - TM_v)(g) \right\|_{L^2(d\mu_\beta)}^2 
\leq C \|v\|_{L^2(d\mu_\beta)}^2 \int_{D} \int_{D} \frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2b}} (1 - |z|^2)^{2b-4-\beta} \, dA(z) \, d\mu_\beta(w)
\leq C \|v\|_{L^2(d\mu_\beta)}^2 \|g\|_{L^2(d\mu_\beta)}^2.
\]

If \( \alpha < 1/2 \), choose a number \( \varepsilon > 0 \) such that those assumptions for Lemma 6 remain true if \( \beta \) is replaced by \( \beta - \varepsilon \). Then

\[
\left| \int_{D} v(z) \frac{g(w) - g(z)}{|1 - \bar{z}w|^b} (1 - |z|^2)^{b-2} \, dA(z) \right|^2 
\leq C \int_{D} \frac{|v(z)|^2}{|1 - \bar{z}w|^{1+\varepsilon-2\alpha}} \, d\mu_\beta(z)
\times \int_{D} \frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2b-1-\varepsilon+2\alpha}} (1 - |z|^2)^{2b-4-\beta} \, dA(z),
\]
by Lemma 5,
\[
\int_D \frac{|v(z)|^2}{|1 - \bar{w}z|^{1+2\alpha}} \, d\mu_{\beta}(z) \leq C(1 - |w|^2)^{-\epsilon} \|v\|^2 \, d\mu_{\beta},
\]
hence by Theorem 1 (\(\sigma = 2b - 4 - \beta, \tau = \beta - \epsilon\))
\[
\left\| (M_{T(v)} - T_M)(g) \right\|_{L^2(d\mu_{\beta})}^2 \leq C \|v\|^2 \, d\mu_{\beta} \int_D \int_D \frac{|g(w) - g(z)|^2}{|1 - \bar{w}z|^{2b-1-\epsilon+2\alpha}(1 - |z|^2)^{2b-4-\beta}} \times dA(z)(1 - |w|^2)^{\beta-\epsilon} \, dA(w) \leq C \|v\|^2 \, d\mu_{\beta} \|g\|^2_{a}.
\]

The proof is complete. \(\square\)

We prove Theorem 2 by showing Theorem 2' stated below. We also need Theorem 2' for proving Theorem 3' later.

**THEOREM 2'**. Let \(\alpha \leq 1/2\) and \(\gamma > -1/2\) if \(\alpha = 1/2\), \(\gamma > \max\{0, -2\alpha\}\) if \(\alpha < 1/2\). Let \(u\) be analytic on \(D\). Then the operator \(\tilde{h}_{u,\gamma}\) is bounded from \(\hat{D}_\alpha\) to \(L^2(dA_\alpha)\) if and only if the measure \(|u(z)|^2 \, dA_\alpha\) is an \(\alpha\)-Carleson measure.

Theorem 2 is then an easy consequence. In fact, let \(\gamma = 1 - 2\alpha\) and \(u = f'\). By (2.1) and (2.2), we have
\[
\frac{\partial}{\partial \bar{w}} (h_{f}(g)) (w) = 0,
\]
\[
\frac{\partial}{\partial \bar{w}} (h_{\alpha}(g)) (w) = \int_D \frac{f'(z)g(z)}{(1 - \bar{w}z)^{2b-2\alpha}} \, dA_\alpha(z) = \tilde{h}_{u,\gamma}(g)(w).
\]

Hence \(h_{f}^{(\alpha)}\) is bounded if and only if \(\tilde{h}_{u,\gamma}\) is bounded from \(\hat{D}_\alpha\) to \(L^2(dA_\alpha)\).

**Proof of Theorem 2'**. If \(u\) is such that \(|u(z)|^2 \, dA_\alpha\) is an \(\alpha\)-Carleson measure and \(g \in \hat{D}_\alpha\), then \(u\bar{g} \in L^2(dA_\alpha)\). By Lemma A, \((b = 2 + \gamma\) and \(\beta = 1 - 2\alpha\),
\[
\tilde{h}_{u,\gamma}(g) \in L^2(dA_\alpha)
\]
and

\[ \| \tilde{h}_{u,\gamma}(g) \|_{L^2(dA_\alpha)} \leq C \| u \tilde{g} \|_{L^2(dA_\alpha)} \leq C \| u \|^2 dA_\alpha \| g \|_\alpha^{1/2}. \]

This implies that \( \tilde{h}_{u,\gamma} \) is bounded from \( D_\alpha \) to \( L^2(dA_\alpha) \).

To prove the converse let \( u \) be analytic on \( D \). We need to show

\[ \| u \tilde{g} \|_{L^2(dA_\alpha)} \leq C \| g \|_\alpha, \quad \forall g \in D_\alpha. \]

Notice that

\[ \| u \tilde{g} \|_{L^2(dA_\alpha)} \leq \| g(0) \|_{L^2(dA_\alpha)} + \| u(g - g(0)) \|_{L^2(dA_\alpha)} \]

and for \( \phi(z) = z \) we have (see also [W2, Lemma 3])

\[ \| u \|_{L^2(dA_\alpha)} \approx |u(0)| + \| \tilde{h}_{u,\gamma}(\phi) \|_{L^2(dA_\alpha)} \leq |u(0)| + C \| \tilde{h}_{u,\gamma} \| \| \phi \|_\alpha < \infty, \]

hence we only need to show

\[ \| u \tilde{g} \|_{L^2(dA_\alpha)} \leq C \| g \|_\alpha, \quad \forall g \in D_\alpha. \]

Using the idea of the proof of Lemma 6 again, we study the difference

\[ u(z)g(z) - \tilde{h}_{u,\gamma}(g)(w) = \int_D \frac{u(z)(\overline{g(w)} - \overline{g(z)})}{(1 - \overline{w}z)^{2+\gamma}} d\mu_\gamma(z). \]

By the boundedness of \( \tilde{h}_{u,\gamma} \), we only need to show that the \( L^2(dA_\alpha) \) norm of this difference is dominated by the \( D_\alpha \) norm of \( g \). In the following, we will use the notation \( B(u) \) to mean the quantity \( \sup_{z \in D} \{|u(z)|(1 - |z|^2)\} \).

If \( \alpha = 1/2 \), then \( dA_\alpha(z) = dA(z) \), by Cauchy’s inequality

\[ \left| \int_D \frac{u(z)(\overline{g(w)} - \overline{g(z)})}{(1 - \overline{w}z)^{2+\gamma}} d\mu_\gamma(z) \right|^2 \]

\[ \leq \int_D |u(z)|^2 dA(z) \int_D \left| \frac{g(w) - g(z)}{|1 - \overline{w}z|^{4+2\gamma}} \right|^2 (1 - |z|^2)^{2\gamma} dA(z); \]

hence, by Theorem 1 (\( \sigma = 2\gamma \) and \( \tau = 0 \)), we have

\[ \| u \tilde{g} - \tilde{h}_{u,\gamma}(g) \|_{L^2(dA_\alpha)}^2 \]

\[ \leq \| u \|^2_{L^2(dA_\alpha)} \int_D \int_D \left| \frac{g(w) - g(z)}{|1 - \overline{w}z|^{4+2\gamma}} \right|^2 (1 - |z|^2)^{2\gamma} dA(z) dA(w) \]

\[ \leq C \| u \|^2_{L^2(dA_\alpha)} \| g \|^2_\alpha; \]
If \( \alpha < 1/2 \), then again by Cauchy's inequality
\[
\left| \int_{\mathbb{D}} \frac{u(z)(g(w) - g(z))}{(1 - \bar{z}w)^{2+\gamma}} (1 - |z|^2)^\gamma \, dA(z) \right|^2
\leq \int_{\mathbb{D}} \frac{|u(z)|^2}{|1 - \bar{z}w|^{2+\gamma}} (1 - |z|^2)^{\gamma+1} \, dA(z)
\times \int_{\mathbb{D}} \frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2+\gamma}} (1 - |z|^2)^{\gamma-1} \, dA(z)
\leq CB(u)^2 \int_{\mathbb{D}} \frac{(1 - |z|^2)^{\gamma-1}}{|1 - \bar{z}w|^{2+\gamma}} \, dA(z)
\times \int_{\mathbb{D}} \frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2+\gamma}} (1 - |z|^2)^{\gamma-1} \, dA(z)
\leq CB(u)^2 (1 - |w|^2)^{-1} \int_{\mathbb{D}} \frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2+\gamma}} (1 - |z|^2)^{\gamma-1} \, dA(z);
\]

hence
\[
\| u\tilde{g} - \tilde{h}_{u,\gamma}(g) \|^2_{L^2(dA_\alpha)}
\leq CB(u)^2 \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2+\gamma}} (1 - |z|^2)^{\gamma-1}
\times dA(z)(1 - |w|^2)^{-2\alpha} \, dA(w)
\leq CB(u)^2 \| g \|^2_{L^2}.
\]

This last inequality is obtained by Theorem 1 (\( \sigma = \gamma - 1 \) and \( \tau = -2\alpha \)). It follows from Lemma 4 that \( B(u) \) is finite. Thus the proof is complete. \( \square \)

Instead of proving Theorem 3, we show the following one. Theorem 3 follows by term by term integration.

**Theorem 3' (Decomposition Theorem).** Let \( \alpha \leq 1/2 \) and \( b > 3/2 \) if \( \alpha = 1/2 \), \( b > 2 \) if \( \alpha < 1/2 \). There exists a \( d_0 > 0 \), so that for any \( d \)-lattice \( \{z_j\}^\infty_{j=0} \) in \( \mathbb{D} \), \( 0 < d < d_0 \), we have:

(a) If \( f \) is analytic in \( \mathbb{D} \) and \( |f(z)|^2 \, dA_\alpha(z) \) is an \( \alpha \)-Carleson measure, then

\[
f(z) = \sum_{j=0}^\infty \lambda_j \frac{(1 - |z|^2)^{b-3/2+\alpha}}{(1 - \overline{z}z)^b},
\]
with
\[ \left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \right\|_\alpha \leq C \left\| f \right\|^2 dA \alpha_\alpha. \]

(b) If \( \{\lambda_j\}^\infty_0 \) satisfies
\[ \left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \right\|_\alpha < \infty, \]
then \( f \), defined by (4.1), is in \( A^{2,1-2\alpha} \) and \( |f(z)|^2 dA_\alpha(z) \) is an \( \alpha \)-Carleson measure, with
\[ \left\| |f|^2 dA \right\|_\alpha \leq C \left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \right\|_\alpha. \]

Remark. The convergence of the series (4.1) is in \( A^{2,1-2\alpha} \). It also converges pointwise.

Proof of Theorem 3'. Without loss of generality, we will assume
\[ b > \max\{2, 2 - 2\alpha\} \quad \text{if} \quad \alpha < 1/2. \]
In fact, for \( \alpha < 0 \), it is easy to check directly that \( |f|^2 dA_\alpha \) is an \( \alpha \)-Carleson measure if and only if \( \sup_{z \in D} (|f(z)||1 - |z|^2|) < \infty \). Pick \( \alpha' < 0 \) so that
\[ b > 2 - 2\alpha'. \]
Hence \( |f|^2 dA_\alpha \) is an \( \alpha \)-Carleson measure if and only if \( |f|^2 dA_{\alpha'} \) is an \( \alpha' \)-Carleson measure.

We show part (b) first. Clearly, by Theorem 2' (\( \gamma = b - 2 \)), we only need to show that the operator \( h_{f,b-2} \) is bounded from \( D_\alpha \) to \( L^2(dA_\alpha) \).

The assumption on the sequence \( \{\lambda_j\}^\infty_0 \) implies that \( \{\lambda_j\}^\infty_0 \) is square summable. Hence by Theorem A, the sum (4.1) converges in \( A^{2,1-2\alpha} \) and then \( f \), defined by (4.1), is in \( A^{2,1-2\alpha} \).

For any \( g \in D_\alpha \), consider the formula
\[ \overline{h}_{f,b-2}(g)(w) = \int_{D} f(z) \frac{\overline{g}(z)}{(1 - \overline{z}w)^b} (1 - |z|^2)^{b-2} dA(z) \]
\[ = \sum_{j=0}^{\infty} \lambda_j (1 - |z_j|^2)^{b-3/2+\alpha} \]
\[ \times \int_{D} \frac{1}{(1 - \overline{z}z_j^b)} \frac{\overline{g}(z)}{(1 - \overline{z}w)^b} (1 - |z|^2)^{b-2} dA(z) \]
\[ = \sum_{j=0}^{\infty} \lambda_j \frac{(1 - |z_j|^2)^{b-3/2+\alpha}}{(1 - \overline{z}_j w)^b} \overline{g}(z_j). \]
By Theorem A (b) \((p = 2, \beta = 1 - 2\alpha)\) we have
\[
\left\| \overline{h_{f, b - 2}}(g) \right\|_{A^{2, 1 - 2\alpha}}^2 \leq C \sum_{j=0}^{\infty} \left| \lambda_j g(z_j) \right|^2 \leq C \left\| \sum_{j=0}^{\infty} \left| \lambda_j \right|^2 \delta_{z_j} \right\|_A \|g\|_A^2.
\]
So (b) is proved.

Now we prove part (a). Let \(g \in D_\alpha\) and \(\{z_j\}^\infty_0\) be a \(d\)-lattice in \(D\). The assumption on \(f\) implies \(fg \in A^{2, 1 - 2\alpha}\) and the discrete version of this is that the sequence
\[
\left\{ f(z_j)g(z_j) \left(1 - \left| z_j \right|^2 \right)^{3/2 - \alpha} \right\}^\infty_0
\]
is square summable (see also [CR] or [R]). This means that the measure (here we use the notation in Lemma B)
\[
\sum_{j=0}^{\infty} \left| f(z_j) \left(1 - \left| z_j \right|^2 \right)^{-1/2 - \alpha} |D_j| \right| \delta_{z_j}
\]
is an \(\alpha\)-Carleson measure and
\[
(4.2) \quad \left\| \sum_{j=0}^{\infty} \left| f(z_j) \left(1 - \left| z_j \right|^2 \right)^{-1/2 - \alpha} |D_j| \right| \delta_{z_j} \right\|_A \leq C \|f\|^2 dA_\alpha.
\]

Let (see Lemma B)
\[
A(f)(z) = C \sum_{j=0}^{\infty} f(z_j)|D_j| \frac{\left(1 - \left| z_j \right|^2 \right)^{b - 2}}{(1 - z_j \bar{z})^b};
\]
then, by part (b) of Theorem 3', \(\|A(f)(z)\|^2 dA_\alpha(z)\) is an \(\alpha\)-Carleson measure. Regarding \(A\) as the operator on the space
\[
\left\{ f \in A^{2, 1 - 2\alpha} : |f(z)|^2 \left(1 - \left| z \right|^2 \right)^{1 - 2\alpha} dA(z) \text{ is an } \alpha\text{-Carleson measure} \right\},
\]
we have, by Lemma B,
\[
|(I - A)(f)(z)| \leq CdT(f)(z).
\]
Let \(d\) be sufficient small. By Lemma 6 \((\beta = 1 - 2\alpha)\), we have the operator norm estimate
\[
\|I - A\| \leq 1/2.
\]
Hence $A^{-1}$ exists and
\[ \|A^{-1}\| \leq \sum_{j=0}^{\infty} \|(I - A)^n\| \leq 2. \]

Now we can write
\[
f(z) = (AA^{-1}f)(z)
\]
\[
= C \sum_{j=0}^{\infty} (A^{-1}f)(z_j)|D_j| \frac{(1 - |z_j|^2)^{b-2}}{(1 - \overline{z}j)^{b}}
\]
\[
= C \sum_{j=0}^{\infty} (A^{-1}f)(z_j)|D_j|(1 - |z_j|^2)^{-1/2-\alpha} \frac{(1 - |z_j|^2)^{b-3/2+\alpha}}{(1 - \overline{z}j)^{b}}.
\]

By the inequality (4.2) and the boundedness of $A^{-1}$, we get
\[
\left\| \sum_{j=0}^{\infty} (A^{-1}f)(z_j)|D_j|(1 - |z_j|^2)^{-1/2-\alpha} \left| \delta_{z_j} \right|_\alpha \right\|
\leq C \|A^{-1}f\|_\alpha \leq C \|A^{-1}\| \|f\|^2 dA_\alpha.
\]

Thus the choice of $\lambda_j = (A^{-1}f)(z_j)|D_j|(1 - |z_j|^2)^{-1/2-\alpha}$ completes the proof. \qed

5. Some questions

(1) Instead of $D$ or $U$, consider more generally any simply connected domain in $C$ (or in $C^n$). It would be nice if we could get a result similar to Theorem 1. The best range of those parameters in Theorem 1 is also unknown. We believe that for nice domains Theorem 1 remains true if $\alpha > 1/2$.

(2) Is it reasonable to consider the sum (1.1) in the Theorem 3 as a series converging in some weak* topology instead of the one in $D_\alpha$?

(3) To answer question (2), maybe we should ask first that what is the predual space of $W_\alpha$ (the predual of $W_0 = BMO$ is $H^1$).

(4) We noted in the introduction that the operators $h_j^{(\alpha)}$ are related to matrices of the form
\[
\left( \frac{f_{k+j} - \alpha j^{1-\alpha}}{(k+j)^{1-2\alpha}} \right).
\]
We know much less about the more symmetric matrix
\[
\begin{pmatrix}
\frac{f_{k+j}}{k^{-\alpha}j^{-\alpha}} \\
\frac{(k+j)^{-2\alpha}}{k^{-\alpha}j^{-\alpha}}
\end{pmatrix}.
\]

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