ILLINOIS JOURNAL OF MATHEMATICS
Volume 37, Number 3, Fall 1993

ON A THEOREM OF AKHIEZER

ELIZABETH KOCHNEFF, YORAM SAGHER AND RUBY TAN

1. Introduction

Akhiezer, [1], showed for $f \in L^2(R)$, $\gamma = a + ib$, $b \neq 0$, that

$$(t-\gamma)H\left(\frac{f(x)}{x-\gamma}\right)(t) = Hf(t) - C(\gamma, f)$$

where H is the Hilbert transform and $C(\gamma, f)$ is a constant depending on f and γ.

If γ is real and both $f \in L^2(R)$ and $(f(t)-\alpha)/(t-\gamma) \in L^2(R)$, Akhiezer showed

$$(t-\gamma)H\left(\frac{f(x)-\alpha}{x-\gamma}\right)(t) = Hf(t) - C(\gamma, f).$$

Akhiezer's proof depends on calculations of Fourier transforms, using complex methods, and therefore does not seem to generalize to $p \neq 2$. A much simpler proof of Akhiezer's theorem in the case $\alpha = 0$ is given in [3]. We prove the theorem under the hypotheses

$$f \in L^1\big(R, dt/(1+t^2)\big) \quad \text{and} \quad (f(t)-\alpha)/(t-\gamma) \in L^1_{loc}.$$

For $\gamma \in R$, if $f \in L^1(R, dt/(1+|t|))$ or if $f \in L^1(T)$, and if $(f(t)-\alpha)/(t-\gamma) \in L^1_{loc}$, we show that $Hf(\gamma)$ exists and equals $C(\gamma, f)$. Since we may

Received September 3, 1991
1991 Mathematics Subject Classification. Primary 44A15; Secondary 26A99.

assume $\alpha = f(\gamma)$ we obtain

$$H\left(\frac{f(x) - f(\gamma)}{x - \gamma}\right)(t) = \frac{Hf(t) - Hf(\gamma)}{t - \gamma}.$$

We further extend the theorem to calculate the commutators with $(x - \gamma)^k$.

Akhiezer's theorem provides a useful tool to calculate the Hilbert transforms of some interesting functions. We give some examples of such calculations.

2. Extensions of Akhiezer's Theorem

Define $k(t) = 1/t$ for $|t| \geq 1$ and $k(t) = 0$ for $|t| < 1$.

If $f \in L^1_{loc}(R)$ and if

$$\lim_{N\to\infty} \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{\varepsilon < |t-x| < N} \frac{f(x)}{t - x}\, dx$$

exists, then this limit is defined to be the Hilbert transform of f and is denoted Hf. This limit exists a.e. for $f \in L^1(R, dt/(1 + |t|))$ and for $f \in L^1(T)$ (see [5]).

If the above limit does not exist, but

$$\lim_{N\to\infty} \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{\varepsilon<|t-x|<N} f(x)\left[\frac{1}{t - x} + k(x)\right] dx$$

exists, then this limit is defined to be the Hilbert transform of f up to an additive constant and is denoted Hf. The definition up to an additive constant is necessary to ensure that the Hilbert transform commutes with translations and dilations. This definition is valid for $f \in L^1(R, dt/(1 + t^2))$; this space includes $BMO(R)$ (see [4]).

We define

$$E_{\alpha,\gamma} f(x) = \frac{f(x) - \alpha}{x - \gamma}. \tag{1}$$

THEOREM 1. *If $f \in L^1(R, dt/(1 + t^2))$ and if $E_{\alpha,\gamma} f \in L^1_{loc}$, then*

$$(t - \gamma) H(E_{\alpha,\gamma} f)(t) = Hf(t) - C(\alpha, \gamma, f) \tag{2}$$

where $C(\alpha, \gamma, f)$ is a constant depending only on f, α, and γ.

For $\gamma \in R$, if $f \in L^1(R, dt/(1 + |t|))$ or if $f \in L^1(T)$, then $C(\alpha, \gamma, f) = Hf(\gamma)$.

Proof. Let us consider simultaneously the cases $f \in L^1(R, dt/(1+|t|))$ and $f \in L^1(T)$. We have

$$
\begin{aligned}
&(t-\gamma)H(E_{\alpha,\gamma}f)(t) - Hf(t)\\
&\quad = \lim_{N\to\infty}\lim_{\varepsilon\to 0}(t-\gamma)\frac{1}{\pi}\int_{\varepsilon<|t-x|<N}\frac{f(x)-\alpha}{x-\gamma}\cdot\frac{1}{t-x}\,dx\\
&\qquad - \lim_{N\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon<|t-x|<N}\frac{f(x)}{t-x}\,dx\\
&\quad = \lim_{N\to\infty}\lim_{\varepsilon\to 0}(t-\gamma)\frac{1}{\pi}\int_{\varepsilon<|t-x|<N}\frac{f(x)-\alpha}{x-\gamma}\cdot\frac{1}{t-x}\,dx\\
&\qquad - \lim_{N\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon<|t-x|<N}\frac{f(x)-\alpha}{t-x}\,dx\\
&\quad = \lim_{N\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon<|t-x|<N}\frac{f(x)-\alpha}{t-x}\cdot\left(\frac{t-\gamma}{x-\gamma}-1\right)dx\\
&\quad = \lim_{N\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon<|t-x|<N}\frac{f(x)-\alpha}{x-\gamma}\,dx\\
&\quad = \lim_{N\to\infty}\frac{1}{\pi}\int_{|t-x|<N}\frac{f(x)-\alpha}{x-\gamma}\,dx.
\end{aligned}
$$

We start by centering the integral at $a = \operatorname{Re}\gamma$. Let $\beta = t - a$, and assume $\beta > 0$. We have

$$
\begin{aligned}
&\int_{|x-t|<N}\frac{f(x)-\alpha}{x-\gamma}\,dx\\
&\quad = \int_{-N+\beta+a<x<N+\beta+a}\frac{f(x)-\alpha}{x-\gamma}\,dx\\
&\quad = \int_{|x-a|<N+\beta}\frac{f(x)-\alpha}{x-\gamma}\,dx - \int_{-N-\beta<x-a<-N+\beta}\frac{f(x)-\alpha}{x-\gamma}\,dx.
\end{aligned}
$$

The second integral converges to 0 as $N \to \infty$ since

$$
\left|\int_{-N-\beta}^{-N+\beta}\frac{f(u+a)-\alpha}{u-ib}\,du\right| \le \int_{-N-\beta}^{-N+\beta}\left|\frac{f(u+a)}{u-ib}\right|du + |\alpha|\int_{-N-\beta}^{-N+\beta}\frac{1}{|u|}\,du.
$$

Thus we have shown

$$(t-\gamma)H(E_{\alpha,\gamma}f)(t) - Hf(t) = \frac{1}{\pi}\lim_{N\to\infty}\int_{|x-a|<N}\frac{f(x)-\alpha}{x-\gamma}\,dx.$$

Consider the case of $\gamma \in R$. We have

$$\begin{aligned}\lim_{N\to\infty}&\int_{|x-\gamma|<N}\frac{f(x)-\alpha}{x-\gamma}\,dx\\ &= \int_{|x-\gamma|\le 1}\frac{f(x)-\alpha}{x-\gamma}\,dx + \lim_{N\to\infty}\int_{1<|x-\gamma|<N}\frac{f(x)}{x-\gamma}\,dx.\end{aligned}$$

It is well known that the above limit exists for $f \in L^1(T)$ (see [5]). Therefore,

$$\begin{aligned}(t-\gamma)&H(E_{\alpha,\gamma}f)(t) - Hf(t)\\ &= \frac{1}{\pi}\int_{|x-\gamma|\le 1}\frac{f(x)-\alpha}{x-\gamma}\,dx + \frac{1}{\pi}\int_{1<|x-\gamma|}\frac{f(x)}{x-\gamma}\,dx\\ &= \lim_{N\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon<|x-\gamma|<N}\frac{f(x)}{x-\gamma}\,dx\\ &= -Hf(\gamma).\end{aligned}$$

Therefore we have shown that if γ is real, then $Hf(\gamma)$ exists and $C(\alpha,\gamma,f) = Hf(\gamma)$.

We now return to the case $\gamma = a + ib$, $b \neq 0$:

$$\begin{aligned}\lim_{N\to\infty}&\frac{1}{\pi}\int_{|x-a|<N}\frac{f(x)-\alpha}{x-\gamma}\,dx\\ &= \lim_{N\to\infty}\frac{1}{\pi}\int_{|x-a|<N}\frac{f(x)}{x-\gamma}\,dx - \alpha\lim_{N\to\infty}\frac{1}{\pi}\int_{|x-a|<N}\frac{1}{x-\gamma}\,dx\\ &= \frac{1}{\pi}\int_R\frac{f(x)}{x-\gamma}\,dx - \alpha\lim_{N\to\infty}\frac{1}{\pi}\int_{|x-a|<N}\frac{(x-a)+ib}{(x-a)^2+b^2}\,dx\\ &= \frac{1}{\pi}\int_R\frac{f(x)}{x-\gamma}\,dx - i\alpha(\operatorname{sgn} b),\end{aligned}$$

where $\operatorname{sgn} b = 1$ for $b > 0$ and $\operatorname{sgn} b = -1$ for $b < 0$. Therefore,

$$(t-\gamma)H(E_{\alpha,\gamma}f)(t) - Hf(t) = \frac{1}{\pi}\int_R\frac{f(x)}{x-\gamma}\,dx - i\alpha(\operatorname{sgn} b) = -C(\alpha,\gamma,f).$$

This proves the theorem if $f \in L^1(R, dt/(1+|t|))$ or $f \in L^1(T)$.

Finally, consider the case $f \in L^1(R, dt/(1+t^2))$. Since $E_{\alpha,\gamma} f \in L^1_{loc}$ we have

$$E_{\alpha,\gamma} f \in L^1(R, dt/(1+|t|)).$$

We have

$$\begin{aligned}
&(t-\gamma) H(E_{\alpha,\gamma} f)(t) - Hf(t) \\
&= \lim_{N\to\infty} \lim_{\varepsilon \to 0} (t-\gamma) \frac{1}{\pi} \int_{\varepsilon<|t-x|<N} \frac{f(x)-\alpha}{x-\gamma} \cdot \frac{1}{t-x}\, dx \\
&\quad - \lim_{N\to\infty} \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon<|t-x|<N} f(x) \left[\frac{1}{t-x} + k(x) \right] dx \\
&= \lim_{N\to\infty} \lim_{\varepsilon \to 0} (t-\gamma) \frac{1}{\pi} \int_{\varepsilon<|t-x|<N} \frac{f(x)-\alpha}{x-\gamma} \cdot \frac{1}{t-x}\, dx \\
&\quad - \lim_{N\to\infty} \lim_{\varepsilon \to 0} \frac{1}{\pi} \left(\int_{\varepsilon<|t-x|<N} (f(x)-\alpha) \left[\frac{1}{t-x} + k(x) \right] dx \right. \\
&\qquad\qquad \left. + \alpha \int_{\varepsilon<|x-t|<N} k(x)\, dx \right).
\end{aligned}$$

As $N \to \infty$ and $\varepsilon \to 0$, the last integral converges to zero, so that

$$\begin{aligned}
&(t-\gamma) H(E_{\alpha,\gamma} f)(t) - Hf(t) \\
&= \lim_{N\to\infty} \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon<|t-x|<N} (f(x)-\alpha) \left[\frac{1}{x-\gamma} - k(x) \right] dx \\
&= \frac{1}{\pi} \int_R (f(x)-\alpha) \left[\frac{1}{x-\gamma} - k(x) \right] dx \\
&= -C(\alpha, \gamma, f).
\end{aligned}$$

This concludes the proof of Theorem 1.

In the proof of Theorem 1, we show that for $\gamma \in R$, if $f \in L^1(R, dt/(1+|t|))$ or if $f \in L^1(T)$, and if $E_{\alpha,\gamma} f \in L^1_{loc}$, then $Hf(\gamma)$ exists. Since f is only defined a.e., we may assume $\alpha = f(\gamma)$. Define

$$E_\gamma f(t) = \frac{f(t)-f(\gamma)}{t-\gamma}.$$

In this case Theorem 1 shows that E_γ commutes with the Hilbert transform

$$H\left(\frac{f(x)-f(\gamma)}{x-\gamma}\right)(t) = \frac{Hf(t)-Hf(\gamma)}{t-\gamma}. \tag{3}$$

For $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{k-1})$ define

$$E_{\alpha,\gamma,k}f(t) = \frac{f(t) - P_{\alpha,k-1}(t-\gamma)}{(t-\gamma)^k} \tag{4}$$

where $P_{\alpha,k-1}(t) = \alpha_0 + \alpha_1 t + \cdots + a_{k-1}t^{k-1}$.

THEOREM 2. *If $f \in L^1(R, dt/(1+t^2))$ and $E_{\alpha,\gamma,k}f(t) \in L^1_{loc}$ then*

$$(t-\gamma)^k H(E_{\alpha,\gamma,k}f)(t) = Hf(t) - Q_{k-1}(t-\gamma)$$

where $Q_{k-1}(t)$ is a polynomial of degree $k-1$ whose coefficients depend only on f, α and γ.

Proof. Define

$$f_0(t) = f(t)$$

and

$$f_j(t) = \frac{f(t) - \alpha_0 - \cdots - \alpha_{j-1}(t-\gamma)^{j-1}}{(t-\gamma)^j}$$

for $j = 1, \dots, k$. Observe that

$$f_j(t) = \frac{f_{j-1}(t) - \alpha_{j-1}}{t-\gamma} = E_{\alpha_{j-1},\gamma}f_{j-1}(t)$$

and

$$f_k(t) = E_{\alpha,\gamma,k}f(t).$$

Since for $j = 1, \dots, k-1$, $f_j \in L^1(R, dt/(1+|t|))$, we have

$$(t-\gamma)Hf_{j+1}(t) = Hf_j(t) - C(\alpha_j, \gamma, f_j). \tag{5}$$

Iterating, we obtain the theorem.

For $\gamma = a + ib$, $b > 0$, if $f \in L^1(R, dt/(1+|t|))$ or $f \in L^1(T)$, and if $\alpha_j = 0$ for all j, we have

$$C(0,\gamma,f_j) = -\frac{1}{\pi}\int_R \frac{f_j(x)}{x-\gamma}\,dx = -\frac{1}{\pi}\int_R \frac{f(x)}{(x-\gamma)^{j+1}}\,dx.$$

Thus, letting

$$F(z) = \frac{1}{2\pi i}\int_R \frac{f(x)}{x-z}\,dx$$

for $z = t + iy$, $y > 0$, we have

$$(t-\gamma)^n H\left(\frac{f(x)}{(x-\gamma)^n}\right)(t) = Hf(t) + 2i\sum_{j=0}^{n-1}\frac{F^{(j)}(\gamma)}{j!}(t-\gamma)^j.$$

For $\gamma \in R$, if $f \in L^1(R, dt/(1+|t|))$ or $f \in L^1(T)$ we have

$$\begin{aligned}(t-\gamma)^k H(E_{\alpha,\gamma,k}f)(t)\\ &= Hf(t) - Hf(\gamma) - (t-\gamma)Hf_1(\gamma) - \cdots - (t-\gamma)^{k-1}Hf_{k-1}(\gamma).\end{aligned}$$

Clearly, for $\gamma \in R$, the coefficients of $P_{\alpha,k-1}(t)$ in the expression $E_{\alpha,\gamma,k}f(t)$ act as generalized derivatives of f at γ. This can also be expressed by the boundary values of the derivatives of the various extensions of f.

THEOREM 3. *Let $\phi(t) \in C^k \cap L^1(R)$ be such that $\int_R \phi(x)\,dx = 1$ and*

$$(1+|t|)^{k+2}\cdot|\phi^{(k)}(t)| \le M < \infty.$$

Let

$$\phi_\varepsilon(t) = \frac{1}{\varepsilon}\phi\left(\frac{t}{\varepsilon}\right).$$

If $f \in L^1(R, dt/(1+t^2))$, if γ is real, and if $E_{\alpha,\gamma,k}f \in L^1_{\mathrm{loc}}$ for $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$, then for $j = 0, 1, \ldots, k-1$ we have

$$j!\alpha_j = \lim_{\varepsilon\to 0}(\phi_\varepsilon * f)^{(j)}(\gamma),$$

where $(\phi_\varepsilon * f)^{(j)}(t)$ is the jth *derivative of $\phi_\varepsilon * f(t)$ with respect to the variable t.*

Proof. We may assume $\gamma = 0$. It is easy to see that for $j = 0, \dots, k-1$ there exists a constant M_j such that

$$(1 + |t|)^{j+2} \cdot |\phi^{(j)}(t)| \leq M_j,$$

Next, observe that $\int_R x^l \phi^{(j)}(-x)\, dx = 0$ for $l = 0, 1, \dots, j-1$ and $\int_R x^j \phi^{(j)}(-x)\, dx = j!$.

Recall the notation

$$f_0(t) = f(t)$$

and, for $j = 1, \dots, k$,

$$f_j(t) = \frac{f(t) - \alpha_0 - \cdots - \alpha_{j-1} t^{j-1}}{t^j}.$$

We have

$$\begin{aligned} |(\phi_\varepsilon * f)^{(j)}(0) - j!\alpha_j| &= \left| \frac{1}{\varepsilon} \int_R \frac{1}{\varepsilon^j} \phi^{(j)}\left(\frac{-x}{\varepsilon}\right) f(x)\, dx - j!\alpha_j \right| \\ &= \left| \frac{1}{\varepsilon} \int_R \frac{1}{\varepsilon^j} \phi^{(j)}\left(\frac{-x}{\varepsilon}\right) \left[f(x) - \alpha_0 - \cdots - \alpha_j x^j \right] dx \right| \\ &= \left| \int_R \left(\frac{x}{\varepsilon}\right)^{j+1} \phi^{(j)}\left(\frac{-x}{\varepsilon}\right) f_{j+1}(x)\, dx \right| \\ &\leq \int_R \left(1 + \left|\frac{x}{\varepsilon}\right|\right)^{j+1} \left|\phi^{(j)}\left(\frac{-x}{\varepsilon}\right)\right| |f_{j+1}(x)|\, dx \\ &\leq M_j \int_R \frac{|f_{j+1}(x)|}{1 + |x/\varepsilon|}\, dx. \end{aligned}$$

Since for $j = 1, \dots, k$, $f_j \in L^1(R, dt/(1 + |t|))$, the last integral converges to zero as $\varepsilon \to 0$ by the dominated convergence theorem. This concludes the proof of the theorem.

Note that the Poisson and Gaussian kernels satisfy the conditions of the theorem.

Observe that for $1 < p < \infty$, Theorem 2 proves that the class of functions f so that both $f \in L^p(R)$ and $E_{\alpha,\gamma,k} f \in L^p(R)$ is preserved by the Hilbert transform. Thus, Theorems 2 and 3 together give us the values of the coefficients of $Q_{k-1}(t)$ in Theorem 2 for $\gamma \in R$. If $Q_{k-1}(t) = \Sigma \beta_j t^j$, then $j!\beta_j = \lim_{\varepsilon \to 0} (H\phi_\varepsilon^{(j)}) * f(\gamma)$.

3. Calculations of Hilbert transforms of some functions

The results of the previous section provide a useful method for calculating the Hilbert transforms of some important functions. We illustrate this method by calculating the Hilbert transform, along an individual coordinate axis, of the n-dimensional Poisson kernel; we also calculate the Hilbert transform of the Gaussian. We have some intermediate results below which may be of independent interest.

LEMMA 4. *If $f \in L^1(R, dt/(1+|t|))$ or $f \in L^1(T)$ we have*

$$(1+t^2)H\left(\frac{f(x)}{1+x^2}\right)(t) = Hf(t) + \frac{1}{\pi}\int_R \frac{(x+t)f(x)}{1+x^2}\,dx.$$

Proof. From Theorem 1 we have

$$(t-i)H\left(\frac{f(x)}{x-i}\right)(t) = Hf(t) + \frac{1}{\pi}\int_R \frac{f(x)}{x-i}\,dx.$$

For $f_1(x) = f(x)/(x-i)$ we have

$$(t+i)H\left(\frac{f_1(x)}{x+i}\right)(t) = Hf_1(t) + \frac{1}{\pi}\int_R \frac{f_1(x)}{x+i}\,dx.$$

Therefore:

$$\begin{aligned}(1+t^2)H\left(\frac{f(x)}{1+x^2}\right)(t) &= (t-i)H\left(\frac{f(x)}{x-i}\right)(t) + \frac{t-i}{\pi}\int_R \frac{f(x)}{1+x^2}\,dx\\ &= Hf(t) + \frac{1}{\pi}\int_R\left[\frac{f(x)}{x-i} + \frac{f(x)(t-i)}{(x-i)(x+i)}\right]dx\\ &= Hf(t) + \frac{1}{\pi}\int_R \frac{x+t}{1+x^2}f(x)\,dx.\end{aligned}$$

THEOREM 5. *For $\alpha > 0$,*

$$H\left(\frac{1}{(1+x^2)^\alpha}\right)(t) = \frac{C(\alpha)}{(1+t^2)^\alpha}\int_0^t \frac{1}{(1+s^2)^{1-\alpha}}\,ds$$

where

$$C(\alpha) = \frac{2\alpha}{\pi}\int_R \frac{du}{(1+u^2)^{\alpha+1}}.$$

Proof. From Theorem 1, we have

$$\begin{aligned} H\left(\frac{x}{(1+x^2)^{\alpha+1}}\right)(t) &= tH\left(\frac{1}{(1+x^2)^{\alpha+1}}\right)(t) + H\left(\frac{x}{(1+x^2)^{\alpha+1}}\right)(0) \\ &= tH\left(\frac{1}{(1+x^2)^{\alpha+1}}\right)(t) - \frac{1}{\pi}\int_R \frac{du}{(1+u^2)^{\alpha+1}} \\ &= tH\left(\frac{1}{(1+x^2)^{\alpha+1}}\right)(t) - \frac{C(\alpha)}{2\alpha}. \end{aligned}$$

From Lemma 4, we have

$$\begin{aligned} H\left(\frac{1}{(1+x^2)^{\alpha+1}}\right)(t) &= \frac{1}{1+t^2}H\left(\frac{1}{(1+x^2)^{\alpha}}\right)(t) \\ &\quad + \frac{t}{1+t^2}\cdot\frac{1}{\pi}\int_R \frac{du}{(1+u^2)^{\alpha+1}} \\ &= \frac{1}{1+t^2}H\left(\frac{1}{(1+x^2)^{\alpha}}\right)(t) + \frac{C(\alpha)}{2\alpha}\cdot\frac{t}{1+t^2}. \end{aligned}$$

Let

$$f(t) = H\left(\frac{1}{(1+x^2)^{\alpha}}\right)(t).$$

Then

$$\begin{aligned} f'(t) &= H\left(\frac{-2\alpha x}{(1+x^2)^{\alpha+1}}\right)(t) \\ &= -2\alpha\left\{t\left[\frac{1}{1+t^2}H\left(\frac{1}{(1+x^2)^{\alpha}}\right)(t) + \frac{C(\alpha)}{2\alpha}\cdot\frac{t}{1+t^2}\right] - \frac{C(\alpha)}{2\alpha}\right\} \\ &= \frac{-2\alpha t}{1+t^2}f(t) + \frac{C(\alpha)}{1+t^2}. \end{aligned}$$

Hence,

$$(1+t^2)^{\alpha} f'(t) + \frac{2\alpha t}{(1+t^2)^{1-\alpha}} f(t) = \frac{C(\alpha)}{(1+t^2)^{1-\alpha}}$$

so that

$$\frac{d}{dt}\left[(1+t^2)^{\alpha} f(t)\right] = \frac{C(\alpha)}{(1+t^2)^{1-\alpha}}.$$

Thus,

$$f(t) = \frac{C(\alpha)}{(1+t^2)^{\alpha}}\left[\int_0^t \frac{1}{(1+s^2)^{1-\alpha}}\, ds + D\right].$$

Since $f(t)$ is odd, $D = 0$ and the theorem is proved.

Note for $\alpha = \frac{1}{2}$ we get

$$H\left(\frac{1}{\sqrt{1+x^2}}\right)(t) = \frac{2}{\pi}\frac{\ln\left(t+\sqrt{1+t^2}\right)}{\sqrt{1+t^2}}.$$

COROLLARY 6. *For* $\alpha > 0$,

$$H\left(\frac{x}{(1+x^2)^{\alpha+1}}\right)(t) = \frac{tC(\alpha+1)}{(1+t^2)^{\alpha+1}}\int_0^t (1+s^2)^{\alpha}\, ds - \frac{C(\alpha)}{2\alpha}.$$

Proof.

$$\begin{aligned} H\left(\frac{x}{(1+x^2)^{\alpha+1}}\right)(t) &= tH\left(\frac{1}{(1+x^2)^{\alpha+1}}\right)(t) - \frac{C(\alpha)}{2\alpha} \\ &= \frac{tC(\alpha+1)}{(1+t^2)^{\alpha+1}}\int_0^t (1+s^2)^{\alpha}\, ds - \frac{C(\alpha)}{2\alpha}. \end{aligned}$$

The Poisson kernel in n-dimensions is defined by

$$P_n(x) = \frac{C_n}{(1+|x|^2)^{(n+1)/2}} = \frac{C_n}{(1+x_1^2+\cdots+x_n^2)^{(n+1)/2}}$$

where

$$C_n = \Gamma\left(\frac{n+1}{2}\right)\pi^{-(n+1)/2}.$$

COROLLARY 7. *For $j = 1, \dots, n$, let*

$$\xi_j = \sqrt{1 + x_1^2 + \cdots + x_{j-1}^2 + x_{j+1}^2 + \cdots + x_n^2}$$

and let $H_j f(x)$ be the Hilbert transform of f with respect to x_j. Then

$$H_j P_n(x) = \frac{C_n \cdot C\left(\frac{n+1}{2}\right)}{\left(1 + |x|^2\right)^{(n+1)/2}} \int_0^{x_j/\xi_j} \left(1 + s^2\right)^{(n-1)/2} ds.$$

Proof. For $\alpha > 0$ and any constant A,

$$\begin{aligned} H\left(\frac{1}{\left(A^2 + u^2\right)^\alpha}\right)(t) &= \frac{1}{A^{2\alpha}} H\left(\frac{1}{\left(1 + \left(\frac{u}{A}\right)^2\right)^\alpha}\right)(t) \\ &= \frac{1}{A^{2\alpha}} \frac{C(\alpha)}{\left(1 + \left(\frac{t}{A}\right)^2\right)^\alpha} \int_0^{t/A} \frac{ds}{\left(1 + s^2\right)^{1-\alpha}} \\ &= \frac{C(\alpha)}{\left(A^2 + t^2\right)^\alpha} \int_0^{t/A} \frac{ds}{\left(1 + s^2\right)^{1-\alpha}}. \end{aligned}$$

Let $\alpha = (n+1)/2$, $t = x_j$, and $A = \xi_j$. This completes the proof.

THEOREM 8. *Assume that $f(z)$ is analytic in a strip $\mathscr{T} = \{z = x + iy: a < y < b\}$ and that $f(\cdot + iy) \in L^p(R)$ for $a < y < b$. Then the Hilbert transform $H(f(\cdot + iy))(x) = Hf(z)$ is analytic in $\mathscr{T}$.*

Proof. Let C be any rectifiable closed curve in $\mathscr{T}$. We have:

$$\begin{aligned} \int_C Hf(z)\, dz &= \int_C \text{p.v.} \int_R \frac{f(x - t + iy)}{t}\, dt\, dz \\ &= \int_C \int_{|t|<1} \frac{f(x - t + iy) - f(x + iy)}{t}\, dt\, dz \\ &\quad + \int_C \int_{|t| \geq 1} \frac{f(x - t + iy)}{t}\, dt\, dz \\ &= \int_{|t|<1} \int_C \frac{f(x - t + iy) - f(x + iy)}{t}\, dz\, dt \\ &\quad + \int_{|t| \geq 1} \int_C \frac{f(x - t + iy)}{t}\, dz\, dt = 0, \end{aligned}$$

since f is analytic. This proves the theorem.

THEOREM 9. *Let* $\mathscr{G}(z) = e^{-z^2/2}/\sqrt{2\pi}$ *be the complex Gaussian. Then*

$$H\mathscr{G}(z) = \frac{1}{\pi} e^{-z^2/2} \int_0^z e^{u^2/2}\, du. \tag{6}$$

Proof. For $z \in R$, this is known (see [2]). However, using the results obtained above we can give a new proof. Let $x \in R$ and let $\mathscr{S}(z) = H\mathscr{G}(z)$. Since $\mathscr{G}'(x) = -x\mathscr{G}(x)$, we have

$$\begin{aligned} \mathscr{S}'(x) &= -H(u\mathscr{G}(u))(x) \\ &= -[xH\mathscr{G}(x) + H(u\mathscr{G}(u))(0)] \\ &= -\left[x\mathscr{S}(x) - \frac{1}{\pi}\int_R \mathscr{G}(u)\, du\right] \\ &= -x\mathscr{S}(x) + \frac{1}{\pi}. \end{aligned}$$

Thus,

$$e^{x^2/2}\mathscr{S}'(x) + xe^{x^2/2}\mathscr{S}(x) = \frac{1}{\pi}e^{x^2/2}$$

so that

$$\frac{d}{dx}\left(e^{x^2/2}\mathscr{S}(x)\right) = \frac{1}{\pi}e^{x^2/2}.$$

Therefore

$$\mathscr{S}(x) = \frac{1}{\pi}e^{-x^2/2}\int_0^x e^{u^2/2}\, du + \frac{C}{\pi}e^{-x^2/2}.$$

Since the Hilbert transform of an even function is an odd function, we have $C = 0$. This proves (6) for $z \in R$.

Since, by Theorem 8, $\mathscr{H}\mathscr{G}(z)$ is an entire function which for real z coincides with $\mathscr{S}(z)$, we have $\mathscr{H}\mathscr{G}(z) = \mathscr{S}(z)$ for all z.

REFERENCES

1. N.I. AKHIEZER, *Theory of approximation*, Frederick Ungar, New York, 1956.
2. A.P. CALERÓN and Y. SAGHER, *The Hilbert transform of the Gaussian*, to appear.
3. J. GARCIA-CUERVA and J.L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North Holland, Amsterdam, 1985.
4. J. GARNETT, *Bounded analytic functions*, Academic Press, San Diego, 1981.
5. A. ZYGMUND, *Trigonometric series*, Vols. I, II, Cambridge University Press, New York, 1959.

THE UNIVERSITY OF ILLINOIS AT CHICAGO
CHICAGO, ILLINOIS