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HARMONIC AND ISOMETRIC ROTATIONS AROUND A CURVE

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1. Introduction

In this paper we initiate the study of *local rotations* around a smooth embedded curve $\sigma: [a, b] \to (M, g)$ in a Riemannian manifold (M, g). These transformations are local diffeomorphisms which generalize in a natural way the rotations around a straight line in Euclidean space E^n. They are determined by means of a field of endomorphisms along the curve, (the so-called *rotation field*), which for each $m \in \sigma$ fix the tangent vectors of σ and when restricted to the fibers of the normal bundle of σ behave like linear isometries.

Reflections with respect to a curve provide a class of examples of such rotations. We refer to [2], [16], [17], [18], [19] for further details about their study.

When σ reduces to a point we obtain the rotations around a point which in turn generalize the geodesic symmetries. Such rotations are used to define different classes of Riemannian manifolds, for example symmetric spaces, generalized symmetric spaces and s-manifolds (see [6], [12], [18]). Moreover, the properties of these rotations may be used to characterize some particular classes of Riemannian spaces. For example, it is proved in [3] that harmonic geodesic symmetries characterize locally symmetric spaces. This result has been extended in [15] to s-regular manifolds by using a special class of rotations around a point. Further, when (M, g, J) is an almost Hermitian manifold, then the field J provides a natural rotation field. The properties of the corresponding rotations may again be used to characterize special classes of almost Hermitian manifolds as is done in [14]. (See also [18] for the use of geodesic symmetries in Hermitian and symplectic geometry.)

In this paper we study similar problems for rotations around a curve σ. The main purpose is to study *harmonic* rotations. In Section 2 we give some preliminaries. Then, in Section 3, we define rotations and derive, in the analytic case, a set of necessary and sufficient conditions for *isometric* rotations. We use this in Section 4 where we consider harmonic rotations and

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investigate their relationship with isometric rotations. In particular, we show that for the so-called free rotations these two concepts coincide for locally symmetric Einstein spaces. Up to now we do not know if this result can be extended to general Riemannian spaces.

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2. Preliminaries

Let σ: $[a, b] \to (M, g)$ be a smooth embedded curve in a smooth Riemannian manifold (M, g). Further, let $N\sigma$ be the normal bundle of σ and denote by $\exp_\sigma$ the exponential map of this normal bundle. It is defined by

$$\exp_\sigma(\sigma(t), v) = \exp_{\sigma(t)} v$$

for any $t \in [a, b]$ and for $v \in T^{\perp}_{\sigma(t)}\sigma$, where $T^{\perp}_{\sigma(t)}\sigma$ denotes the fiber of $N\sigma$ over $\sigma(t)$, i.e., the orthogonal complement of the tangent space $T_{\sigma(t)}\sigma$ of σ at $\sigma(t)$ in $T_{\sigma(t)}M$.

Next, consider the tubular neighborhood $U(s)$ of radius s about σ, that is,

$$U(s) = \left\{\exp_{\sigma(t)} v \,|\, v \in T^{\perp}_{\sigma(t)}\sigma,\ \|v\| < s,\ a \le t \le b\right\}.$$

Let

$$B^{\perp}_{\sigma(t)}(s) = \left\{v \in T^{\perp}_{\sigma(t)}\sigma \,|\ \|v\| < s\right\}$$

denote the $(n - 1)$-dimensional ball of radius s in $T^{\perp}_{\sigma(t)}\sigma$ and consider

$$N_\sigma(s) = \bigcup_{t \in [a,b]} B^{\perp}_{\sigma(t)}(s),$$

the open solid tube of radius s about the zero section of the normal bundle $N\sigma$ of σ.

Since $[a, b]$ is compact and since σ: $[a, b] \to M$ is an embedding we can choose $s > 0$ to be so small that $\exp_\sigma$ is a C^∞ diffeomorphism of $N_\sigma(s)$ onto $U(s)$ (see for example [10, p. 114]).

On any sufficiently small tubular neighborhood U of the curve σ there is a special type of coordinates, namely Fermi coordinates, which are particularly convenient to study the geometry in a neighborhood of the curve. We briefly describe such a system (see [8], [9], [17], [18] for more details).

So, let ∇ be the Levi Civita connection and R the Riemann curvature tensor defined by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all tangent vectors X, Y. Further, let $\sigma: [a, b] \to M$ be a unit speed curve as above and let $\{e_1 = \dot{\sigma}(a), e_2, \ldots, e_n;\ n = \dim M\}$ be an orthonormal basis of $T_{\sigma(a)}M$. Next, let E_1 be the unit tangent field $\dot{\sigma}$ and $E_2, \ldots, E_n$ the normal vector fields along σ which are parallel with respect to the normal connection $\nabla^{\perp}$ of the normal bundle $N\sigma$ and such that $E_i(a) = e_i$, $i = 2, \ldots, n$. Then the *Fermi coordinates* $(x^1, \ldots, x^n)$ with respect to $\sigma(a)$ and the frame field $(E_1, \ldots, E_n)$ are defined by

$$x^1\left(\exp_{\sigma(t)} \sum_{j=2}^{n} t^j E_j\right) = t - a,$$

$$x^i\left(\exp_{\sigma(t)} \sum_{j=2}^{n} t^j E_j\right) = t^i, \quad 2 \le i \le n.$$

For $p \in U$, we have $p = \exp_{\sigma(t)} v$, where

$$v = \sum_{i=2}^{n} x^i E_i(t) = ru \in T^{\perp}_{\sigma(t)}\sigma$$

and

$$\|u\| = 1, \quad r^2 = \sum_{i=2}^{n} (x^i)^2.$$

In general σ is not a geodesic and we put

$$\kappa_u = g(\ddot{\sigma}, u)$$

where

$$\ddot{\sigma}(t) = \left(\nabla_{\dot{\sigma}(t)}\dot{\sigma}\right)(t)$$

is the *(mean) curvature vector* of σ normal to $\dot{\sigma}$ at t. If u is chosen to be parallel along σ (with respect to $\nabla^{\perp}$), we have

$$\nabla_{\dot{\sigma}} u = g\left(\nabla_{\dot{\sigma}} u, \dot{\sigma}\right)\dot{\sigma}.$$

Therefore, since $g(u, \dot{\sigma}) = 0$, we have

$$\nabla_{\dot{\sigma}} u = -g(u, \ddot{\sigma})\dot{\sigma} = -\kappa_u \dot{\sigma}.$$

In Section 4 we shall need the expressions of g and g^{-1} with respect to Fermi coordinates. Put

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad i, j = 1, \ldots, n.$$

Then we have (see for example [8], [18], [19]):

LEMMA 2.1. *Let $m = \sigma(t)$ and $p = \exp_{\sigma(t)}(ru)$, $\|u\| = 1$. With respect to Fermi coordinates $(x^1, \ldots, x^n)$ we have*

$$\begin{aligned}
g_{11}(p) &= 1 - 2\kappa_u(m)r + \left(\kappa_u^2 - R_{1u1u}\right)(m)r^2 \\
&\quad -\tfrac{1}{3}(\nabla_u R_{1u1u} - 4\kappa_u R_{1u1u})(m)r^3 + O(r^4), \\
g_{1a}(p) &= -\tfrac{2}{3}R_{1uau}(m)r^2 - \tfrac{1}{12}(3\nabla_u R_{1uau} - 4\kappa_u R_{1uau})(m)r^3 + O(r^4), \\
g_{ab}(p) &= \delta_{ab} - \tfrac{1}{3}R_{uaub}(m)r^2 - \tfrac{1}{6}(\nabla_u R_{uaub})(m)r^3 + O(r^4), \\
g^{11}(p) &= 1 + 2\kappa_u(m)r + (3\kappa_u + R_{1u1u})(m)r^2 \\
&\quad +\tfrac{1}{3}\left(\nabla_u R_{1u1u} + 8\kappa_u R_{1u1u} + 12\kappa_u^3\right)(m)r^3 + O(r^4), \\
g^{1a}(p) &= \tfrac{2}{3}R_{1uau}(m)r^2 + \tfrac{1}{4}(\nabla_u R_{1uau} + 4\kappa_u R_{1uau})(m)r^3 + O(r^4), \\
g^{ab}(p) &= \delta_{ab} + \tfrac{1}{3}R_{uaub}(m)r^2 + \tfrac{1}{6}(\nabla_u R_{uaub})(m)r^3 + O(r^4),
\end{aligned}$$

$a, b = 2, \ldots, n$. Here we let

$$R_{uiuj}(m) = R_{uE_i(t)uE_j(t)}(\sigma(t)) = g\left(R_{uE_i(t)}u, E_j(t)\right)(\sigma(t)), \text{ etc.}$$

for $i, j = 1, \ldots, n$.

3. Rotations and isometries of tubular neighborhoods

We start with some motivating considerations. Let f be an isometry of (M,g) whose (totally geodesic) fixed point set has positive dimension and let σ be a curve as in Section 2 contained in this fixed point set. Then we have:

LEMMA 3.1. *On a sufficiently small tubular neighborhood U of σ the isometry f can be expressed by*

$$f = \exp_\sigma \circ f_{*|\sigma} \circ \exp_\sigma^{-1}. \tag{3.1}$$

Proof. For each point $p \in U$ there exists a unique geodesic $\gamma\colon [0,1] \to M$ of minimal length such that $p = \gamma(1)$ and $\sigma(t) = \gamma(0)$ for some $t \in [a, b]$.

Furthermore,

$$\dot\gamma(0) = \exp^{-1}_{\sigma(t)}(p).$$

Now, the curve $f \circ \gamma$ is also a geodesic emanating from the same point $\sigma(t)$ and with initial velocity $f_{*|\sigma(t)}(\dot\gamma(0))$. Hence

$$f(p) = f(\gamma(1)) = \exp_{\sigma(t)}\big(f_{*|\sigma(t)}\dot\gamma(0)\big).$$

This implies (3.1).

Remark. There are several examples of Riemannian manifolds endowed with isometries as described above. For example, let (M, g) be a homogeneous Riemannian manifold and let K be the isotropy group at some point of M. Since the linear isotropy representation of K in T_pM is faithful the isotropy group at p can be identified with a subgroup of $O(T_pM)$, the linear isotropy group at p. Now, if we suppose that $\dim M$ is odd, then any orientation-preserving element $f_{*|p}$ of the linear isotropy group admits the eigenvalue 1. Let v be a unit tangent vector corresponding to this eigenvalue and consider the geodesic through p given by $\exp_p(tv)$. Then $f(\exp_p(tv))$ is also a geodesic with the same initial conditions as those of $\exp_p(tv)$ and hence

$$f\big(\exp_p(tv)\big) = \exp_p(tv).$$

Motivated by these considerations, in particular by (3.1), we now turn to the definition of rotations.

DEFINITIONS. Let $S(t)$ be a field of linear endomorphisms

$$S(t)\colon T_{\sigma(t)}M \to T_{\sigma(t)}M$$

along the curve σ such that $S(t)$ restricted to $T_{\sigma(t)}\sigma$ is the identity map and on each fiber $T^{\perp}_{\sigma(t)}\sigma$ of the normal bundle $N\sigma$ it is a linear isometry, that is,

$$S(t)\dot\sigma = \dot\sigma, \quad g(S(t)x, S(t)y) = g(x, y)$$

for all $x, y \in T^{\perp}_{\sigma(t)}\sigma$. Then $S(t)$ is said to be a *rotation field along* σ. (In what follows we shall use the same notation $S(t)$ to indicate the operator on $T_{\sigma(t)}M$ as well as its restriction to the fiber of $N\sigma$ at $\sigma(t)$.)

Now, let U be a tubular neighborhood of σ with sufficiently small radius. Then the local diffeomorphism s_σ defined by

$$s_\sigma = \exp_\sigma \circ S \circ \exp_\sigma^{-1}$$

is called a (*local*) *S-rotation around* σ. Moreover, if $S - I$ is non-singular in the normal bundle, we say that s_σ is a *free S-rotation*.

For $S = -I$, s_σ defines the *reflection with respect to* σ.

Note that we have

$$s_\sigma: U \to U: \exp_\sigma(\sigma(t), v) \mapsto \exp_\sigma(\sigma(t), S(t)v).$$

Furthermore, σ is contained in the fixed point set of s_σ.

The analytic expressions of s_σ follow easily by using a system of Fermi coordinates. We have

$$x^1 \circ s_\sigma = x^1, \quad x^i \circ s_\sigma = S^i_j x^j \tag{3.2}$$

where $S^i_j(t)$ are the components of $S(t)$ at $\sigma(t)$ with respect to the basis $\{E_2(t), \ldots, E_n(t)\}$ defined in Section 2. Moreover, we have $s_{\sigma *|\sigma(t)} = S(t)$ for all $t \in [a, b]$.

From the expressions (3.2) it is clear that the study of S-rotations is different from and somewhat more complicated than that of rotations around a point due to the special role played by the x^1-coordinate.

Remark. Note that S is *parallel* along σ if and only if S is parallel with respect to $\nabla^\perp$ and $S\ddot{\sigma} = \ddot{\sigma}$. In this case it follows that each higher order derivative of σ is also an eigenvector of S with eigenvalue $+1$, that is,

$$S\sigma^{(k)} = \sigma^{(k)}, \quad k \in \mathbf{N}_0.$$

So, once a parallel rotation field S is given, we have restrictions on σ. For example, if S defines a reflection, i.e. $S = -I$ in $N\sigma$, then $\ddot{\sigma} = 0$ and hence, σ is a geodesic. The same holds when S is a free rotation field.

Note that Lemma 3.1. yields that each isometry f is a rotation around any curve σ contained in the fixed point set and its rotation field is $f_{*|\sigma}$. As may be checked directly, this rotation field is parallel. We stress the fact that the isometric rotations around σ are exactly the isometries which have a (totally geodesic) fixed point set of positive dimension containing σ and this is the only relation between the curve and the isometry.

Now, we will look for the conditions under which a rotation field S along σ defines an *isometric rotation*. This criterion will be used in Section 4.

THEOREM 3.2. *Let σ: $[a, b] \to M$ be a C^∞ embedded curve in a Riemannian manifold (M, g) and suppose that the S-rotation s_σ is an isometry. Then*

(3.3) *S is parallel along σ*

and

$$(3.4) \qquad \left(\nabla^k_{u\cdots u}R\right)_{uxuy} = \left(\nabla^k_{Su\cdots Su}R\right)_{SuSxSuSy}$$

for all $u \in T^{\perp}_{\sigma(t)}\sigma$, *all* $x, y \in T_{\sigma(t)}M$, *all* $t \in [a, b]$ *and all* $k \in N$.

Conversely, if (M, g) *is analytic and* S *is a rotation field along* σ *such that* (3.3) *and* (3.4) *hold, then the corresponding* S*-rotation is an isometry.*

Proof. First, let s_σ be an isometry. Then $s_{\sigma *|\sigma}$ is parallel along σ and since $s_{\sigma *|\sigma} = S$, S is parallel. Finally, (3.4) follows since any isometry preserves the curvature tensor and its covariant derivatives.

To prove the converse one may use one of the methods, as developed in [8], [18] (see also [9]), to write down power series expansions for the components of analytic tensor fields with respect to a Fermi coordinate system. Then it is not difficult to see that the coefficients in the expansions of the components of the metric tensor g only depend on the subset

$$\left\{\left(\nabla^k_{u\cdots u}R\right)_{u\cdot}u, \quad u \in T^{\perp}_{\sigma(t)}\sigma, k \in \mathbf{N}\right\}$$

of the set of all covariant derivatives of the curvature tensor R and on the (mean) curvature vector $\ddot{\sigma}$ of σ. Then

$$S\ddot{\sigma} = \ddot{\sigma}$$

since $S\dot{\sigma} = \dot{\sigma}$ and S is parallel and this together with (3.4) shows that s_σ is an isometry. This finishes the proof.

Remark. A similar theorem has been proved in [13] by using an alternative method for rotations around a point. That result is an immediate weaker version of the classical theorem of Cartan concerning the existence of local isometries on normal neighborhoods. Theorem 3.2 may be viewed as a generalization of Cartan's theorem to rotations around a curve.

The criterion given in Theorem 3.2 becomes considerably simpler for locally symmetric spaces. In this case we have:

COROLLARY 3.3. *Let* (M, g) *be a locally symmetric Riemannian manifold and* σ *a curve as in Theorem* 3.2. *Then the* S*-rotation* s_σ *is an isometry if and only if*

$$(3.5) \qquad S \text{ is parallel along } \sigma$$

and

$$(3.6) \qquad R_{uxuy} = R_{SuSxSuSy}$$

for all $u \in T^{\perp}_{\sigma(t)}\sigma$, *all* $x, y \in T_{\sigma(t)}M$ *and all* $t \in [a, b]$.

To finish this section we shall apply this criterion to consider isometric rotations in real, complex and quaternionic space forms.

(a) First, let (M, g) be a space of constant curvature c. Since

$$R_{XYZW} = c\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\},$$

we see that (3.6) is always satisfied. Hence s_σ *is an isometry if and only if the rotation field* S *is parallel.*

As a consequence of this we see that a reflection is an isometry if and only if σ is a geodesic. (This result was also obtained in [2], [16].) To see this, we first suppose that s_σ is an isometry. Then the result follows from the second remark in this section. Conversely, suppose σ is a geodesic. Then $S\dot\sigma = \dot\sigma$ gives

$$\dot S\dot\sigma = 0.$$

Moreover, for $u \in T^{\perp}_{\sigma(t)}\sigma$ we have $Su = -u$. Hence

$$\dot S u = -S\dot u - \dot u = \dot u - \dot u = 0$$

and hence S is parallel.

(b) Next, let (M, g, J) be a Kähler manifold of constant holomorphic sectional curvature $c \neq 0$. Then we have

$$\begin{aligned} R_{XYZW} = \tfrac{1}{4}c\{&g(X,Z)g(Y,W) - g(X,W)g(Y,Z) \\ &+g(JX,Z)g(JY,W) - g(JX,W)g(JY,Z) \\ &+2g(JX,Y)g(JZ,W)\}. \end{aligned}$$

From this we easily derive that (3.6) is equivalent to the following conditions for the rotation field S:

(3.7) $$SJ\dot\sigma = J\dot\sigma \quad \textit{and} \quad SJu = JSu$$

or

(3.8) $$SJ\dot\sigma = -J\dot\sigma \quad \textit{and} \quad SJu = -JSu$$

for all u orthogonal to $\dot\sigma$.

When s_σ is a reflection we derive from (3.7) and (3.8) that s_σ can never be an isometry except when $\dim M = 2$ in which case (M, g) has constant curvature and we return to the case (a).

(c) Finally, let (M, g) be a quaternionic Kähler manifold of constant quaternionic sectional curvature $c \neq 0$. In this case the Riemannian curvature tensor has the special form

$$R_{XYZW} = \tfrac{1}{4}c\Big\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z) + \sum_{\alpha=1}^{3}\big[g(J_\alpha X,Z)g(J_\alpha Y,W) - g(J_\alpha X,W)g(J_\alpha Y,Z) + 2g(J_\alpha X,Y)g(J_\alpha Z,W)\big]\Big\}$$

(see [11]). From this one derives that (3.6) is equivalent to

$$SJ_\alpha = \sum_{\beta=1}^{3} a_{\alpha\beta}J_\beta S, \quad \alpha = 1,2,3,$$

where $A = (a_{\alpha\beta}) \in SO(3)$ and $a_{\alpha\beta}$ are functions of t.

As for the complex case one derives that, when s_σ is a reflection with respect to a geodesic, then it can never be an isometry except for dim $M = 4$ in which case we have again a space of constant curvature and hence again case (a).

4. Harmonic rotations

Let (M, g) and (N, h) be Riemannian manifolds with metrics g and h and let

$$\varphi\colon (M,g) \to (N,h)$$

be a smooth map. The covariant differential $\nabla(\varphi_*)$ is a symmetric tensor of order two which is called the *second fundamental form* of φ. The trace of $\nabla(\varphi_*)$ is denoted by $\tau(\varphi)$ and is called the *tension field* of φ. A *harmonic map* φ is a map with vanishing $\tau(\varphi)$ (see [4], [5]).

To express this condition analytically, let $U \subset M$ be a domain with coordinates $(x^1,\dots,x^m)$ and $V \subset N$ a domain with coordinates $(y^1,\dots,y^n)$ such that $\varphi(U) \subseteq V$. Then φ can be locally represented by $y^\alpha = \varphi^\alpha(x^1,\dots,x^m)$, $\alpha = 1,\dots,n$. Further, we have

$$\nabla(\varphi_*)^\gamma_{ij} = \frac{\partial^2\varphi^\gamma}{\partial x^i\,\partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial\varphi^\gamma}{\partial x^k} + {}^N\Gamma^\gamma_{\alpha\beta}(\varphi)\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}, \tag{4.1}$$

$i, j = 1,\dots,m$ and $\gamma = 1,\dots,n$. Here ${}^M\Gamma^k_{ij}$ and ${}^N\Gamma^\gamma_{\alpha\beta}$ denote the Christoffel symbols for (M, g) and (N, h), respectively. Hence, φ is harmonic if and only if

$$\tau(\varphi)^\gamma = g^{ij}(\nabla(\varphi_*))^\gamma_{ij} = 0. \tag{4.2}$$

In the rest of this section we focus on *harmonic S-rotations* around a curve. Our aim is to prove:

THEOREM 4.1. *Let $\sigma\colon [a,b] \to (M,g)$ be a smooth embedded curve in a Riemannian manifold M and s_σ an S-rotation around σ. If s_σ is harmonic, then S is parallel along σ. Moreover, if s_σ is a free S-rotation, then σ is a geodesic.*

THEOREM 4.2. *Let s_σ be a harmonic free rotation on a locally symmetric space such that the Ricci tensor is S-invariant. Then s_σ is an isometry and conversely.*

Then we get easily the following corollaries.

COROLLARY 4.3. *A free rotation s_σ on a locally symmetric Einstein space is harmonic if and only if it is an isometry.*

COROLLARY 4.4. *A rotation around a geodesic in a locally symmetric Einstein space is harmonic if and only if it is an isometry.*

From (4.2) and (3.2) we get that s_σ is harmonic if and only if

$$\tag{4.3} \begin{aligned} \tau(s_\sigma)^c(p) &= \{g^{11}(\nabla s_{\sigma *})^c_{11} + 2g^{1a}(\nabla s_{\sigma *})^c_{1a} + g^{ab}(\nabla s_{\sigma *})^c_{ab}\}(p) \\ &= 0, \end{aligned}$$

$$\tag{4.4} \begin{aligned} \tau(s_\sigma)^1(p) &= \{g^{11}(\nabla s_{\sigma *})^1_{11} + 2g^{1a}(\nabla s_{\sigma *})^1_{1a} + g^{ab}(\nabla s_{\sigma *})^1_{ab}\}(p) \\ &= 0, \end{aligned}$$

with $a,b,c = 2,\dots,n$, $p = \exp_{\sigma(t)}(ru)$, $\|u\| = 1$ and where

(4.5)

$$\begin{aligned}
(\nabla s_{\sigma *})^1_{11}(p) &= -\Gamma^1_{11}(p) + \Gamma^1_{\alpha\beta}(s_\sigma(p))\dot S^\alpha_\delta \dot S^\beta_\mu x^\delta x^\mu + \Gamma^1_{\alpha 1}(s_\sigma(p))\dot S^\alpha_\delta x^\delta \\
&\quad + \Gamma^1_{1\beta}(s_\sigma(p))\dot S^\beta_\mu x^\mu + \Gamma^1_{11}(s_\sigma(p)), \\
(\nabla s_{\sigma *})^1_{1a}(p) &= -\Gamma^1_{1a}(p) + \Gamma^1_{\alpha\beta}(s_\sigma(p))\dot S^\alpha_\gamma S^\beta_a x^\gamma + \Gamma^1_{1\beta}(s_\sigma(p))S^\beta_a, \\
(\nabla s_{\sigma *})^1_{ab}(p) &= -\Gamma^1_{ab}(p) + \Gamma^1_{\alpha\beta}(s_\sigma(p))S^\alpha_a S^\beta_b, \\
(\nabla s_{\sigma *})^c_{11}(p) &= \ddot S^c_\gamma x^\gamma - \Gamma^1_{11}(p)\dot S^c_\delta x^\delta - \Gamma^k_{11}(p)S^c_k \\
&\quad + \Gamma^c_{\alpha\beta}(s_\sigma(p))\dot S^\alpha_\mu \dot S^\beta_\nu x^\mu x^\nu + \Gamma^c_{\alpha 1}(s_\sigma(p))\dot S^\alpha_\mu x^\mu \\
&\quad + \Gamma^c_{1\beta}(s_\sigma(p))\dot S^\beta_\nu x^\nu + \Gamma^c_{11}(s_\sigma(p)), \\
(\nabla s_{\sigma *})^c_{1a}(p) &= \dot S^c_a - \Gamma^k_{1a}(p)S^c_k - \Gamma^1_{1a}(p)\dot S^c_\mu x^\mu + \Gamma^c_{\alpha\beta}(s_\sigma(p))\dot S^\alpha_\mu S^\beta_a x^\mu \\
&\quad + \Gamma^c_{1\beta}(s_\sigma(p))S^\beta_a, \\
(\nabla s_{\sigma *})^c_{ab}(p) &= -\Gamma^k_{ab}(p)S^c_k - \Gamma^1_{ab}(p)\dot S^c_\mu x^\mu + \Gamma^c_{\alpha\beta}(s_\sigma(p))S^\alpha_a S^\beta_b.
\end{aligned}$$

Next, we put

$$\tau(s_\sigma)^c(p) = \sum_{t=0}^{3} A_t^c r^t + O(r^4), \quad c = 2, \dots, n, \tag{4.6}$$

$$\tau(s_\sigma)^1(p) = \sum_{t=0}^{3} A_t^1 r^t + O(r^4). \tag{4.7}$$

Then (4.3) and (4.4) give the following necessary conditions to have a harmonic rotation s_σ:

$$A_t^c = 0, \quad A_t^1 = 0, \quad t = 0, 1, 2, 3, \quad c = 2, \dots, n. \tag{4.8}$$

Hence, we have to compute the expressions for A_t^c and A_t^1. To do this we use the classical formula for the Christoffel symbols in terms of the components of the metric tensor:

$$\Gamma_{ij}^k(p) = \frac{1}{2}\left\{\sum_{l=1}^{n} g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)\right\}(p).$$

Now, using the expressions in Lemma 2.1, (4.5) and (4.3), (4.4), we get the desired coefficients A_t^c and A_t^1, $c = 2, \dots, n$. (We omit the lengthy but straightforward computations.) Then we are ready for the proofs.

Proof of Theorem 4.1. Using $A_0^c, c = 2, \dots, n$, we obtain

$$g(\ddot\sigma, E_c) - g(\ddot\sigma, SE_c) = 0, \quad c = 2, \dots, n.$$

This yields

$$({}^tS - I)\ddot\sigma = k\dot\sigma$$

for some k. On the other hand, $({}^tS - I)\ddot\sigma$ is orthogonal to $\dot\sigma$ and hence

$$S\ddot\sigma = \ddot\sigma. \tag{4.9}$$

Next, from the conditions $A_1^c = 0$, we obtain, taking into account (4.9),

$$g(\ddot{S}u, E_c) - (R_{1u1S^{-1}c} - R_{1Su1c}) - \frac{2}{3}\sum_{a=2}^{n}(R_{uaS^{-1}ca} - R_{SuSacSa}) = 0 \tag{4.10}$$

or, equivalently,

$$g\big(\ddot{S}u, SE_c\big) - (R_{1u1c} - R_{1Su1Sc}) \\ - \frac{2}{3}\sum_{a=2}^{n}(R_{uaca} - R_{SuSaScSa}) = 0 \tag{4.11}$$

where Sc and $S^{-1}c$ denote the vectors

$$(S(\partial/\partial x^c))(\sigma(t)) \quad \text{and} \quad (S^{-1}(\partial/\partial x^c))(\sigma(t)),$$

respectively, for $c = 2, \dots, n$. Now, put $E_c = u$ in (4.11). Then we obtain

$$g(\ddot{S}u, Su) - (R_{1u1u} - R_{1Su1Su}) - \frac{2}{3}\sum_{a=2}^{n}(R_{uaua} - R_{SuSaSuSa}) = 0. \tag{4.12}$$

Since $\|u\| = 1$ we have $g(\dot{S}u, Su) = 0$. Differentiating once again we get

$$\begin{aligned} 0 &= g(\ddot{S}u, Su) + g(\dot{S}\dot{u}, Su) + g(\dot{S}u, \dot{S}u) + g(\dot{S}u, S\dot{u}) \\ &= g(\ddot{S}u, Su) + g(\dot{S}u, \dot{S}u) + g\big(({}^t S\dot{S} + {}^t\dot{S}S)u, \dot{u}\big) \\ &= g(\ddot{S}u, Su) + g(\dot{S}u, \dot{S}u) \end{aligned} \tag{4.13}$$

because ${}^t S\dot{S} + {}^t\dot{S}S = 0$ on normal vectors to σ. Using this in (4.12), then putting $u = E_c$ and summing up with respect to $c = 2, \dots, n$, we get with $\dot{S}\dot{\sigma} = 0$,

$$\|\dot{S}\|^2 + \frac{2}{3}\sum_{a,c=2}^{n}(R_{caca} - R_{ScSaScSa}) = 0.$$

This implies

$$\nabla_{\dot{c}} S = 0; \text{ i.e., } S \textit{ is parallel}.$$

Finally, it is clear from (4.9) that, if s_σ is free, then $\ddot{\sigma} = 0$ and this finishes the proof.

Proof of Theorem 4.2. Since any isometry is harmonic we only have to prove the direct part of the theorem. So, let s_σ be harmonic. Following Theorem 3.2 we have to prove

$$R_{1u1u} = R_{1Su1Su}, \quad R_{1uau} = R_{1SuSaSu}, \quad R_{aubu} = R_{SaSuSbSu},$$

$a, b = 2, \dots, n.$

From (4.12) and the fact that S is parallel we get

$$R_{1u1u} - R_{1Su1Su} + \frac{2}{3}\sum_{a=2}^{n}(R_{uaua} - R_{SuSaSuSa}) = 0 \tag{4.14}$$

and since ρ is S-invariant this yields at once

$$R_{1u1u} = R_{1Su1Su} \tag{4.15}$$

for all $u \in T^{\perp}_{\sigma(t)}\sigma$.

Further, we consider the conditions $A_3^c = 0$. Since s_σ is free, σ is a geodesic. By putting $S^{-1}c = u$, the conditions becomes

$$\begin{aligned}(4.16)\quad & -60\sum_{a=2}^{n} R_{1uau}(R_{1uau} - R_{1SuSaSu}) + 36\sum_{a=2}^{n}\left(R^2_{1uau} - R^2_{1SuSaSu}\right)\\ & + 6\sum_{a,b=2}^{n}\left(R^2_{uaub} - R^2_{SuSaSuSb}\right)\\ & - 10\sum_{a,b=2}^{n} R_{uaub}(R_{uaub} - R_{SuSaSuSb}) = 0.\end{aligned}$$

To handle this condition we integrate (4.16) over the unit sphere $S^{n-2}(1)$ in $T^{\perp}_{\sigma(t)}\sigma$. (See [1], [7], [8], [9] for more details.) First, note that the integrals of

$$\sum_{a=2}^{n}\left(R^2_{1uau} - R^2_{1SuSaSu}\right) \quad \text{and} \quad \sum_{a=2}^{n}\left(R^2_{uaub} - R^2_{SuSaSuSb}\right)$$

are zero. Next, let

$$A = \sum_{a=2}^{n} R_{1uau}(R_{1uau} - R_{1SuSaSu}),$$

$$B = \sum_{a,b=2}^{n} R_{uaub}(R_{uaub} - R_{SuSaSuSb})$$

and let

$$c_{n-2} = \frac{(n-1)\pi^{(n-1)/2}}{\left(\frac{n-1}{2}\right)!}$$

denote the volume of the unit sphere in Euclidean space E^{n-1}. Then we have

$$\int_{S^{n-2}(1)} A\,du = \frac{c_{n-2}}{(n-1)(n+1)} \sum_{a,i,j=2}^{n} \big[R_{1iai}R_{1jaj} + R_{1iaj}R_{1iaj} + R_{1iaj}R_{1jai} - R_{1iai}R_{1SjSaSj} - R_{1iaj}R_{1SiSaSj} - R_{1iaj}R_{1SjSaSi}\big]$$

$$= \frac{c_{n-2}}{(n-1)(n+1)}\left[\sum_{\alpha=1}^{n}\sum_{a,j=2}^{n}\left(R^2_{1\alpha aj} - R_{1\alpha aj}R_{1S\alpha SaSj}\right) + \sum_{a=2}^{n}\sum_{\alpha,\beta=1}^{n}\left(R_{1\alpha a\beta}R_{1\beta a\alpha} - R_{1\alpha a\beta}R_{1S\beta SaS\alpha}\right)\right]$$

$$= \frac{c_{n-2}}{(n-1)(n+1)}\left[\sum_{\alpha,\beta,\gamma=1}^{n}\left(R^2_{1\alpha\beta\gamma} + R_{1\alpha\beta\gamma}R_{1\gamma\beta\alpha} - R_{1\alpha\beta\gamma}R_{1S\alpha S\beta S\gamma} - R_{1\alpha\beta\gamma}R_{1S\gamma S\beta S\alpha}\right) - 3\sum_{\alpha,\beta=1}^{n}\left(R^2_{1\alpha 1\beta} - R_{1\alpha 1\beta}R_{1S\alpha 1S\beta}\right)\right].$$

Now we use the following identities [7]:

$$\sum_{\alpha,\beta,\gamma=1}^{n} R_{1\alpha\beta\gamma}R_{1\gamma\beta\alpha} = \frac{1}{2}\sum_{\alpha,\beta,\gamma=1}^{n} R^2_{1\alpha\beta\gamma}, \tag{4.17}$$

$$\sum_{\alpha,\beta,\gamma=1}^{n} R_{1\alpha\beta\gamma}R_{1S\gamma S\beta S\alpha} = \frac{1}{2}\sum_{\alpha,\beta,\gamma=1}^{n} R_{1\alpha\beta\gamma}R_{1S\alpha S\beta S\gamma} \tag{4.18}$$

to obtain

$$\int_{S^{n-2}(1)} A\,du = \frac{3c_{n-2}}{2(n-1)(n+1)} \sum_{\alpha,\beta,\gamma=1}^{n} R_{1\alpha\beta\gamma}\left(R_{1\alpha\beta\gamma} - R_{1S\alpha S\beta S\gamma}\right). \tag{4.19}$$

Further, we use the same procedure to compute the integral of B. We get

$$\int_{S^{n-2}(1)} B\,du = \frac{c_{n-2}}{(n-1)(n+1)} \sum_{a,b=2}^{n}\left[\left(\rho_{ab} - R_{1a1b}\right)^2 + \frac{3}{2}\sum_{i,j=2}^{n} R^2_{iajb} - \left(\rho_{ab} - R_{1a1b}\right)\left(\rho_{SaSb} - R_{1Sa1Sb}\right) + \frac{3}{2}\sum_{i,j=2}^{n} R_{iajb}R_{SiSaSjSb}\right].$$

Using (4.15) and the S-invariance of ρ we obtain

(4.20)
$$\int_{S^{n-2}(1)} B\,du = \frac{3c_{n-2}}{2(n-1)(n+1)} \sum_{a,b,i,j=2}^{n} R_{iajb}\left(R_{iajb} - R_{SiSaSjSb}\right).$$

Finally, (4.16), (4.19) and (4.20) yield

$$6 \sum_{\alpha,\beta,\gamma=1}^{n} R_{1\alpha\beta\gamma}\left(R_{1\alpha\beta\gamma} - R_{1S\alpha S\beta S\gamma}\right) + \sum_{a,b,i,j=2}^{n} R_{iajb}\left(R_{iajb} - R_{SiSaSjSb}\right) = 0$$

and hence also

(4.21)
$$6 \sum_{\alpha,\beta,\gamma=1}^{n} \left(R_{1\alpha\beta\gamma} - R_{1S\alpha S\beta S\gamma}\right)^2 + \sum_{a,b,i,j=2}^{n} \left(R_{iajb} - R_{SiSaSjSb}\right)^2 = 0.$$

So, this gives

$$R_{1\alpha\beta\gamma} = R_{1S\alpha S\beta S\gamma}, \qquad \alpha,\beta,\gamma = 1,\dots,n,$$
$$R_{iajb} = R_{SiSaSjSb}, \qquad i,j,a,b = 2,\dots,n.$$

Hence the required result is obtained.

Remark. In our theorems we restricted to the case of locally symmetric spaces with S-invariant Ricci tensor because in the more general case the conditions $A_t^c = 0$, $A_t^1 = 0$ become more complicated and up to now we do not know if our results can be extended to the case of general Riemannian manifolds.

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