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A NEW CHARACTERIZATION OF DIRICHLET TYPE SPACES AND APPLICATIONS

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1. Introduction

Let $\mathbf{D}$ be the unit disk of the complex plane $\mathbf{C}$ and $dA(z) = 1/\pi\, dx\, dy$ be the normalized Lebesgue measure on $\mathbf{D}$. For $\alpha < 1$, let

$$dA_\alpha(z) = (2 - 2\alpha)(1 - |z|^2)^{1-2\alpha}\, dA(z).$$

The Sobolev space $L^{2,\alpha}$ is the Hilbert space of functions $u\colon \mathbf{D} \to \mathbf{C}$, for which the norm

$$\|u\| = \left(\left| \int_{\mathbf{D}} u\, dA_\alpha(z) \right|^2 + \int_{\Delta} \left(|\partial u/\partial z|^2 + |\partial u/\partial \bar{z}|^2 \right) dA_\alpha(z) \right)^{1/2}$$

is finite. The space D_α is the subspace of all analytic functions in $L^{2,\alpha}$. This scale of spaces includes the Dirichlet type spaces ($\alpha > 0$), the Hardy space ($\alpha = 0$) and the Bergman spaces ($\alpha < 0$). (The Hardy and Bergman spaces are usually described differently, however see Lemma 3 of Section 3.) Let

$$\dot{D}_\alpha = \{ g \in D_\alpha : g(0) = 0 \}$$

and let

$$\dot{P} = \{ g \text{ is a polynomial on } \mathbf{D} : g(0) = 0 \}.$$

Clearly $\dot{P}$ is dense in $\dot{D}_\alpha$. Let P_α denote the orthogonal projection from $L^{2,\alpha}$ onto $\dot{D}_\alpha$. For a function $f \in L^{2,\alpha}$ it is possible to define the (small) Hankel operator with symbol f, $h_f^{(\alpha)}$, on $\dot{P}$ by (see also [W1])

$$h_f^{(\alpha)} = \overline{P_\alpha(f\bar{g})}.$$

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When we say $h_f^{(\alpha)}$ is bounded, we mean there exists a constant $C > 0$ such that

$$\left\| h_f^{(\alpha)}(g) \right\|_\alpha \le C\|g\|_\alpha, \quad \forall g \in \dot{P}.$$

If we use the normalized monomials as a basis for $\dot{D}_\alpha$ and their conjugates as a basis for $\overline{\dot{D}_\alpha}$, then the matrix of $h_f^{(\alpha)}$ is

$$\left(\overline{f_{k+j}} \frac{j\sqrt{\beta_{k+j,\alpha}}}{(k+j)\sqrt{\beta_{k,\alpha}}\sqrt{\beta_{j,\alpha}}} \right)_{k,j\ge 1} \sim \left(\overline{f_{k+j}} \frac{k^{-\alpha} j^{1-\alpha}}{(k+j)^{1-2\alpha}} \right).$$

Here

$$\beta_{n,\alpha} = \frac{n^2}{2-2\alpha} B(n, 2-2\alpha) \approx n^{2\alpha}$$

($B(\cdot,\cdot)$ is the classical Beta function) and $\{f_n\}$ are the Taylor coefficients of the analytic part of the symbol f:

$$P_\alpha(f)(z) = \sum_{n=0}^{\infty} f_n z^n.$$

For $\alpha < 1$, define the space W_α to be the space of all analytic functions f on $\mathbf{D}$ for which

$$\|f\|_{W_\alpha} = \sup_{\|g\|_\alpha \le 1} \left(\int_{\mathbf{D}} |g(z)|^2 |f'(z)|^2 \, dA_\alpha(z) \right)^{1/2} < \infty.$$

Clearly $W_\alpha \subseteq D_\alpha$. And it is easy to see that $W_\alpha = B$ (Bloch space) if $\alpha < 0$; $W_0 = BMO$ and $W_\alpha = D_\alpha$ if $\alpha > 1/2$. (See [W1] and [W3] for more about W_α.)

There are many equivalent norm characterizations of D_α. The one that we are going to present here can be viewed as a generalization of one of the results in [AFP, Proposition 3.6] (see also [AFJP]).

The question of characterizing the symbol functions on $\mathbf{D}$ for which the Hankel operators on the Dirichlet type space D_α are bounded was raised in [W1]. The space W_α is related to the boundedness of the Hankel operators (See [Ax], [P], [RS], [AFP] and [J] for $\alpha \le 0$; [W2] for $\alpha > 1/2$). Our decomposition theorem for W_α (Theorem 3 below) includes theorems similar to those proved in [R] and [RS] for the Bloch space ($= W_\alpha$, $\alpha < 0$) and the space BMO ($= W_0$).

Throughout this paper, we will use the symbol C to denote a positive constant which may vary at each occurrence, but will not depend on any function or measure that we deal with. We also use the symbol $\approx$ to mean comparable.

Our main results are:

THEOREM 1. *Suppose g is an analytic function on* $\mathbf{D}$, $\alpha \leq 1/2$, $\sigma, \tau > -1$ *and* $\min(\sigma, \tau) + 2\alpha > -1$. *Then we have*

$$\int_{\mathbf{D}}\int_{\mathbf{D}} \frac{|g(z) - g(w)|^2}{|1 - \bar{z}w|^{3+\sigma+\tau+2\alpha}} (1 - |z|^2)^{\sigma} (1 - |w|^2)^{\tau} \, dA(z) \, dA(w)$$
$$\approx \int_{\mathbf{D}} |g'(z)|^2 (1 - |z|^2)^{1-2\alpha} \, dA(z).$$

THEOREM 2. *Assume f is analytic on* $\mathbf{D}$ *and* $\alpha \leq 1/2$, *then* $h_f^{(\alpha)}$ *is bounded if and only if* $f \in W_\alpha$.

For any fixed z in $\mathbf{D}$, δ_z is the point measure on $\mathbf{D}$ defined by

$$\delta_z(w) = \begin{cases} 1 & \text{if } w = z; \\ 0 & \text{if } w \neq z. \end{cases}$$

THEOREM 3. *Let* $\alpha \leq 1/2$ *and* $b > 1/2$ *if* $\alpha = 1/2$, $b > 1$ *if* $\alpha < 1/2$. *There exists a* $d_0 > 0$, *so that for* $0 < d < d_0$ *and any d-lattice* $\{z_j\}_0^\infty$ *in* $\mathbf{D}$, *we have*:

(a) *If* $f \in W_\alpha$ *then*

$$f(z) = \sum_{j=0}^{\infty} \lambda_j \frac{\left(1 - |z_j|^2\right)^{b-1/2+\alpha}}{\left(1 - \overline{z_j}z\right)^b} \tag{1.1}$$

with

$$\left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \right\|_\alpha \leq C|f|_{W_\alpha}^2.$$

(b) *If* $\{\lambda_j\}_0^\infty$ *satisfies*

$$\left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \right\|_\alpha < \infty,$$

then f, defined by (1.1), *converges in* D_α *with*

$$\|f\|_{W_\alpha}^2 \leq C \left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \right\|_\alpha.$$

(*The d-lattice and the norm* $\| \cdot \|_\alpha$ *will be defined in Section* 2.)

For $\alpha = 1/2$ and $\sigma = \tau$, Theorem 1 is proved in [AFP, Proposition 3.6] (see also [AFJP]) with "$=$." Notice that (see [AFJP]) we can't prove Theorem 1 by using the identity

$$|f(z) - f(w)|^2 = |f(z)|^2 - f(z)\overline{f(w)} - \overline{f(z)}f(w) + |f(w)|^2$$

and then integrating each term; that will simply give $\infty - \infty - \infty + \infty$. We should be very careful when we use Fubini's theorem. Theorem 2 is also true for $\alpha > 1/2$ (see [W2] or [W4]). Theorem 3 has its root in [CR], [R] and [RS]. Proofs for Theorem 2 (or Theorem 3) for the case of $\alpha \leq 0$ can be found, for example, in [P], [R] and [W4] (or [CR], [R] and [RS]). The difficulties, for the case of $0 < \alpha \leq 1/2$, are that the reproducing kernel of the space D_α, unlike the other case, can't give us sufficient information (see for example [RW] and [W4]) and, unlike the 0-Carleson measure, the α-Carleson measure can't be characterized by a single box (see [G], [A], [S] and [J]). Our method, however, works for all $\alpha \leq 1/2$.

In Section 2 we will give the background and the preliminaries needed for the rest part of this paper. In Section 3, we will prove Theorem 1. In Section 4, we will apply Theorem 1 to get Theorem 2 and 3. Finally we will end this paper with some questions.

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2. Background and preliminaries

For $\beta > -1$ and $0 < p < \infty$, let

$$d\mu_\beta(z) = (1 + \beta)(1 - |z|^2)^\beta \, dA(z).$$

The Bergman space $A^{p,\beta}$ is the space of all analytic functions in $L^p(d\mu_\beta)$. $L^2(d\mu_\beta)$ and $A^{2,\beta}(= D_{-(1+\beta)/2})$ are Hilbert spaces. The orthogonal projection from $L^2(d\mu_\beta)$ to $A^{2,\beta}$ is (see [Z])

$$u \to \int_{\mathbf{D}} \frac{u(z)}{(1 - \bar{z}w)^{\beta+2}} \, d\mu_\beta(z).$$

In particular if $u \in A^{2,\beta}$, then

$$u(w) = \int_{\mathbf{D}} \frac{u(z)}{(1 - \bar{z}w)^{\beta+2}} \, d\mu_\beta(z).$$

This formula is sometimes called the reproducing formula of $A^{2,\beta}$.

Denote by $K_\alpha(z,w)$ the reproducing kernel of the space $\dot{D}_\alpha$. We know $K_\alpha(\cdot,w) \in \dot{D}_\alpha$ and the orthogonal projection P_α: $L^{2,\alpha} \to \dot{D}_\alpha$ is (see also [W2])

$$(2.1) \qquad P_\alpha(u)(w) = \int_{\mathbf{D}} \frac{\partial u}{\partial z}(z) \overline{\frac{\partial K_\alpha}{\partial z}(z,w)}\, dA_\alpha(z).$$

It has the property

$$(2.2) \qquad \frac{\partial}{\partial w}(P_\alpha(u))(w) = \int_{\mathbf{D}} \frac{\frac{\partial u}{\partial z}(z)}{(1-\bar{z}w)^{3-2\alpha}}\, dA_\alpha(z), \quad u \in L^{2,\alpha}.$$

The Bloch space B and the space BMO, on $\mathbf{D}$, are defined respectively to be the functions f which are analytic in $\mathbf{D}$ and satisfy (see [G] or [Z])

$$\|f\|_B = \sup_{z\in\mathbf{D}} \left\{|f'(z)|(1-|z|^2)\right\} < \infty;$$

$$\|f\|_{BMO} = \sup_{z\in\mathbf{D}} \left\{\int_{\mathbf{D}} \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2} |f'(w)|^2\, dA(w)\right\} < \infty.$$

Let $w \in \mathbf{D}$, let ϕ_w be the function defined by $\phi_w(z) = (w-z)/(1-\bar{w}z)$. We know ϕ_w: $\mathbf{D} \to \mathbf{D}$ is an analytic, 1-1, and onto map. The hyperbolic distance on $\mathbf{D}$, which is Moebius invariant, is defined by

$$d(z,w) = \log\frac{1+|\phi_w(z)|}{1-|\phi_w(z)|}.$$

A sequence $\{z_j\}_0^\infty$ in $\mathbf{D}$ is called a d-lattice, (see [R]), if every point of $\mathbf{D}$ is within hyperbolic distance $5d$ of some z_j and no two points of this sequence are within hyperbolic distance $d/5$ of each other.

A nonnegative measure μ on $\mathbf{D}$ is called an α-Carleson measure if

$$\int_{\mathbf{D}} |g(z)|^2\, d\mu(z) \le C\|g\|_\alpha^2, \quad \forall g \in D_\alpha.$$

The best constant C, denoted by $\|\mu\|_\alpha$, is said to be the α-Carleson measure norm of μ.

0-Carleson measures are just the classical Carleson measures (see [G]). There are many equivalent characterizations on α-Carleson measure (see [A], [KS], [S] and [J]). In this paper, however, we don't need them. The above definition seems easier to work with in our proofs. The space W_α can also be defined as the space of all analytic functions f on $\mathbf{D}$ for which the measure $|f'(z)|^2\, dA_\alpha(z)$ is an α-Carleson measure.

The following results can be found in [R, Theorems 2.2, 2.10] (see also [CR] and [RS]).

THEOREM A. *Suppose* $0 < p < \infty$, $-1 < \beta$ *and* $b > (1+\beta)/p + \max(1, 1/p)$. *There is a positive number* d_0 *such that for any* $0 < d < d_0$ *and any d-lattice* $\{z_j\}_0^\infty$, *there is a* $C = C(\beta, p, b, d)$ *so that*:

(a) *If* $f \in A^{p,\beta}$ *then*

$$f(z) = \sum_{j=0}^{\infty} \lambda_j \frac{\left(1 - |z_j|^2\right)^{b-(2+\beta)/p}}{\left(1 - \bar{z}_j z\right)^b}, \tag{2.3}$$

with

$$\sum_{j=0}^{\infty} |\lambda_j|^p \leq C \|f\|_{A^{p,\beta}}^p.$$

(b) *Conversely, if* $\{\lambda_j\}_0^\infty$ *satisfies* $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$, *then* f, *defined by* (2.3), *converges in* $A^{p,\beta}$ *with*

$$\|f\|_{A^{p,\beta}}^p \leq C \sum_{j=0}^{\infty} |\lambda_j|^p.$$

THEOREM B. *Suppose* $b > 1$. *There is a positive* d_0 *such that for any* d, $0 < d < d_0$, *and any d-lattice* $\{z_j\}_0^\infty$, *there is a* $C > 0$ *so that*:

(a) *If* $f \in B$ *(or BMO), then*

$$f(z) = \sum_{j=0}^{\infty} \lambda_j \frac{\left(1 - |z_j|^2\right)^b}{\left(1 - \bar{z}_j z\right)^b}, \tag{2.4}$$

with

$$\sup_{j \geq 0} \left\{|\lambda_j|\right\} \leq C \|f\|_B$$

$$\left(or \left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \left(1 - |z_j|^2\right) \delta_{z_j} \right\|_0 \leq C \|f\|_{BMO} \right).$$

(b) *Conversely, if* $\{\lambda_j\}_0^\infty$ *satisfies*

$$\sup_{j \geq 0} \left\{|\lambda_j|\right\} < \infty$$

$$\left(or \left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \left(1 - |z_j|^2\right) \delta_{z_j} \right\|_0 < \infty \right),$$

then f, defined by (2.4), *converges in the weak* topology in B (or BMO) with*

$$\|f\|_B \leq C \sup_{j \geq 0} \left\{ |\lambda_j| \right\}$$

$$\left(or \ \|f\|_{BMO} \leq C \left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \left(1 - |z_j|^2\right) \delta_{z_j} \right\|_0 \right).$$

Remark. The assumption on (b) of Theorem A in [R] is $b > (2 + \beta)\max(1, 1/p)$. It is easy to check that we can change to the above assumption (for the detail see [W1]). The original form of Theorem B in [R] also contains the results for Besov spaces.

The ideas of the proofs of Theorem A and B in [CR], [R] and [RS], which we also need here, are to start with the reproducing formula

$$f(w) = (b-1) \int_{\mathbf{D}} \frac{f(z)}{(1 - \bar{z}w)^b} \left(1 - |z|^2\right)^{b-2} dA(z), \quad b > 1,$$

and then to approximate this integral by a Riemann sum

$$(Af)(w) = C \sum_{j=0}^{\infty} f(z_j) |D_j| \frac{\left(1 - |z_j|^2\right)^{b-2}}{\left(1 - \bar{z}_j w\right)^b}.$$

Here $\{D_j\}_0^\infty$ is a proper disjoint cover of $\mathbf{D}$, and $|D_j| = \int_{D_j} dA(z)$ is the normalized area of D_j.

The key steps using these ideas are summarized as the following lemmas (see [CR, pp. 22–25] or [R] and [RS]):

Lemma A. (1) *If $\beta > -1$ and $b > 1 + (1 + \beta)/2$, then the operator*

$$(Tf)(w) = \int_{\mathbf{D}} \frac{f(z)}{|1 - \bar{z}w|^b} \left(1 - |z|^2\right)^{b-2} dA(z)$$

is bounded on $L^2(d\mu_\beta(z))$.

(2) *If $b > 2$, then the operator T is bounded on the space*

$$\left\{ u : \left\| \, |u(z)|^2 (1 - |z|^2) \, dA(z) \right\|_0 < \infty \right\}.$$

Lemma B. *Let $\{z_j\}_0^\infty$ be a d-lattice in $\mathbf{D}$, then there exists a disjoint decomposition $\{D_j\}_0^\infty$ of $\mathbf{D}$, i.e., $\bigcup_{j=0}^{\infty} D_j = \mathbf{D}$, such that $|D_j| \approx (1 - |z_j|^2)^2$, $z_j \in D_j$ and*

$$|f(w) - (Af)(w)| \leq Cd(Tf)(w).$$

3. A characterization of D_α

In this section, we will prove Theorem 1 which generalizes a result in [AFP], which says ($\beta > -1$)

$$\int_{\mathbf{D}}\int_{\mathbf{D}} \frac{|f(z)-f(w)|^2}{|1-\bar{z}w|^{4+2\beta}}\, d\mu_\beta(z)\, d\mu_\beta(w) = \int_{\mathbf{D}} |f'(z)|^2\, dA(z).$$

Notice that $1/(1-\bar{z}w)^{2+\beta}$ is the Bergman reproducing kernel of $\mathbf{D}$ with respect to the measure $d\mu_\beta(z)$. If we consider any "good" plane domain and the corresponding Bergman kernel with respect to a more general nonnegative measure $d\nu(z)$, then a similar formula is still true (see [AFJP]).

We need some lemmas for proving Theorem 1.

LEMMA 1. *For $x, y > 0$, the Gamma and Beta function are defined as*

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt, \qquad B(x,y) = \int_0^1 r^{x-1}(1-r)^{y-1}\, dr.$$

For fixed x and y, we have for any natural numbers j and k

$$\Gamma(j+x)/\Gamma(j+y) \approx (j+1)^{x-y}, \quad B(k,x) \approx k^{-x}; \tag{3.1}$$

$$B(j+1+x,y) - B(j+k+1+x,y) \approx (j+1)^{-y} - (k+j+1)^{-y}. \tag{3.2}$$

Here "$\approx$" is independent of j and k.

Proof. (3.1) can be found in [T, section 1.87]. For (3.2), we have

$$\begin{aligned} B(j+1+x,y) - B(j+k+1+x,y) &= \int_0^1 (r^{j+x} - r^{j+k+x})(1-r)^{y-1}\, dr \\ &= \int_0^1 r^{j+x}\left(\sum_{n=0}^{k-1} r^n\right)(1-r)^y\, dr \\ &= \sum_{n=0}^{k-1} B(n+j+x+1, y+1) \\ &\approx \sum_{n=0}^{k-1} (n+j+1)^{-y-1} \\ &\approx \int_0^k (t+j+1)^{-y-1}\, dt \\ &\approx (j+1)^{-y} - (k+j+1)^{-y}. \end{aligned}$$

The proof is complete. □

LEMMA 2. *For $\alpha \le 1/2, \sigma + 2\alpha > -1$, we have*

$$\int_0^\infty t^{\sigma+2\alpha}(1+t)^{\alpha-\sigma-3/2}\left((1+t)^{1/2-\alpha} - t^{1/2-\alpha}\right) dt < \infty.$$

Proof. Obvious. □

LEMMA 3. *If $f(z) = \sum_{j=0}^\infty a_j z^j \in D_\alpha$, then $\|f\|_\alpha^2 \approx \sum_{j=0}^\infty (j+1)^{2\alpha}|a_j|^2$*

Proof. Obvious. □

Before proving Theorem 1, notice that if $\sigma \neq \tau$, say, $\sigma \ge \tau$, then by the fact that $(1-|z|), (1-|w|) \le |1-\bar{z}w|$, for $z, w \in \mathbf{D}$, we have

$$\begin{aligned}\frac{(1-|z|^2)^\sigma(1-|w|^2)^\sigma}{|1-\bar{z}w|^{3+2\sigma+2\alpha}} &\le \frac{(1-|z|^2)^\sigma(1-|w|^2)^\tau}{|1-\bar{z}w|^{3+\sigma+\tau+2\alpha}} \\ &\le \frac{(1-|z|^2)^\tau(1-|w|^2)^\tau}{|1-\bar{z}w|^{3+2\tau+2\alpha}}.\end{aligned}$$

Hence, in Theorem 1, the case $\sigma \neq \tau$ can be obtained from the case $\sigma = \tau$.

Proof of Theorem 1. We only need to consider the case of $\sigma = \tau$ and $\alpha < 1/2$.

For convenience, let $\beta = 3/2 + \sigma + \alpha$ and

$$f(z) = \sum_{k=0}^\infty a_k z^k, \quad \chi(k) = \begin{cases} 1, & \text{if } k \ge 0; \\ 0, & \text{if } k < 0. \end{cases}$$

By setting $z = re^{i\theta}$, $w = se^{i\phi}$, $t = \phi - \theta$ and $\zeta = se^{it}$, we can write

$$\begin{aligned}&\int_{\mathbf{D}}\int_{\mathbf{D}} \frac{|f(z)-f(w)|^2}{|1-\bar{z}w|^{2\beta}}(1-|z|^2)^\sigma(1-|w|^2)^\sigma\, dA(z)\, dA(w) \\ &= \frac{1}{\pi^2}\int_0^1\int_{\partial\mathbf{D}}\int_0^1\int_{\partial\mathbf{D}} \frac{|f(re^{i\theta}) - f(se^{i(\theta+t)})|^2}{|1-rse^{it}|^{2\beta}} \\ &\quad \times (1-r^2)^\sigma(1-s^2)^\sigma\, d\theta\, r\, dr\, dt\, s\, ds \\ &= \frac{2}{\pi}\int_0^1\int_{\partial\mathbf{D}}\int_0^1 \sum_{k=1}^\infty |a_k|^2 \frac{|r^k - s^k e^{ikt}|^2}{|1-rse^{it}|^{2\beta}}(1-r^2)^\sigma(1-s^2)^\sigma\, r\, dr\, dt\, s\, ds \\ &= 2\sum_{k=1}^\infty |a_k|^2 \int_0^1\int_{\mathbf{D}} \frac{|r^k-\zeta^k|^2}{|1-r\zeta|^{2\beta}}(1-r^2)^\sigma(1-|\zeta|^2)^\sigma\, dA(\zeta)\, r\, dr.\end{aligned}$$

Let $g_r(\zeta) = (r^k - \zeta^k)/(1 - r\zeta)^\beta$. By Lemma 3, we only need to show

$$I(k) = \int_0^1 \int_{\mathbf{D}} |g_r(\zeta)|^2 (1 - r^2)^\sigma (1 - |\zeta|^2)^\sigma \, dA(\zeta) r \, dr \approx k^{2\alpha}, \quad k \geq 1.$$

Notice that for $r \in [0, 1)$, $g_r(\zeta)$ is in the Bergman space $A^{2,\sigma}$. Hence the reproducing formula for $A^{2,\sigma}$ allows us to write

$$\begin{aligned} g_r(\zeta) &= (1 + \sigma) \int_{\mathbf{D}} \frac{g_r(\eta)}{(1 - \overline{\eta}\zeta)^{2+\sigma}} (1 - |\eta|^2)^\sigma \, dA(\eta) \\ &= \sum_{j=0}^{\infty} \frac{1}{B(j+1, \sigma+1)} \int_{\mathbf{D}} g_r(\eta) \overline{\eta}^j \zeta^j (1 - |\eta|^2)^\sigma \, dA(\eta). \end{aligned}$$

Thus we have

$$\begin{aligned} &\int_{\mathbf{D}} |g_r(\zeta)|^2 (1 - |\zeta|^2)^\sigma \, dA(\zeta) \\ &\quad = \sum_{j=0}^{\infty} \frac{1}{B(j+1, \sigma+1)} \left| \int_{\mathbf{D}} g_r(\eta) \overline{\eta}^j (1 - |\eta|^2)^\sigma \, dA(\eta) \right|^2; \end{aligned}$$

i.e.,

$$\begin{aligned} I(k) = \sum_{j=0}^{\infty} \frac{1}{B(j+1, \sigma+1)} \int_0^1 \left| \int_{\mathbf{D}} g_r(\eta) \overline{\eta}^j (1 - |\eta|^2)^\sigma \, dA(\eta) \right|^2 \\ \times (1 - r^2)^\sigma r \, dr. \end{aligned}$$

We now compute the integral above by observing that

$$g_r(\eta) = (r^k - \eta^k) \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta)}{\Gamma(\beta)\Gamma(n+1)} r^n \eta^n,$$

hence

$$\begin{aligned} &\int_{\mathbf{D}} g_r(\eta) \overline{\eta}^j (1 - |\eta|^2)^\sigma \, dA(\eta) \\ &\quad = \Gamma(\beta)^{-1} \left(\frac{\Gamma(j+\beta) B(j+1, \sigma+1)}{\Gamma(j+1)} r^{k+j} \right. \\ &\qquad\qquad \left. - \chi(j-k) \frac{\Gamma(j-k+\beta) B(j+1, \sigma+1)}{\Gamma(j-k+1)} r^{j-k} \right), \end{aligned}$$

and

$$\int_0^1 \left| \int_{\mathbf{D}} g_r(\eta) \bar{\eta}^j (1 - |\eta|^2)^\sigma \, dA(\eta) \right|^2 (1 - r^2)^\sigma r \, dr$$

$$= \Gamma(\beta)^{-2} B(j+1, \sigma+1)^2 \left(\frac{\Gamma(j+\beta)^2 B(k+j+1, \sigma+1)}{\Gamma(j+1)^2} - 2 \frac{\chi(j-k)\Gamma(j+\beta)\Gamma(j-k+\beta)B(j+1,\sigma+1)}{\Gamma(j+1)\Gamma(j-k+1)} + \frac{\chi(j-k)\Gamma(j-k+\beta)^2 B(j-k+1,\sigma+1)}{\Gamma(j-k+1)^2} \right).$$

So

$$I(k) = \Gamma(\beta)^{-2} \lim_{m\to\infty} \sum_{j=0}^{m} B(j+1,\sigma+1) \left(\frac{\Gamma(j+\beta)^2 B(k+j+1,\sigma+1)}{\Gamma(j+1)^2} - 2 \frac{\chi(j-k)\Gamma(j+\beta)\Gamma(j-k+\beta)B(j+1,\sigma+1)}{\Gamma(j+1)\Gamma(j-k+1)} + \frac{\chi(j-k)\Gamma(j-k+\beta)^2 B(j-k+1,\sigma+1)}{\Gamma(j-k+1)^2} \right)$$

$$\approx \lim_{m\to\infty} \left\{ \sum_{j=m-k+1}^{m} \frac{B(j+1,\sigma+1)B(j+k+1,\sigma+1)\Gamma(j+\beta)^2}{\Gamma(j+1)^2} + 2 \sum_{j=0}^{m-k} \left(\frac{B(j+1,\sigma+1)B(k+j+1,\sigma+1)\Gamma(j+\beta)^2}{\Gamma(j+1)^2} - \frac{B(j+k+1,\sigma+1)^2 \Gamma(j+k+\beta)\Gamma(j+\beta)}{\Gamma(j+k+1)\Gamma(j+1)} \right) \right\}.$$

For any j, by (3.1) of Lemma 1, we have

$$\frac{B(j+1,\sigma+1)B(j+k+1,\sigma+1)\Gamma(j+\beta)^2}{\Gamma(j+1)^2} \approx j^{-1-\sigma}(k+j)^{-1-\sigma} j^{2\beta-2} \leq j^{2\alpha-1},$$

hence for $\alpha < 1/2$

$$\lim_{m\to\infty} \sum_{j=m-k+1}^{m} \frac{B(j+1,\sigma+1)B(j+k+1,\sigma+1)\Gamma(j+\beta)^2}{\Gamma(j+1)^2} = 0;$$

If we let $x = \beta - 1$ and $y = 1/2 - \alpha$, then by Lemma 1, we get

$$\begin{aligned}
&\frac{B(j+1,\sigma+1)B(k+j+1,\sigma+1)\Gamma(j+\beta)^2}{\Gamma(j+1)^2} \\
&\quad - \frac{B(j+k+1,\sigma+1)^2\Gamma(j+k+\beta)\Gamma(j+\beta)}{\Gamma(j+k+1)\Gamma(j+1)} \\
&\quad = \frac{B(k+j+1,\sigma+1)\Gamma(j+\beta)\Gamma(\sigma+1)}{\Gamma(j+1)\Gamma(1/2-\alpha)} \\
&\quad \times (B(j+\beta,1/2-\alpha) - B(j+k+\beta,1/2-\alpha)). \\
&\quad \approx (j+1)^{\beta-1}(k+j+1)^{-\sigma-1}\left((j+1)^{\alpha-1/2} - (j+k+1)^{\alpha-1/2}\right).
\end{aligned}$$

Combine these computations to get (using Lemma 2)

$$\begin{aligned}
I(k) &\approx \lim_{m\to\infty} \sum_{j=0}^{m-k} (j+1)^{\beta-1}(k+j+1)^{-\sigma-1} \\
&\quad \times \left((j+1)^{\alpha-1/2} - (j+k+1)^{\alpha-1/2}\right) \\
&\approx \int_0^\infty x^{\beta-1}(k+x)^{-\sigma-1}\left(x^{\alpha-1/2} - (k+x)^{\alpha-1/2}\right) dx \\
&= k^{2\alpha}\int_0^\infty t^{\sigma+2\alpha}(1+t)^{\alpha-\sigma-3/2}\left((1+t)^{1/2-\alpha} - t^{1/2-\alpha}\right) dt \\
&\approx k^{2\alpha}.
\end{aligned}$$

The proof of Theorem 1 is now complete. □

Theorem 1 has a version on the upper half plane, **U**, which can't be obtained by using Cayley transform on Theorem 1 (except for the case $\sigma = \tau$ and $\alpha = 1/2$). One may prove it by applying the Fourier transform on horizontal lines and then using Plancherel's Theorem (see [AFP, page 1024]).

THEOREM 1′. *Suppose g is analytic on* **U**, $0 < \alpha < 1$ *and* $\sigma, \tau > -1$. *Then*

$$\int_{\mathbf{U}}\int_{\mathbf{U}} \frac{|g(z) - g(w)|^2}{|z - \overline{w}|^{3+\sigma+\tau+2\alpha}} y^\sigma v^\tau \, dx\, dy\, du\, dv \approx \int_{\mathbf{U}} |g'(z)|^2 y^{1-2\alpha}\, dx\, dy.$$

4. Applications

In this section we prove Theorem 2 and 3. We first need the following lemmas.

For $\gamma > -1$ and $u \in L^2(dA_\alpha)$, define the operator

$$\tilde{h}_{u,\gamma}(g)(w) = \overline{\int_{\mathbf{D}} \frac{u(z)\overline{g(z)}}{(1-\bar{z}w)^{2+\gamma}}\, d\mu_\gamma(z)}, \quad \forall g \in \dot{P}.$$

LEMMA 4. *Suppose $\alpha < 1$, $\gamma > -\alpha$, $u \in A^{2,1-2\alpha}$ and $\tilde{h}_{u,\gamma}$ is bounded from $\dot{D}_\alpha$ to $L^2(dA_\alpha)$, then* $\sup_{z\in\mathbf{D}}\{|u(z)|(1-|z|^2)\} < \infty$.

Proof. (cf. [W2, Theorem 1]). Let $[\alpha]$ be the greatest integer in α and set $n = -[\alpha]$. We consider the functions

$$f_a(z) = (1-|a|^2)^{1/2+\alpha+n} \frac{z^{n+1}}{(1-\bar{a}z)^{n+1}},$$

$$e_a(z) = \frac{(1-|a|^2)^{3/2-\alpha}(1-|z|^2)^{\gamma-1+2\alpha}}{(1-\bar{a}z)^{2+\gamma}}.$$

Clearly for any $a \in \mathbf{D}$, f_a is in $\dot{D}_\alpha$ with $\|f_a\|_\alpha \approx 1$ and e_a is in $L^2(dA_\alpha)$ with $\|e_a\|_{L^2(dA_\alpha)} \approx 1$. It is easy to check that

$$\begin{aligned}
&\int_{\mathbf{D}} \overline{\tilde{h}_{u,\gamma}(f_a)(w) e_a(w)}\, dA_\alpha(w) \\
&= (1-|a|^2)^{2+n} \int_{\mathbf{D}}\int_{\mathbf{D}} \frac{u(z)\bar{z}^{n+1}}{(1-\bar{z}w)^{2+\gamma}(1-a\bar{z})^{n+1}} \\
&\quad \times \frac{(1-|w|^2)^{\gamma-1+2\alpha}}{(1-a\bar{w})^{2+\gamma}}\, d\mu_\gamma(z)\, dA_\alpha(w) \\
&= \frac{2-2\alpha}{\gamma+1}(1-|a|^2)^{n+2} \int_{\mathbf{D}} \frac{u(z)\bar{z}^{n+1}}{(1-\bar{z}a)^{3+\gamma+n}}\, d\mu_\gamma(z) \\
&= \frac{(2-2\alpha)}{(1+\gamma)(2+\gamma)\cdots(n+2+\gamma)}(1-|a|^2)^{n+2} u^{(n+1)}(a).
\end{aligned}$$

This implies

$$\sup_{a\in\mathbf{D}} \left\{(1-|a|^2)^{n+2}\left|u^{(n+1)}(a)\right|\right\} \le C\|\tilde{h}_{u,\gamma}\|\, \|f_a\|_\alpha \|e_a\|_{L^2(dA_\alpha)}.$$

Recall

$$\sup_{a\in\mathbf{D}}\left\{|u(a)|(1-|a|^2)\right\}\approx\sum_{j=0}^{n}|u^{(j)}(0)|+\sup_{a\in\mathbf{D}}\left\{\left(1-|a|^2\right)^{n+2}|u^{(n+1)}(a)|\right\}.$$

Hence the proof is complete. □

LEMMA 5. *Let $\alpha\le 1/2$ and $\varepsilon>0$. If μ is an α-Carleson measure, then for any $w\in\mathbf{D}$,*

$$\int_{\mathbf{D}}\frac{\left(1-|w|^2\right)^{\varepsilon}}{|1-\bar{z}w|^{1+\varepsilon-2\alpha}}\,d\mu(z)\le C\|\mu\|_{\alpha}.$$

Remark. For $\alpha=0$ and $\varepsilon=1$, this condition is also sufficient (see [G, p. 239]).

Proof. For fixed $w\in\mathbf{D}$, a straightforward computation shows that

$$g(z)=\left(1-|w|^2\right)^{\varepsilon/2}(1-\overline{w}z)^{\alpha-1/2-\varepsilon/2}$$

is in D_α and $\|g\|_\alpha\le C$ independently of w. Hence

$$\int_{\mathbf{D}}\frac{\left(1-|w|^2\right)^{\varepsilon}}{|1-\bar{z}w|^{1+\varepsilon-2\alpha}}\,d\mu(z)=\int_{\mathbf{D}}|g(z)|^2\,d\mu(z)\le\|\mu\|_\alpha\|g\|_\alpha^2\le C\|\mu\|_\alpha.$$

The proof is now complete. □

For $b>1$, consider the operator

$$(Tf)(w)=\int_{\mathbf{D}}\frac{f(z)}{|1-\bar{z}w|^{b}}\left(1-|z|^2\right)^{b-2}dA(z).$$

LEMMA 6. *Let $\alpha\le 1/2$, $\beta>-1$, $\beta+2\alpha>-1$ and*

$$b>\max\left\{\frac{\beta+3}{2},\frac{\beta+3}{2}-\alpha\right\}.$$

Suppose $v(z)$ is a function in $L^2(d\mu_\beta)$. If the measure $|v(z)|^2\,d\mu_\beta(z)$ is an α-Carleson measure, then the measure $|T(v)(z)|^2\,d\mu_\beta(z)$ is also an α-Carleson measure.

Remark. For the case of $\alpha=0$ and $\beta=1$ (which is part 2) of Lemma A), Lemma 6 is proved in [RS]. The method we are going to use here is quite different from theirs (which is based on the fact that the 0-Carleson measure can be characterized by a single box). Also it seems very hard (at least for us) to prove this lemma by using the results in [A], [S], [J] and [KS], because the corresponding conditions in there are hard to verify.

Proof of Lemma 6. Notice that $|w(z)|^2\, d\mu_\beta(z)$ is an α-Carleson measure if and only if the multiplier $M_w\colon D_\alpha \to L^2(d\mu_\beta)$ is bounded. We only need to prove that the multiplier $M_{T(v)}$ is bounded from D_α to $L^2(d\mu_\beta)$. Because T is bounded on $L^2(d\mu_\beta)$, by Lemma A, we have TM_v is bounded from D_α to $L^2(d\mu_\beta)$, hence we only need to show the difference $M_{T(v)} - TM_v$ is bounded from D_α to $L^2(d\mu_\beta)$.

In fact, $\forall g \in D_\alpha$, we have

$$\left|(M_{T(v)} - TM_v)(g)(w)\right|^2 = \left|\int_{\mathbf{D}} v(z)\frac{g(w) - g(z)}{|1 - \bar{z}w|^b}(1 - |z|^2)^{b-2}\, dA(z)\right|^2.$$

If $\alpha = 1/2$, then

$$\begin{aligned}&\left|\int_{\mathbf{D}} v(z)\frac{g(w) - g(z)}{|1 - \bar{z}w|^b}(1 - |z|^2)^{b-2}\, dA(z)\right|^2\\ &\qquad\le C\|v\|^2_{L^2(d\mu_\beta)}\int_{\mathbf{D}}\frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2b}}(1 - |z|^2)^{2b-4-\beta}\, dA(z);\end{aligned}$$

hence, by Theorem 1 ($\sigma = 2b - 4 - \beta, \tau = \beta$),

$$\begin{aligned}&\left\|(M_{T(v)} - TM_v)(g)\right\|^2_{L^2(d\mu_\beta)}\\ &\quad\le C\|v\|^2_{L^2(d\mu_\beta)}\int_{\mathbf{D}}\int_{\mathbf{D}}\frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2b}}(1 - |z|^2)^{2b-4-\beta}\, dA(z)\, d\mu_\beta(w)\\ &\quad\le C\|v|^2_{L^2(d\mu_\beta)}\|g\|^2_{1/2}.\end{aligned}$$

If $\alpha < 1/2$, choose a number $\varepsilon > 0$ such that those assumptions for Lemma 6 remain true if β is replaced by $\beta - \varepsilon$. Then

$$\begin{aligned}&\left|\int_{\mathbf{D}} v(z)\frac{g(w) - g(z)}{|1 - \bar{z}w|^b}(1 - |z|^2)^{b-2}\, dA(z)\right|^2\\ &\qquad\le C\int_{\mathbf{D}}\frac{|v(z)|^2}{|1 - \bar{z}w|^{1+\varepsilon-2\alpha}}\, d\mu_\beta(z)\\ &\qquad\qquad\times\int_{\mathbf{D}}\frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2b-1-\varepsilon+2\alpha}}(1 - |z|^2)^{2b-4-\beta}\, dA(z),\end{aligned}$$

by Lemma 5,

$$\int_{\mathbf{D}} \frac{|v(z)|^2}{|1-\bar{z}w|^{1+\varepsilon-2\alpha}}\, d\mu_\beta(z) \le C(1-|w|^2)^{-\varepsilon} \big\| |v|^2\, d\mu_\beta \big\|_\alpha;$$

hence by Theorem 1 ($\sigma = 2b-4-\beta, \tau = \beta - \varepsilon$)

$$\begin{aligned}
&\big\|(M_{T(v)} - TM_v)(g)\big\|^2_{L^2(d\mu_\beta)} \\
&\quad \le C\big\| |v|^2\, d\mu_\beta \big\|_\alpha \int_{\mathbf{D}}\int_{\mathbf{D}} \frac{|g(w)-g(z)|^2}{|1-\bar{z}w|^{2b-1-\varepsilon+2\alpha}} (1-|z|^2)^{2b-4-\beta} \\
&\qquad \times dA(z)(1-|w|^2)^{\beta-\varepsilon}\, dA(w) \\
&\quad \le C\big\| |v|^2\, d\mu_\beta \big\|_\alpha \|g\|^2_\alpha .
\end{aligned}$$

The proof is complete. □

We prove Theorem 2 by showing Theorem 2′ stated below. We also need Theorem 2′ for proving Theorem 3′ later.

THEOREM 2′. *Let $\alpha \le 1/2$ and $\gamma > -1/2$ if $\alpha = 1/2$, $\gamma > \max\{0, -2\alpha\}$ if $\alpha < 1/2$. Let u be analytic on $\mathbf{D}$. Then the operator $\tilde{h}_{u,\gamma}$ is bounded from $\dot{D}_\alpha$ to $L^2(dA_\alpha)$ if and only if the measure $|u(z)|^2\, dA_\alpha$ is an α-Carleson measure.*

Theorem 2 is then an easy consequence. In fact, let $\gamma = 1-2\alpha$ and $u = f'$. By (2.1) and (2.2), we have

$$\frac{\partial}{\partial w}\big(h_f^{(\alpha)}(g)\big)(w) = 0,$$

$$\frac{\partial}{\partial \bar{w}}\big(h_f^{(\alpha)}(g)\big)(w) = \overline{\int_{\mathbf{D}} \frac{f'(z)\overline{g(z)}}{(1-\bar{z}w)^{3-2\alpha}}\, dA_\alpha(z)} = \tilde{h}_{u,\gamma}(g)(w).$$

Hence $h_f^{(\alpha)}$ is bounded if and only if $\tilde{h}_{u,\gamma}$ is bounded from $\dot{D}_\alpha$ to $L^2(dA_\alpha)$.

Proof of Theorem 2′. If u is such that $|u(z)|^2\, dA_\alpha$ is an α-Carleson measure and $g \in \dot{D}_\alpha$, then $u\bar{g} \in L^2(dA_\alpha)$. By Lemma A, ($b = 2+\gamma$ and $\beta = 1-2\alpha$),

$$\tilde{h}_{u,\gamma}(g) \in L^2(dA_\alpha)$$

and

$$\left\|\tilde{h}_{u,\gamma}(g)\right\|_{L^2(dA_\alpha)} \le C\|u\bar{g}\|_{L^2(dA_\alpha)} \le C\left\||u|^2\,dA_\alpha\right\|_\alpha^{1/2}\|g\|_\alpha.$$

This implies that $\tilde{h}_{u,\gamma}$ is bounded from $\dot{D}_\alpha$ to $L^2(dA_\alpha)$.

To proof the converse let u be analytic on $\mathbf{D}$. We need to show

$$\|ug\|_{L^2(dA_\alpha)} \le C\|g\|_\alpha, \qquad \forall g \in D_\alpha.$$

Notice that

$$\|ug\|_{L^2(dA_\alpha)} \le |g(0)|\|u\|_{L^2(dA_\alpha)} + \left\|u(g - g(0))\right\|_{L^2(dA_\alpha)}$$

and for $\phi(z) = z$ we have (see also [W2, Lemma 3])

$$\|u\|_{L^2(dA_\alpha)} \approx |u(0)| + \left\|\tilde{h}_{u,\gamma}(\phi)\right\|_{L^2(dA_\alpha)} \le |u(0)| + C\|\tilde{h}_{u,\gamma}\|\;\|\phi\|_\alpha < \infty,$$

hence we only need to show

$$\|ug\|_{L^2(dA_\alpha)} \le C\|g\|_\alpha, \qquad \forall g \in \dot{D}_\alpha.$$

Using the idea of the proof of Lemma 6 again, we study the difference

$$u(w)\overline{g(w)} - \overline{\tilde{h}_{u,\gamma}(g)(w)} = \int_{\mathbf{D}} \frac{u(z)(\overline{g(w)} - \overline{g(z)})}{(1 - \bar{z}w)^{2+\gamma}}\,d\mu_\gamma(z).$$

By the boundedness of $\tilde{h}_{u,\gamma}$, we only need to show that the $L^2(dA_\alpha)$ norm of this difference is dominated by the D_α norm of g. In the following, we will use the notation $B(u)$ to mean the quantity $\sup_{z\in\mathbf{D}}\{|u(z)|(1 - |z|^2)\}$.

If $\alpha = 1/2$, then $dA_\alpha(z) = dA(z)$, by Cauchy's inequality

$$\left|\int_{\mathbf{D}} \frac{u(z)(\overline{g(w)} - \overline{g(z)})}{(1 - \bar{z}w)^{2+\gamma}}\,d\mu_\gamma(z)\right|^2 \le \int_{\mathbf{D}} |u(z)|^2\,dA(z) \int_{\mathbf{D}} \frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{4+2\gamma}}(1 - |z|^2)^{2\gamma}\,dA(z);$$

hence, by Theorem 1 ($\sigma = 2\gamma$ and $\tau = 0$), we have

$$\begin{aligned}\left\|u\bar{g} - \overline{\tilde{h}_{u,\gamma}(g)}\right\|^2_{L^2(dA_\alpha)} &\le \|u\|^2_{L^2(dA_\alpha)} \int_{\mathbf{D}}\int_{\mathbf{D}} \frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{4+2\gamma}}(1 - |z|^2)^{2\gamma}\,dA(z)\,dA(w) \\ &\le C\|u\|^2_{L^2(dA_\alpha)}\|g\|^2_\alpha;\end{aligned}$$

If $\alpha < 1/2$, then again by Cauchy's inequality

$$\begin{aligned}
&\left|\int_{\mathbf{D}} \frac{u(z)(\overline{g(w)} - \overline{g(z)})}{(1 - \bar{z}w)^{2+\gamma}} (1 - |z|^2)^{\gamma} \, dA(z)\right|^2 \\
&\quad \leq \int_{\mathbf{D}} \frac{|u(z)|^2}{|1 - \bar{z}w|^{2+\gamma}} (1 - |z|^2)^{\gamma+1} \, dA(z) \\
&\quad\quad \times \int_{\mathbf{D}} \frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2+\gamma}} (1 - |z|^2)^{\gamma-1} \, dA(z) \\
&\quad \leq CB(u)^2 \int_{\mathbf{D}} \frac{(1 - |z|^2)^{\gamma-1}}{|1 - \bar{z}w|^{2+\gamma}} \, dA(z) \\
&\quad\quad \times \int_{\mathbf{D}} \frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2+\gamma}} (1 - |z|^2)^{\gamma-1} \, dA(z) \\
&\quad \leq CB(u)^2 (1 - |w|^2)^{-1} \int_{\mathbf{D}} \frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2+\gamma}} (1 - |z|^2)^{\gamma-1} \, dA(z);
\end{aligned}$$

hence

$$\begin{aligned}
&\left\| u\bar{g} - \overline{\tilde{h}_{u,\gamma}(g)} \right\|^2_{L^2(dA_\alpha)} \\
&\quad \leq CB(u)^2 \int_{\mathbf{D}} \int_{\mathbf{D}} \frac{|g(w) - g(z)|^2}{|1 - \bar{z}w|^{2+\gamma}} (1 - |z|^2)^{\gamma-1} \\
&\quad\quad \times dA(z)(1 - |w|^2)^{-2\alpha} \, dA(w) \\
&\quad \leq CB(u)^2 \|g\|^2_\alpha.
\end{aligned}$$

This last inequality is obtained by Theorem 1 ($\sigma = \gamma - 1$ and $\tau = -2\alpha$). It follows from Lemma 4 that $B(u)$ is finite. Thus the proof is complete. □

Instead of proving Theorem 3, we show the following one. Theorem 3 follows by term by term integration.

THEOREM 3′ (DECOMPOSITION THEOREM). *Let $\alpha \leq 1/2$ and $b > 3/2$ if $\alpha = 1/2$, $b > 2$ if $\alpha < 1/2$. There exists a $d_0 > 0$, so that for any d-lattice $\{z_j\}_0^\infty$ in* **D**, *$0 < d < d_0$, we have:*

(a) *If f is analytic in* **D** *and $|f(z)|^2 \, dA_\alpha(z)$ is an α-Carleson measure, then*

$$f(z) = \sum_{j=0}^{\infty} \lambda_j \frac{\left(1 - |z_j|^2\right)^{b-3/2+\alpha}}{\left(1 - \overline{z_j}z\right)^b} \tag{4.1}$$

with

$$\left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \right\|_\alpha \leq C \left\| |f|^2 \, dA_\alpha \right\|_\alpha .$$

(b) *If* $\{\lambda_j\}_0^\infty$ *satisfies*

$$\left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \right\|_\alpha < \infty,$$

then f, *defined by* (4.1), *is in* $A^{2,1-2\alpha}$ *and* $|f(z)|^2 \, dA_\alpha(z)$ *is an* α*-Carleson measure, with*

$$\left\| |f|^2 \, dA_\alpha \right\|_\alpha \leq C \left\| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \right\|_\alpha .$$

Remark. The convergence of the series (4.1) is in $A^{2,1-2\alpha}$. It also converges pointwise.

Proof of Theorem 3′. Without loss of generality, we will assume

$$b > \max\{2, 2 - 2\alpha\} \quad \text{if } \alpha < 1/2.$$

In fact, for $\alpha < 0$, it is easy to check directly that $|f|^2 \, dA_\alpha$ is an α-Carleson measure if and only if $\sup_{z \in \mathbf{D}}\{|f(z)|(1 - |z|^2)\} < \infty$. Pick $\alpha' < 0$ so that $b > 2 - 2\alpha'$. Hence $|f|^2 \, dA_\alpha$ is an α-Carleson measure if and only if $|f|^2 \, dA_{\alpha'}$ is an α'-Carleson measure.

We show part (b) first. Clearly, by Theorem 2′ ($\gamma = b - 2$), we only need to show that the operator $\tilde{h}_{f,b-2}$ is bounded from D_α to $L^2(dA_\alpha)$.

The assumption on the sequence $\{\lambda_j\}_0^\infty$ implies that $\{\lambda_j\}_0^\infty$ is square summable. Hence by Theorem A, the sum (4.1) converges in $A^{2,1-2\alpha}$ and then f, defined by (4.1), is in $A^{2,1-2\alpha}$.

For any $g \in D_\alpha$, consider the formula

$$\begin{aligned}
\overline{\tilde{h}_{f,b-2}(g)(w)} &= \int_{\mathbf{D}} f(z) \frac{\overline{g(z)}}{(1 - \bar{z}w)^b} (1 - |z|^2)^{b-2} \, dA(z) \\
&= \sum_{j=0}^{\infty} \lambda_j \left(1 - |z_j|^2\right)^{b-3/2+\alpha} \\
&\quad \times \int_{\mathbf{D}} \frac{1}{\left(1 - \overline{z_j}z\right)^b} \frac{\overline{g(z)}}{(1 - \bar{z}w)^b} (1 - |z|^2)^{b-2} \, dA(z) \\
&= \sum_{j=0}^{\infty} \lambda_j \frac{\left(1 - |z_j|^2\right)^{b-3/2+\alpha}}{\left(1 - \overline{z_j}w\right)^b} \overline{g(z_j)}.
\end{aligned}$$

By Theorem A (b) ($p = 2, \beta = 1 - 2\alpha$) we have

$$\left\|\overline{\tilde{h}_{f,b-2}(g)}\right\|^2_{A^{2,1-2\alpha}} \le C\sum_{j=0}^{\infty}\left|\lambda_j g(z_j)\right|^2 \le C\left\|\sum_{j=0}^{\infty}|\lambda_j|^2\delta_{z_j}\right\|_\alpha \|g\|^2_\alpha.$$

So (b) is proved.

Now we prove part (a). Let $g \in D_\alpha$ and $\{z_j\}_0^\infty$ be a d-lattice in $\mathbf{D}$. The assumption on f implies $fg \in A^{2,1-2\alpha}$ and the discrete version of this is that the sequence

$$\left\{f(z_j)g(z_j)\left(1 - |z_j|^2\right)^{3/2-\alpha}\right\}_0^\infty$$

is square summable (see also [CR] or [R]). This means that the measure (here we use the notation in Lemma B)

$$\sum_{j=0}^{\infty}\left|f(z_j)\left(1 - |z_j|^2\right)^{-1/2-\alpha}|D_j|\right|^2\delta_{z_j}$$

is an α-Carleson measure and

$$\left\|\sum_{j=0}^{\infty}\left|f(z_j)\left(1 - |z_j|^2\right)^{-1/2-\alpha}|D_j|\right|^2\delta z_j\right\|_\alpha \le C\left\||f|^2\, dA_\alpha\right\|_\alpha. \tag{4.2}$$

Let (see Lemma B)

$$A(f)(z) = C\sum_{j=0}^{\infty} f(z_j)|D_j|\frac{\left(1 - |z_j|^2\right)^{b-2}}{\left(1 - \overline{z}_j z\right)^b};$$

then, by part (b) of Theorem 3′, $|A(f)(z)|^2\, dA_\alpha(z)$ is an α-Carleson measure. Regarding A as the operator on the space

$$\left\{f \in A^{2,1-2\alpha}\colon |f(z)|^2(1 - |z|^2)^{1-2\alpha}\, dA(z) \text{ is an } \alpha\text{-Carleson measure}\right\},$$

we have, by Lemma B,

$$|(I - A)(f)(z)| \le CdT(f)(z).$$

Let d be sufficient small. By Lemma 6 ($\beta = 1 - 2\alpha$), we have the operator norm estimate

$$\|I - A\| \le 1/2.$$

Hence A^{-1} exists and

$$\|A^{-1}\| \le \sum_{j=0}^{\infty} \left\|(I-A)^n\right\| \le 2.$$

Now we can write

$$\begin{aligned} f(z) &= (AA^{-1}f)(z) \\ &= C\sum_{j=0}^{\infty} (A^{-1}f)(z_j)|D_j| \frac{\left(1-|z_j|^2\right)^{b-2}}{\left(1-\overline{z_j}z\right)^b} \\ &= C\sum_{j=0}^{\infty} (A^{-1}f)(z_j)|D_j|\left(1-|z_j|^2\right)^{-1/2-\alpha} \frac{\left(1-|z_j|^2\right)^{b-3/2+\alpha}}{\left(1-\overline{z_j}z\right)^b}. \end{aligned}$$

By the inequality (4.2) and the boundedness of A^{-1}, we get

$$\begin{aligned} &\left\| \sum_{j=0}^{\infty} \left|(A^{-1}f)(z_j)|D_j|\left(1-|z_j|^2\right)^{-1/2-\alpha}\right|^2 \delta_{z_j} \right\|_\alpha \\ &\qquad \le C\left\| |A^{-1}f|^2\, dA_\alpha \right\|_\alpha \le C\|A^{-1}\| \left\| |f|^2\, dA_\alpha \right\|_\alpha. \end{aligned}$$

Thus the choice of $\lambda_j = (A^{-1}f)(z_j)|D_j|(1-|z_j|^2)^{-1/2-\alpha}$ completes the proof. □

5. Some questions

(1) Instead of **D** or **U**, consider more generally any simply connected domain in **C** (or in $\mathbf{C}^n$). It would be nice if we could get a result similar to Theorem 1. The best range of those parameters in Theorem 1 is also unknown. We believe that for nice domains Theorem 1 remains true if $\alpha > 1/2$.

(2) Is it reasonable to consider the sum (1.1) in the Theorem 3 as a series converging in some weak* topology instead of the one in D_α?

(3) To answer question (2), maybe we should ask first that what is the predual space of W_α (the predual of $W_0 = BMO$ is H^1).

(4) We noted in the introduction that the operators $h_f^{(\alpha)}$ are related to matrices of the form

$$\left(\overline{f_{k+j}} \frac{k^{-\alpha} j^{1-\alpha}}{(k+j)^{1-2\alpha}}\right).$$

We know much less about the more symmetric matrix

$$\left(\overline{f_{k+j}}\frac{k^{-\alpha}j^{-\alpha}}{(k+j)^{-2\alpha}}\right).$$

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