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THE SYMMETRIC GENUS OF FINITE ABELIAN GROUPS[1]

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1. Introduction

A finite group G can be represented as a group of automorphisms of a compact Riemann surface [3]. In other words, there is a compact Riemann surface on which G acts and each non-identity element of G acts non-trivially on the surface. The *symmetric genus* $\sigma(G)$ is the minimum genus of any Riemann surface on which G acts faithfully. The *strong symmetric genus* $\sigma^{\circ}(G)$ is the minimum genus of any surface on which G acts faithfully and preserves the orientation. This terminology was introduced by Tucker [11].

Here we consider abelian groups acting on Riemann surfaces. Let A be a finite abelian group. The strong symmetric genus $\sigma^{\circ}(A)$ has been completely determined by Maclachlan [5]. Also the abelian groups of symmetric genus zero and one are well-known. We will calculate the symmetric genus $\sigma(A)$ in the case where $\sigma(A) \geq 2$ by using non-euclidean crystallographic groups (NEC groups). Our basic approach is to represent A as a quotient of an NEC group Γ by a surface group K, so that A acts on the surface U/K, where U is the open upper half-plane. We show that there is an action of A on a surface of least genus induced by an NEC group with a signature of one of three types. Groups of type I are Fuchsian groups and the corresponding action is orientation preserving. Groups of types II and III contain reflections. We denote by $\tau(A)$ the minimum genus of any action of A induced by an NEC group of type II. The number $\tau(A)$ depends on the relative sizes of the ranks of certain parts of A. The size of the largest elementary abelian 2-group direct summand of A determines whether $\sigma(A)$ is given by an action induced by a group of type I, II, or III. Our main result is the following.

THEOREM 5.7. *Let A be an abelian group of even order with canonical form $(Z_2)^a \times Z_{m_1} \times \cdots \times Z_{m_d}$ where $m_1 > 2$. If the symmetric genus $\sigma(A) \geq 2$,*

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then

(i) $\sigma(A) = 1 + |A| \cdot (a + 3d - 4)/8$ *if* $a \geq d + 2$
(ii) $\sigma(A) = \tau(A)$ *if* $1 \leq a \leq d + 1$
(iii) $\sigma(A) = \min\{\sigma^\circ(A), \tau(A)\}$ *if* $a = 0$.

We emphasize here that the numbers $\sigma^\circ(A)$ and $\tau(A)$ are easily calculated for a particular group A, and we will indicate how to calculate $\tau(A)$ in the appropriate section.

There are, of course, other genus parameters for a finite group G. The most important is the graph theoretic genus $\gamma(G)$ [13]. The graph theoretic genus $\gamma(A)$ of an abelian group A was first studied by White [12] and he developed genus formulas in special cases. Jungerman and White [2] later found $\gamma(A)$ for "most" of the remaining abelian groups. There is an interesting similarity between the formula in Theorem 5.7(i) and the corresponding formula for the graph theoretic genus [12, p. 208, 209]. If $a > 0$, then the graph theoretic genus is

$$\gamma(A) = 1 + |A|(a + 2d - 4)/8.$$

The symmetric genus is also naturally related to the real genus [7]. The real genus $\rho(G)$ is the minimum algebraic genus of any bordered surface on which G acts. The real genus of an abelian group A was investigated in [8] with techniques similar to those employed here.

2. Preliminaries

We will use the following notation:

- $[a, b]$ The commutator, $aba^{-1}b^{-1}$
- Z_n The cyclic group of order n
- $[a]$ The greatest integer in a
- $|x|$ The order of the element x
- $|G|$ The order of the group G
- $\mu(\Gamma)$ The non-euclidean area of NEC group Γ
- $\mathscr{P}_A$ The set of all NEC groups Γ which map onto the group A where the kernel is a Fuchsian surface group
- $A[p]$ The subgroup of A generated by the elements of order p

We shall also assume that all surfaces are compact. Let G be a group of automorphisms of the Riemann surface X, and let G^+ be the subgroup of G consisting of the orientation-preserving automorphisms. Clearly, G^+ has index at most two in G. Consequently, if the group G has no subgroup of index two, then $G = G^+$ and G acts on X preserving orientation. In particular, if A is a finite abelian group of odd order, then $\sigma(A) = \sigma^\circ(A)$. Thus we shall concentrate on abelian groups of even order.

There are infinite families of groups with genus $\sigma \leq 1$, and some of these are abelian. The groups of symmetric genus zero are well known. Indeed the classification of these groups is a classical result that is sometimes credited to Maschke. The abelian group A has symmetric genus zero if and only if A is Z_n, $Z_2 \times Z_{2n}$, or $(Z_2)^3$; see [1, §6.3.2].

The groups of symmetric genus one have also been classified, in a sense. If $\sigma(G) = 1$, then G is a quotient of a plane Euclidean space group and thus G has one of 17 partial presentations [1, pp. 291, 292]. The abelian group A has symmetric genus one if and only if A is $Z_m \times Z_{mn}$ with $m \geq 3$, $Z_2 \times Z_2 \times Z_{2n}$ with $n \geq 2$, or $(Z_2)^4$. The book [1] has a good discussion of the work on groups of small symmetric genus and graph-theoretic genus.

Non-euclidean crystallographic groups (NEC groups) have been quite useful in investigating group actions on surfaces. Let $\mathscr{L}$ denote the group of automorphisms of the open upper half-plane U, and let $\mathscr{L}^+$ denote the subgroup of index 2 consisting of the orientation-preserving automorphisms. An NEC group is a discrete subgroup Γ of $\mathscr{L}$ (with the quotient space U/Γ compact). If $\Gamma \subseteq \mathscr{L}^+$, then Γ is called a *Fuchsian* group. Otherwise Γ is called a *proper NEC group*; in this case Γ has a canonical Fuchsian subgroup $\Gamma^+ = \Gamma \cap \mathscr{L}^+$ of index 2.

Associated with the NEC group Γ in its *signature*, which has the form

$$(2.1) \qquad \left(p, \pm, [\lambda_1, \ldots, \lambda_r]; \left\{(\nu_{11}, \ldots, \nu_{1s_1}), \ldots, (\nu_{k1}, \ldots, \nu_{ks_k})\right\}\right).$$

The quotient space $X = U/\Gamma$ is a surface with topological genus p and k boundary components. The surface is orientable if the plus sign is used and non-orientable if the minus sign is used. The integers $\lambda_1, \ldots, \lambda_r$, called the *ordinary periods*, are the ramification indices of the natural quotient mapping from U to X in fibers above interior points of X. The integers $\nu_{i1}, \ldots, \nu_{is_i}$, called the *link periods*, are the ramification indices in fibers above points on the ith boundary component of X.

Associated with the signature (2.1) is a presentation for the NEC group Γ, although the form of the presentation depends upon whether the plus or minus sign is used. If the plus sign is used, then Γ has generators

(i) $x_1, \ldots, x_r$
(ii) $c_{10}, \ldots, c_{1s_1}, \ldots, c_{k0}, \ldots, c_{ks_k}$
(iii) $e_1, \ldots, e_k$
(iv) $a_1, b_1, \ldots, a_p, b_p$

and relations
(a) $(x_i)^{\lambda_i} = 1$ for $i = 1, \ldots, r$
(b) $(c_{i,j-1})^2 = (c_{i,j})^2 = (c_{i,j-1}c_{i,j})^{\nu_{ij}} = 1$ for $i = 1, \ldots, k$ and $j = 1, \ldots, s_i$
(c) $e_i c_{i0} (e_i)^{-1} = c_{is_i}$ for $i = 1, \ldots, k$
(d) $x_1 \cdots x_r e_1 \cdots e_k [a_1, b_1] \cdots [a_p, b_p] = 1$.

If there is a minus sign in the signature, then the generators (iv) are replaced by generators

(iv′) $a_1, \ldots, a_p$

and the relation (d) is replaced by the relation

(d′) $x_1 \cdots x_r e_1 \cdots e_k (a_1)^2 \cdots (a_p)^2 = 1.$

For more information about signatures, see [4] and [9].

Let Γ be an NEC group with signature (2.1). The non-euclidean area $\mu(\Gamma)$ of a fundamental region Γ can be calculated directly from its signature [9, p. 235]:

$$(2.2)\quad \mu(\Gamma)/2\pi = \alpha p + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \sum_{i=1}^{k}\sum_{j=1}^{s_i}\frac{1}{2}\left(1 - \frac{1}{n_{ij}}\right),$$

where $\alpha = 2$ if the plus sign is used and $\alpha = 1$ otherwise.

A Fuchsian group K is called a *surface group* if the quotient map from U to U/K is unramified. These groups are especially important in studying Riemann surfaces. Let X be a Riemann surface of genus $g \geq 2$. Then X can be represented as U/K where K is a Fuchsian surface group with $\mu(K) = 4\pi(g - 1)$. Let G be a group of dianalytic automorphisms of the Riemann surface X. Then there is an NEC group Γ and a homomorphism $\phi\colon \Gamma \to G$ onto G such that kernel $(\phi) = K$.

Now let G be a finite group. If we can find an NEC group Γ and a homomorphism $\phi\colon \Gamma \to G$ onto G such that kernel (ϕ) is a Fuchsian surface group, then G acts on the Riemann surface U/K. A subgroup Δ of an NEC group is a (Fuchsian) surface group if and only if $\Delta \subseteq \mathscr{L}^+$ and it has no elements of finite order. Macbeath [4, p. 1198] has shown that an element of finite order in an NEC group Γ is conjugate to one of the following:

(i) a power of x_i for $i = 1, \ldots, r$

(ii) a power of some $c_{i,j-1}c_{i,j}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, s_i$

(iii) some $c_{i,j}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, s_i$.

It is usually easy to see that none of these elements are in the kernel of ϕ. If Γ is a proper NEC group, then it is also necessary to check that kernel$(\phi) \subseteq \mathscr{L}^+$ or equivalently, $\phi(\Gamma^+)$ has index two in G [10, Theorem 1, p. 52]. Thus it is straightforward to verify that kernel(ϕ) is a Fuchsian surface group, and we will omit this part of the proof from all subsequent arguments.

If Λ is a subgroup of finite index in Γ, then

$$[\Gamma\colon \Lambda] = \mu(\Lambda)/\mu(\Gamma).$$

It follows that the genus of the surface U/K on which $G = \Gamma/K$ acts is given by

$$g = 1 + |G| \cdot \mu(\Gamma)/4\pi. \tag{2.3}$$

Minimizing g is therefore equivalent to minimizing $\mu(\Gamma)$. Among the NEC groups Γ for which G is a quotient of Γ by a surface group, we want to identify the one for which $\mu(\Gamma)$ is as small as possible; then equation (2.3) will give the symmetric genus of the group G.

Every finite abelian group of rank r has a unique *canonical form*

$$A = Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}$$

such that m_i divides m_{i+1} for $i = 1, \ldots, r-1$ and $m_1 > 1$ [6, p. 387]. We will relabel the invariants so that we may exhibit the Z_2 factors explicitly. Thus the canonical form can be written

$$A = (Z_2)^a \times Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_n} \tag{2.4}$$

where $r = n + a$. Notice that if $a > 0$, then all of the invariants m_i are even. This canonical form is very useful for studying genus parameters; see [1], [2], [5], [8], and [13]. We shall need another canonical form that we shall call the *alternate canonical form*. Let $m_1, \ldots, m_k$ be the invariants from the canonical form (2.4) that are not divisible by 4. Then let

$$C = Z_{m_{k+1}} \times \cdots \times Z_{m_n}$$

where C is trivial in case $k = n$. Now let E be the Sylow 2-subgroup of $(Z_2)^a \times Z_{m_1} \times \cdots \times Z_{m_k}$; E is an elementary abelian 2-group. The Primary Decomposition Theorem implies that there is an odd order group B satisfying $E \times B = (Z_2)^a \times Z_{m_1} \times \cdots \times Z_{m_k}$. We define the alternate canonical form for the abelian group A as

$$A = E \times B \times C.$$

The abelian group A is completely described by the rank of E and the invariants of B and C.

Now suppose the abelian group A acts on the Riemann surface X of genus $g \geq 2$. Represent X as U/K where K is a surface group. We then obtain an NEC group Γ with signature (2.1) and a homomorphism $\phi\colon \Gamma \to A$ onto A such that kernel $(\phi) = K$. The following is basic.

PROPOSITION. *Suppose A has even order. Then each link period in the signature of Γ is two. Further, each non-empty period cycle has at least two link periods.*

Proof. Let Γ have canonical presentation associated with (2.1). The surface group K contains no elements of finite order. Suppose $n = n_{ij}$ is a link period, and write $c = c_{i,j-1}$, and $d = c_{ij}$ so that $c^2 = d^2 = (cd)^n = 1$ in Γ.

If n is odd, then since A is abelian, $cd \in K = \text{kernel}(\phi)$. If n is even and $n \geq 4$, then $(cd)^{n/2} \in K$. In either case, K would contain an analytic element of finite order. Hence $n = 2$.

Suppose there were a period cycle with exactly one link period (equal to 2). This period cycle has corresponding generators c, d, and e satisfying $c^2 = d^2 = (cd)^2 = 1$ and $ece^{-1} = d$. It follows that $\phi(c) = \phi(d)$ since A is abelian, and again $cd \in K$. Thus a non-empty period cycle must have at least two periods.

3. Reduction of signatures

Let A be a finite abelian group of even order with $\sigma(A) \geq 2$. Define $\mathscr{P}_A$ to be the set of all NEC groups Γ with homomorphism ϕ: $\Gamma \to A$ onto A such that $K = \text{kernel}(\phi)$ is a Fuchsian surface group. All link periods in the signature of Γ equal 2. The group A acts on $X = U/K$, a Riemann surface whose genus is given by (2.3). We will find the symmetric genus of G by minimizing $\mu(\Gamma)$.

In this section, we show that we need only consider elements of $\mathscr{P}_A$ with certain types of signatures. Given any $\Gamma \in \mathscr{P}_A$, we will construct $\Gamma' \in \mathscr{P}_A$ having signature of a certain type and satisfying $\mu(\Gamma') \leq \mu(\Gamma)$. The groups Γ and Γ' will have similar structures, and we shall specify Γ' by giving the generators of Γ which are not in Γ' and a list of new generators (indicated by primes). We will construct the homomorphism ϕ': $\Gamma' \to A$ onto A by specifying the images of the new generators in Γ' and indicating any difference between the action of ϕ' and that of ϕ. The homomorphism ϕ' will act in the same way as ϕ on all generators that both groups have in common. If Γ' is a proper NEC group, then we shall construct ϕ' such that if $y \in (\Gamma')^+$, then $\phi'(y) \in \phi(\Gamma^+)$. It is then clear that $\phi'((\Gamma')^+)$ has index two in G, and $\text{kernel}(\phi') \subseteq \mathscr{L}^+$. We omit the proof that kernel(ϕ) is a surface group, as explained in §2. Subsequently, we will use the notation that a bar over an element of $\Gamma \in \mathscr{P}_A$ will indicate its image under ϕ in A.

We begin by showing that the ordinary periods may be arranged so that each period is divisible by all preceding periods.

LEMMA 3.1. *Suppose $\Gamma \in \mathscr{P}_A$ and $z_1, z_2 \in \Gamma$ are either elliptic or connecting generators (x_i or e_i). Let p be any prime and suppose $|\bar{z}_i| = m_i n_i$ where m_i is a power of p and $p \nmid n_i$. Suppose $m_2 < m_1$. There exists $\Gamma' \in \mathscr{P}_A$ where the corresponding generators z_1' and z_2' have images with order $m_2 n_1$ and $m_1 n_2$ respectively. Furthermore $\mu(\Gamma') \leq \mu(\Gamma)$ whenever*

(a) *$z_1 = x_i$ and $z_2 = e_j$, or*

(b) *$z_1 = x_i$ and $z_2 = x_j$ and $|\bar{z}_1| \leq |\bar{z}_2|$, or*

(c) *$z_1 = e_i$ and $z_2 = e_j$.*

Proof. There exist integers a_1, b_1, a_2, and b_2 such that $a_1 m_1 + b_1 n_1 = 1$ and $a_2 m_2 + b_2 n_2 = 1$. We replace generators z_1 and z_2 in Γ by generators z_1' and z_2' in Γ' of the same type. If either z_1 or z_2 is an elliptic generator, then let the ordinary period of its replacement be given by $|z_1'| = m_2 n_1$ or $|z_2'| = m_1 n_2$. The homomorphism $\phi'\colon \Gamma' \to A$ is given by

$$\phi'\colon \begin{array}{l} z_1' \to \bar{z}_1^{m_1 a_1} \bar{z}_2^{n_2 b_2} \\ z_2' \to \bar{z}_1^{n_1 b_1} \bar{z}_2^{m_2 a_2} \end{array}$$

We use equation (2.2) to compute the areas. In case (a),

$$\mu(\Gamma') = \mu(\Gamma) + 2\pi(m_1 - m_2)/m_1 m_2 n_1 < \mu(\Gamma).$$

In case (b), $n_1 < n_2$ and

$$\mu(\Gamma') = \mu(\Gamma) + 2\pi(1/m_1 - 1/m_2)(1/n_1 - 1/n_2) < \mu(\Gamma).$$

Finally, in case (c), $\mu(\Gamma') = \mu(\Gamma)$.

LEMMA 3.2. *Suppose $\Gamma \in \mathscr{P}_A$. Then there exists $\Gamma' \in \mathscr{P}_A$ with the following properties.*

(a) *$\mu(\Gamma') \leq \mu(\Gamma)$.*

(b) *The generators of Γ' have images in A such that $|\bar{x}_i'|$ divides $|\overline{x_{i+1}'}|$ for $i = 1, \ldots, r-1$, $|\overline{x_r'}|$ divides $|\overline{e_1'}|$, and $|\overline{e_j'}|$ divides $|\overline{e_{j+1}'}|$ for $j = 1, \ldots, k-1$.*

Proof. Let π be the set of all primes which divide $|A|$. Arrange the generators $x_1, \ldots, x_r$ in increasing order. Let $p \in \pi$ and use Lemma 3.1 to construct a new group Γ' in which the divisibility condition holds for the p-part of the order of the image in A of the elliptic and connecting generators. We may do this for every prime in π and the result will be the divisibility condition of the lemma.

Notice that since the homomorphism is one to one on $\langle x_i \rangle$, Lemma 3.2 asserts that the ordinary periods of the signature may be arranged so that each one divides its successor.

We are now in a position to show that we need only consider groups with a plus sign in their signature.

LEMMA 3.3. *Let* $\Gamma \in \mathscr{P}_A$ *with signature*

$$(g, -, [\lambda_1, \ldots, \lambda_r], \{C_1, \ldots, C_k\}).$$

Then there is a group $\Gamma' \in \mathscr{P}_A$ *with a plus sign in its signature which satisfies* $\mu(\Gamma') \leq \mu(\Gamma)$.

Proof. We begin by assuming the divisibility condition from Lemma 3.2. The genus of the new group Γ' will be $h = [g/2]$. It is obvious that we will have to replace the generators $a_1, \ldots, a_g$ by new generators $a'_1, b'_1, \ldots, a'_h, b'_h$. When we define $\phi'\colon \Gamma' \to A$ the images of these new generators will be

$$\phi'\colon \begin{matrix} a'_i \to \bar{a}_{2i-1} \\ b'_i \to \bar{a}_{2i} \end{matrix} \qquad \text{for } i = 1, \ldots, h \text{ and } k = 0,$$

$$\phi'\colon \begin{matrix} a'_i \to \bar{a}_{2i-1}\bar{c}_1 \\ b'_i \to \bar{a}_{2i}\bar{c}_1 \end{matrix} \qquad \text{for } i = 1, \ldots, h \text{ and } k \neq 0.$$

Suppose that g is even. If there are no period cycles ($k = 0$), then redefine the ordinary period

$$\lambda'_r = |\bar{x}_r \bar{a}_1^2 \cdots \bar{a}_g^2| = \left|(\bar{x}_1 \cdots \bar{x}_{r-1})^{-1}\right| \leq \lambda_r.$$

These are the only new generators and the other change in the homomorphism is given by

$$\phi'\colon e'_k \to \bar{e}_k \bar{a}_1^2 \cdots \bar{a}_g^2 \quad \text{when } k \neq 0,$$

$$\phi'\colon x'_r \to \bar{x}_r \bar{a}_1^2 \cdots \bar{a}_g^2 \quad \text{when } k = 0.$$

When $k \neq 0$, we see that $\mu(\Gamma') = \mu(\Gamma)$, otherwise $\mu(\Gamma') \leq \mu(\Gamma)$.

Now suppose g is odd. If $k = 0$, we need to redefine the ordinary period $\lambda'_r = |\bar{x}_r \bar{a}_1 \cdots \bar{a}_g| \leq 2\lambda_r$ and add an elliptic element x'_{r+1} with order $\lambda'_{r+1} = |\bar{a}_1 \cdots \bar{a}_g| \leq 2\lambda_r$. If $k \neq 0$, we add the connecting generator e'_{k+1} and the reflection c'_{k+1}. This corresponds to adding an empty period cycle to the signature. The homomorphism $\phi'\colon \Gamma' \to A$ is given by

$$\phi'\colon \begin{array}{l} x'_r \quad \to \bar{x}_r \bar{a}_1 \cdots \bar{a}_g \\ x'_{r+1} \to \bar{a}_1 \cdots \bar{a}_g \end{array} \quad \text{if } k = 0,$$

$$\phi'\colon \begin{array}{l} e'_k \quad \to \bar{e}_k \bar{a}_1 \cdots \bar{a}_g \bar{c}_1 \\ e'_{k+1} \to \bar{a}_1 \cdots \bar{a}_g \bar{c}_1 \end{array} \quad \text{if } k \neq 0.$$

If $k = 0$, then $\mu(\Gamma') = \mu(\Gamma) + 2\pi(1/\lambda_r - 1/\lambda'_r - 1/\lambda'_{r+1}) \leq \mu(\Gamma)$. In addition, if $k \neq 0$, then we send the reflection c'_{k+1} to some element of order two in A, and clearly, $\mu(\Gamma') = \mu(\Gamma)$.

We now prove several lemmas which cumulatively show that we need only consider NEC-groups with certain types of signature.

LEMMA 3.4. *Suppose* $\Gamma \in \mathscr{P}_A$ *has signature*

$$(p, +, [\lambda_1, \ldots, \lambda_r], \{C_1, \ldots, C_k\})$$

with $p > 0$ *and* $k > 0$. *Then there exists* $\Gamma' \in \mathscr{P}_A$ *with signature*

$$\left(0, +, \left[\lambda_1, \ldots, \lambda_{r+2p}\right], \{C_1, \ldots, C_k\}\right)$$

satisfying $\mu(\Gamma') < \mu(\Gamma)$.

Proof. We will find $\Gamma' \in \mathscr{P}_A$ with signature

$$(p - 1, +, [\lambda_1, \ldots, \lambda_{r+2}], \{C_1, \ldots, C_k\})$$

satisfying $\mu(\Gamma') < \mu(\Gamma)$. A simple induction will complete the proof of the lemma. We construct Γ' by deleting generators a_p and b_p and replacing them with elliptic generators x'_{r+1} and x'_{r+2} with orders $\lambda_{r+1} = |\bar{a}_p|$ and $\lambda_{r+2} = |\bar{b}_p|$. The homomorphism $\phi'\colon \Gamma' \to A$ is given by

$$\phi'\colon \begin{array}{l} x'_{r+1} \to \bar{a}_p \\ x'_{r+2} \to \bar{b}_p \\ e'_k \quad \to \bar{e}_k(\bar{a}_p \bar{b}_p)^{-1}. \end{array}$$

It is easy to see that $\mu(\Gamma') = \mu(\Gamma) - 2\pi(1/\lambda_{r+1} + 1/\lambda_{r+2}) < \mu(\Gamma)$.

Notice that this reduction fails if there are no period cycles (i.e., if $k = 0$).

LEMMA 3.5. *Suppose* $\Gamma \in \mathscr{P}_A$ *has signature*

$$(0, +, [\lambda_1, \dots, \lambda_r], \{C_1, \dots, C_l, (2^n), (2^m)\})$$

with $n \geq 2$ *and* $m \geq 2$. *Then there exists* $\Gamma' \in \mathscr{P}_A$ *with signature*

$$(0, +, [\lambda_1, \dots, \lambda_{r+1}], \{C_1, \dots, C_l, (2^{n+m})\})$$

satisfying $\mu(\Gamma') < \mu(\Gamma)$.

Proof. Let the period cycles (2^n) and (2^m) correspond to reflections $c_0, \dots, c_n$ and $d_0, \dots, d_m$ respectively. We will replace these reflections by the reflections $u'_0, \dots, u'_{n+m}$. The connecting generator e_{l+2} will be replaced by the elliptic generator x'_{r+1} with order $\lambda_{r+1} = |\bar{e}_{l+2}|$. The homomorphism $\phi'\colon \Gamma' \to A$ will be given by

$$\phi'\colon \begin{array}{lll} x'_{r+1} \to \bar{e}_{l+2} & \\ u'_i \to \bar{c}_i & \text{for } i = 1, \dots, n \\ u'_{n+i} \to \bar{d}_i & \text{for } i = 1, \dots, m \\ u'_0 \to \bar{d}_0 = \bar{d}_m & \end{array}$$

It is easily checked that $\mu(\Gamma') = \mu(\Gamma) - 2\pi/\lambda_{r+1} < \mu(\Gamma)$.

Now a simple induction allows us to assume that there is at most one non-empty period cycle. We have shown that among the NEC groups in $\mathscr{P}_A$ with minimal area, there is either one with no period cycles or one with genus zero and at most one non-empty period cycle.

LEMMA 3.6. *Suppose* $\Gamma \in \mathscr{P}_A$ *has signature*

$$(0, +, [\lambda_1, \dots, \lambda_r], \{(\ )^l, (2^t)\})$$

with $\lambda_r \geq 4$ *and* $t \geq 3$. *Then there exists* $\Gamma' \in \mathscr{P}_A$ *with signature*

$$(0, +, [\lambda_1, \dots, \lambda_{r-1}], \{(\ )^{l+1}, (2^{t-1})\})$$

and $\mu(\Gamma') \leq \mu(\Gamma)$.

Proof. Let the period cycle (2^t) correspond to reflections $c_0, \dots, c_t$ and connecting generator f. We replace c_t and x_r by a reflection d'_{l+1} and a connecting generator e'_{l+1} associated with a new empty period cycle. The homomorphism $\phi'\colon \Gamma' \to A$ is given by

$$\phi'\colon \begin{array}{l} e'_{l+1} \to \bar{x}_r \\ d'_{l+1} \to \bar{c}_t \\ c'_0 \quad \to \bar{c}_{t-1}. \end{array}$$

We see that $\mu(\Gamma') = \mu(\Gamma) - 2\pi(1/4 - 1/\lambda_r) \le \mu(\Gamma)$.

LEMMA 3.7. *Suppose $\Gamma \in \mathscr{P}_A$ has signature*

$$\left(0, +, [\lambda_1, \dots, \lambda_r], \{(\ \ )^l, (2^t)\}\right)$$

with λ_r odd and $t \ge 3$. Then there exists $\Gamma' \in \mathscr{P}_A$ with signature

$$\left(0, +, [\lambda_1, \dots, \lambda_{r-1}, 2\lambda_r], \{(\ \ )^l, (2^{t-1})\}\right)$$

and $\mu(\Gamma') < \mu(\Gamma)$.

Proof. Let the period cycle (2^t) be associated with reflections $c_0, \dots, c_t$. We will replace x_r and c_t with a new elliptic generator x'_r of order $2\lambda_r$. The homomorphism $\phi'\colon \Gamma' \to A$ is given by

$$\phi'\colon \begin{array}{l} x'_r \quad \to \bar{c}_{t-1}\bar{c}_t\bar{x}_r \\ e'_{l+1} \to \bar{c}_{t-1}\bar{c}_t\bar{e}_{l+1} \\ c'_0 \quad \to \bar{c}_{t-1}. \end{array}$$

Clearly, $\mu(\Gamma') = \mu(\Gamma) - 2\pi(1/4 - 1/2\lambda_r) < \mu(\Gamma)$.

LEMMA 3.8. *Suppose $\Gamma \in \mathscr{P}_A$ has signature*

$$\left(0, +, [\lambda_1, \dots, \lambda_r], \{(\ \ )^l, (2,2)\}\right)$$

with $r \ge 1$. Then there exists $\Gamma' \in \mathscr{P}_A$ with signature

$$\left(0, +, [\lambda_1, \dots, \lambda_{r-1}], \{(\ \ )^{l+2}\}\right)$$

and $\mu(\Gamma') \le \mu(\Gamma)$.

Proof. The period cycle $(2,2)$ corresponds to reflections d_0, d_1, and d_2 and connecting generator f. Replace the generators x_r, d_0, d_1, d_2 and f with new connecting generators e'_{l+1} and e'_{l+2} and new reflections c'_{l+1} and

c'_{l+2}. The homomorphism ϕ' is defined by

$$\phi': \begin{array}{l} e'_{l+1} \to \bar{x}_r \\ e'_{l+2} \to \bar{f} \\ c'_{l+1} \to \bar{d}_1 \\ c'_{l+2} \to \bar{d}_2. \end{array}$$

We see that $\mu(\Gamma') = \mu(\Gamma) - 2\pi(1/2 - 1/\lambda_r) \leq \mu(\Gamma)$.

LEMMA 3.9. *Suppose $\Gamma \in \mathscr{P}_A$ has signature*

$$\left(0, +, [2^r], \left\{(\quad)^l, (2^t)\right\}\right)$$

with $r \geq 1$ and $t \geq 2$. Then there exists $\Gamma' \in \mathscr{P}_A$ with signature

$$\left(0, +, [2^{r-1}], \left\{(\quad)^l, (2^{t+1})\right\}\right)$$

and $\mu(\Gamma') < \mu(\Gamma)$.

Proof. We must delete the elliptic element x_r and replace the reflections $c_0, \ldots, c_t$ corresponding to the period cycle (2^t) by new reflections $c'_0, \ldots, c'_{t+1}$. The homomorphism ϕ' is defined by

$$\phi': \begin{array}{l} c'_{t+1} \to \bar{x}_r \\ e'_{l+1} \to \bar{x}_r \bar{e}_{l+1} \\ c'_0 \to \bar{x}_r. \end{array}$$

Clearly, $\mu(\Gamma') = \mu(\Gamma) - \pi/2 < \mu(\Gamma)$.

We use Lemmas 3.6 and 3.7 to reduce the number of link periods in the non-empty period cycle, as long as there are ordinary periods larger than 2. (Either lemma may be used when λ_r is odd and at least five.) Therefore, we obtain a signature of one of the following two types.

(A) $(0, +, [2^r], \{(\quad)^s, (2^t)\})$
(B) $(0, +, [\lambda_1, \ldots, \lambda_r], \{(\quad)^s, (2, 2)\})$

If we have a signature of type (B), then we apply Lemma 3.8 to eliminate the final link periods and obtain one of the form

(B′) $(0, +, [\lambda_1, \ldots, \lambda_r], \{(\ )^s\})$.

If we have a signature of type (A) with $r \geq 1$, then we use Lemma 3.9 (r times) to obtain a signature of type (A) with $r = 0$. Therefore, in order to minimize $\mu(\Gamma)$, we only need to consider three types of signatures. These signatures are summarized in the main result of this section, Theorem 3.10.

THEOREM 3.10. *Among the NEC groups in $\mathcal{P}_A$ with minimal non-euclidean area, there is a group Γ whose signature has one of the following forms.*

(I) $(g, +, [\lambda_1, \ldots, \lambda_r], \{\ \})$

(II) $(0, +, [\lambda_1, \ldots, \lambda_r], \{(\ )^k\})$

(III) $(0, +, [\ ], \{(\ )^s, (2^t)\})$ $(t \geq 2)$

Furthermore, in cases (I) *and* (II), λ_i *divides* λ_{i+1} *for* $1 \leq i \leq r-1$.

Henceforth, we will refer to groups with these signatures as groups of Type I, II, III.

4. Groups of Type II

Let A be an abelian group with even order and $\sigma(A) \geq 2$. In this section, we find a group with minimal area from among the Type II groups in $\mathcal{P}_A$ so that we can determine the number of its empty period cycles (which is equal to k). The value of k will depend on the ranks of the groups in the alternate canonical form, introduced in Section 2. We begin with the following upper bound on the value of k.

LEMMA 4.1. *Let $E \times B \times C$ be the alternate canonical form for the abelian group A. Let Γ be a group with minimal non-euclidean area from among the groups in $\mathcal{P}_A$ with Type II signature. Let $c_0, \ldots, c_k$ be the reflections associated with the empty period cycles in the signature. Then the projections on E of the images in A of $c_1, \ldots, c_k$ are linearly independent, and hence $k \leq$* rank(E).

Proof. Clearly $\bar{c}_0 = \bar{c}_k$. Let $\bar{\omega}$ be an involution in C. Since C has no Z_2 factors, there exists $\bar{z} \in C$ such that $\bar{z}^2 = \bar{\omega}$. There exists an element z in Γ whose image is $\bar{z}$. It follows that $\bar{\omega}$ is the image of an element (namely z^2) of Γ which does not involve any reflections.

Suppose that the projections into E of $\bar{c}_1, \ldots, \bar{c}_k$ are linearly dependent. Then some linear combination (written multiplicatively) of $\bar{c}_1, \ldots, \bar{c}_k$ is in C. Let $\omega = \bar{c}_1^{r_1} \cdots \bar{c}_k^{r_k}$, where $r_i = 0$ or 1 for all i, be the linear combination in C. We may suppose that $r_k = 1$ by reordering the reflections if necessary. Now we define a new NEC group Γ' with one less empty period cycle and one more ordinary period ($\lambda_{r+1} = |\bar{e}_k|$) than Γ. Define the homomorphism $\phi'\colon \Gamma \to A$ by

$$\phi'\colon \begin{array}{l} x'_{r+1} \to \bar{e}_k \\ c'_0 \quad\ \to \bar{c}_{k-1}. \end{array}$$

Since $\bar{c}_k$ is the image of a linear combination of reflections $c_1, \dots, c_{k-1}$ and ω (which involves no reflections), ϕ' is onto A. It is elementary that $\Gamma' \in \mathscr{P}_A$. The fact that $\mu(\Gamma') + 2\pi/\lambda_{r+1} = \mu(\Gamma)$ contradicts the minimality of $\mu(\Gamma)$ and the conclusion follows.

Next we show that when the number of Z_2 factors in A is small, there is an NEC group with a particular value of k among the Type II groups in $\mathscr{P}_A$ with minimal area.

LEMMA 4.2. *Let $A = E \times B \times C$ be an abelian group in alternate canonical form and suppose* $\operatorname{rank}(E) \le \operatorname{rank}(C) + 1$. *Among the NEC groups in $\mathscr{P}_A$ with Type II signature, there is a group Γ with minimal non-euclidean area which satisfies $k =$* $\operatorname{rank}(E)$.

Proof. Let $\Gamma \in \mathscr{P}_A$ be a group with minimal non-euclidean area from among the groups of Type II signature. We may suppose that λ_i divides λ_{i+1} for $i = 1, \dots, r-1$ by Lemma 3.2. Since $k \le \operatorname{rank}(E)$ by Lemma 4.1, it follows that $k - 1 \le \operatorname{rank}(C)$. Therefore, Lemma 3.2 and a rank argument shows that 4 divides $|\bar{e}_i|$ for all i and hence 4 divides at least $(\operatorname{rank}(C) - k + 1)$ of the λ_i. We may assume that $\operatorname{rank}(E) \ge 1$, since the lemma is easy if $\operatorname{rank}(E) = 0$. Therefore, there exists some index $t < r$ such that $\lambda_t = 2u$ where u is odd. The projections of $\bar{c}_1, \dots, \bar{c}_k$ onto E are linearly independent by Lemma 4.1.

Case 1. $\bar{x}_t^u \in C$. We construct Γ' by replacing the ordinary period λ_t by u and $\mu(\Gamma') < \mu(\Gamma)$.

Case 2. $pr_E(\bar{x}_t^u) \ne 1$. If the projection is not linearly independent of the projections $pr_E(\bar{c}_i)$ for $i = 1, \dots, k$ then we replace λ_t by u as in case 1. Hence we may assume that the projections are linearly independent. Define Γ' as the NEC group with signature

$$\left(0, +, [\lambda_1, \dots, \lambda_{t-1}, u, \lambda_{t+1}, \dots, \lambda_{r-1}], \{(\ )^{k+1}\}\right).$$

We define the homomorphism $\phi'\colon \Gamma' \to A$ by

$$\phi'\colon \begin{array}{ll} x'_t & \to \bar{x}_t^{2q} \\ c'_{k+1} & \to \bar{x}_t^{u} \\ e'_{k+1} & \to \bar{x}_r \bar{x}_t^{uq} \end{array}$$

where q is the unique solution of $q(u+2) \equiv 1 \pmod{2u}$. It follows that $\mu(\Gamma') = \mu(\Gamma) + 2\pi(1/\lambda_r - 1/\lambda_t) \leq \mu(\Gamma)$. Now we continue this process until $k = \operatorname{rank}(E)$.

Notice that in this case we may assume that if λ_i is even, then it is divisible by 4.

Finally, we show that when the number of Z_2 factors in A is large, there is an NEC group with a particular value of k among the Type II groups in $\mathscr{P}_A$ with minimal area.

LEMMA 4.3. *Let $A = E \times B \times C$ be an abelian group in alternate canonical form with* $\operatorname{rank}(E) > \operatorname{rank}(C) + 1$. *Among the NEC groups which have minimal non-euclidean area in the subset of $\mathscr{P}_A$ consisting of groups with Type II signature, there is a group with*

$$k = [(\operatorname{rank}(E) + \operatorname{rank}(C) + 1)/2].$$

Proof. Let Γ be a group with minimal non-euclidean area in the subset of $\mathscr{P}_A$ consisting of groups with Type II signature. We may assume that $|\bar{e}_i|$ divides $|\bar{e}_{i+1}|$ for all i and that $|\bar{x}_j|$ divides $|\bar{e}_i|$ for all i and j, by Lemma 3.2. Suppose that $|\bar{e}_1|$ were odd. Therefore, $|\bar{x}_j|$ would be odd and since $\operatorname{rank}(E) \geq 2$, we would have $k \geq 2$. We define a new Type II NEC group Γ' by replacing e_1, c_1, and e_2 by x'_{r+1} and e'_2 where $\lambda'_{r+1} = 2|\bar{e}_1|$ (hence, Γ' has $k-1$ empty period cycles with connecting generators $e'_2, \ldots, e'_k$). Define a homomorphism $\phi'\colon \Gamma' \to A$ by

$$\phi'\colon \begin{array}{ll} x'_{r+1} & \to \bar{e}_1\bar{c}_1\bar{c}_2 \\ e'_2 & \to \bar{e}_2\bar{c}_1\bar{c}_2. \end{array}$$

The minimality of $\mu(\Gamma)$ and the fact that $\mu(\Gamma') < \mu(\Gamma)$ would give a contradiction. Therefore, $|\bar{e}_1|$ is even, and so is $|\bar{e}_i|$ for all i, by the divisibility condition. Now let T_2 be the 2-primary part of

$$\langle \bar{x}_1, \ldots, \bar{x}_r, \bar{e}_1, \ldots, \bar{e}_k \rangle.$$

The group T_2 contains a maximal elementary abelian direct summand T which is a subgroup of E. We can show that T is linearly independent of the reflections and that its rank is at least $(k - 1 - \text{rank}(C))$. Hence we have $k + (k - 1 - \text{rank}(C)) \leq \text{rank}(E)$ and

$$2k \leq \text{rank}(E) + \text{rank}(C) + 1.$$

Now suppose that $2k < \text{rank}(E) + \text{rank}(C)$. Since the reflections and connecting generators account for at most $(2k - 1)$ of the rank of the 2-primary part of A (which has rank equal to $\text{rank}(E) + \text{rank}(C)$), we see that there must be at least two elliptic generators whose images in A have even order. Since $k \leq \text{rank}(E) - 1$ and $|\bar{x}_i|$ divides $|\bar{e}_j|$ for all i and j, we may assume that at least one of these elliptic generators has image in A whose order is not divisible by 4. We may suppose that these elliptic generators are x_i and x_r. Now construct an NEC group Γ' by replacing x_i and x_r by x_i', e_1' and c_1' where $\lambda_i' = u = |\bar{x}_i|/2$ and renumbering the connecting generators and reflections so that the new ones are listed first. Define the homomorphism $\phi'\colon \Gamma' \to A$ by

$$\phi'\colon \begin{aligned} x_i' &\to \bar{x}_i^{2y} \\ e_1' &\to \bar{x}_r \bar{x}_i^{uz} \\ c_1' &\to \bar{x}_i^{uz}, \end{aligned}$$

where y and z are positive integers satisfying the congruence

$$2y + uz \equiv 1 \pmod{2u}.$$

Finally, it follows that

$$\mu(\Gamma') = \mu(\Gamma) - 2\pi(1/\lambda_i - 1/\lambda_r) \leq \mu(\Gamma)$$

and equality holds if and only if $\lambda_i = \lambda_r$. Therefore, we see that if k has the value stated, then Γ has minimal area in this subset, although there may be other groups in $\mathscr{P}_A$ of Type II signature with the same area.

Note that the strong symmetric genus $\sigma^\circ(A)$ is the genus of the group $\Gamma \in \mathscr{P}_A$ which has minimal non-euclidean area from among the Type I groups in $\mathscr{P}_A$. Similarly, we define a Type II genus $\tau(A)$ as the genus of the group $\Gamma \in \mathscr{P}_A$ which has minimal non-euclidean area from among the groups in $\mathscr{P}_A$ of Type II. The Type II genus may be computed in the following way.

Let $A = E \times B \times C$ be an abelian group in alternate canonical form with $e = \text{rank}(E)$, $b = \text{rank}(B)$, and $c = \text{rank}(C)$. Suppose that the invariants of

B are $\beta_1, \ldots, \beta_b$ and the invariants of C are $\gamma_1, \ldots, \gamma_c$. Let Γ be a group of minimal area from among the Type II groups in $\mathscr{P}_A$ and set $M = \mu(\Gamma)/2\pi$. There are two cases.

Case 1. Suppose $e \leq c + 1$. Then $k = e$ in the Type II signature and a simple computation yields the following formula.

$$M = b + c - 1 - \sum_{i=1}^{b} \frac{1}{\beta_i} - \sum_{i=1}^{c+1-e} \frac{1}{\gamma_i}. \tag{4.4}$$

Case 2. Suppose $e > c + 1$. Then $k = [(e + c + 1)/2]$ in the Type II signature. Now let $n = \text{rank}(A)$, $r = n - k + 1$,

$$\kappa = \max\{k + b - e, k\}$$

and $\delta = 1$ if $e + c$ is even and 0 if it is odd. We derive the following formula for M.

$$M = n - 1 - \sum_{i=1}^{\kappa} \frac{1}{\beta_i} - \frac{\delta}{2\beta_r}. \tag{4.5}$$

Taken together these two formulas allow us to compute the Type II genus as shown in the following theorem.

THEOREM 4.6. *Let A be an abelian group of even order with $\sigma(A) \geq 2$ and let $\tau(A)$ be the Type II genus. Then $\tau(A) = 1 + |A|M/2$ where M is given by either Formula* (4.4) *or* (4.5).

5. The main theorem

In this section we state and prove the main theorem. Let A be an abelian group with even order and $\sigma(A) \geq 2$. We begin by showing that if A has enough Z_2 factors, then the minimum area will occur for a group of Type III and we obtain an explicit formula for $\sigma(A)$.

We begin by establishing the following upper bound on the symmetric genus of A.

LEMMA 5.1. *Let the abelian group A have the canonical form*

$$(Z_2)^a \times Z_{m_1} \times \cdots \times Z_{m_d}$$

where $m_1 > 2$. *If* $a \geq d + 2$, *then*

$$\sigma(A) \leq 1 + |A|(a + 3d - 4)/8.$$

Proof. Let Γ be the NEC group with signature

$$\left(0, +, [\ ], \left\{(\ )^d, (2^{a-d})\right\}\right).$$

We calculate the area by (2.2) as

$$\mu(\Gamma)/2\pi = d - 1 + (a - d)/4 = (3d + a - 4)/4.$$

Since $\sigma(A) \geq 2$, $\mu(\Gamma) > 0$. Therefore, if we can show that $\Gamma \in \mathscr{P}_A$, we will have the upper bound. Let $t = a - d$. The group Γ has generators $c_1, \dots, c_d$, $d_0, \dots, d_t$, $e_1, \dots, e_d$, f and relations $c_i^2 = d_j^2 = [e_i, c_i] = e_1 \dots e_d f = 1$, $f \cdot d_0 \cdot f^{-1} = d_t$, and $(d_0 d_1)^2 = \cdots = (d_{t-1} d_t)^2 = 1$ for all i and j. Let $v_1, \dots, v_a$ be generators for $(Z_2)^a$ and w_j be a generator for the factor Z_{m_j} of A. Define a homomorphism $\phi\colon \Gamma \to A$ by

$$\phi\colon \begin{array}{ll} c_i \to v_i & \text{for } i = 1, \dots, d \\ d_j \to v_{d+j} & \text{for } j = 1, \dots, (a - d) \\ e_i \to w_i & \text{for } i = 1, \dots, d \\ d_0 \to v_a & \\ f \to (w_1 \cdots w_d)^{-1}. & \end{array}$$

It is easily checked that ϕ is a homomorphism onto A and that the kernel is a surface group. Therefore $\Gamma \in \mathscr{P}_A$.

LEMMA 5.2. *Let the abelian group A have the canonical form*

$$(Z_2)^a \times Z_{m_1} \times \cdots \times Z_{m_d}$$

where $m_1 > 2$. *If* $a \geq d + 2$, *then*

$$\sigma(A) \geq 1 + |A|(a + 3d - 4)/8.$$

Proof. Suppose that A acts on the Riemann surface X of genus $\sigma = \sigma(A) \geq 2$. We may represent X as U/K where K is a surface group. Then we obtain an NEC group Γ and a homomorphism ϕ from Γ onto A with kernel(ϕ) $= K$. By theorem 3.10 we may assume that Γ is a group of Type I, II, or III. We will consider each of the three types separately.

First suppose that Γ has Type II signature

$$\left(0, +, [\lambda_1, \ldots, \lambda_r], \{(\ )^k\}\right)$$

with $k \geq 1$. The canonical generating set for Γ has $r + 2k$ generators, one of which is obviously redundant. Since A is a quotient group of Γ,

$$\text{rank}(A) = a + d \leq r + 2k - 1.$$

Since each $\lambda_i \geq 2$, we may use equation (2.2) to derive the inequality

$$\mu(\Gamma)/2\pi \geq k - 2 + r/2 = ((2k + r - 1) - 3)/2.$$

It follows that $\mu(\Gamma)/2\pi \geq ((a + d) - 3)/2$. Since $a \geq d + 2$ by hypothesis, we see that $a + d \geq (a + 3d)/2 + 1$. Therefore, we obtain

$$\mu(\Gamma)/2\pi \geq (a + 3d - 4)/4,$$

and by equation (2.3) we derive the inequality for the genus $\sigma(A)$.

Next suppose that Γ has Type III signature

$$\left(0, +, [\ ], \{(\ )^s, (2^t)\}\right)$$

with $t \geq 2$. After removing all redundant generators from the canonical presentation of Γ, the simplified presentation has $(2s + t)$ generators. At most s of these generators have order larger than 2. Since A is a quotient of Γ, it follows that $2s + t \geq a + d$ and $s \geq d$. Now by equation (2.2) $\mu(\Gamma)/2\pi = s - 1 + t/4$. Thus, the inequality for $\sigma(A)$ follows by

$$\begin{aligned} 4(\mu(\Gamma)/2\pi) &= 4s + t - 4 = (2s + t) + 2s - 4 \geq (a + d) + 2d - 4 \\ &= a + 3d - 4. \end{aligned}$$

Finally, suppose Γ has Type I signature

$$(p, +, [\lambda_1, \ldots, \lambda_r], \{\ \}).$$

The argument is more delicate in this case. Let r_2 be the number of ordinary periods equal to 2, r_3 the number equal to 3, and r_h the number larger than 3. Now from (2.2)

$$\mu(\Gamma)/2\pi \geq 2p - 2 + r_2/2 + 2r_3/3 + 3r_h/4$$

$$12 \cdot (\mu(\Gamma)/2\pi) \geq 24p + 6r_2 + 8r_3 + 9r_h - 24 \tag{5.1}$$

$$12 \cdot (\mu(\Gamma)/2\pi) \geq 18p + 3r_2 + 8r_3 + 9r_h + (6p + 3r_2) - 24 \tag{5.2}$$

We will use these inequalities to obtain a lower bound for $\mu(\Gamma)$.

After removing a redundant generator from the canonical presentation for Γ, the generating set has $2p + r - 1$ generators. Also for p equal to 2 or 3, let $A[p]$ denote the subgroup of A generated by the elements of order p. There are two cases, depending on whether r_h is zero.

Case 1. Suppose $r_h \neq 0$. Then we have

$$\begin{aligned} &2p + r_h + r_3 - 1 \geq d = \operatorname{rank}(A/A[2]), \\ (5.3)\qquad &2p + r_h + r_2 - 1 \geq a + d = \operatorname{rank}(A/A[3]). \end{aligned}$$

If either $p \geq 1$ or $r_2 \geq 1$, then $6p + 3r_2 \geq 3$ and from (5.2) we have

$$\begin{aligned} 12 \cdot (\mu(\Gamma)/2\pi) &\geq 18p + 3r_2 + 8r_3 + 9r_h - 21 \\ &= 6 \cdot (2p + r_h + r_3 - 1) + 3 \cdot (2p + r_h + r_2 - 1) \\ &\quad + 2r_3 - 12 \\ &\geq 6d + 3 \cdot (a + d) - 12 = 9d + 3a - 12. \end{aligned}$$

Now, if $p = r_2 = 0$, then $r_h - 1 \geq a + d$ by (5.3). Using this, (5.1) and the fact that $a \geq 2$, we get the inequalities

$$\begin{aligned} 12 \cdot (\mu(\Gamma)/2\pi) &\geq 9 \cdot (r_h - 1) + 8r_3 - 15 \geq 9 \cdot (a + d) - 15 \\ &= 9d + 3a + (6a - 15) \geq 9d + 3a - 3. \end{aligned}$$

Case 2. Suppose $r_h = 0$. First assume that $r_3 > 0$, so that there is a redundant generator of order 3 in the canonical presentation of Γ. Here we have

$$\begin{aligned} 2p + r_3 - 1 &\geq d = \operatorname{rank}(A/A[2]), \\ 2p + r_2 &\geq a + d = \operatorname{rank}(A/A[3]). \end{aligned}$$

If $p \geq 1$, then from (5.2)

$$\begin{aligned} 12 \cdot (\mu(\Gamma)/2\pi) &\geq 18p + 6r_2 + 8r_3 - 18 \\ &= 6 \cdot (2p + r_3 - 1) + 3 \cdot (2p + r_2) + 2r_3 - 12 \\ &\geq 6d + 3 \cdot (a + d) + 2 - 12 = 9d + 3a - 10. \end{aligned}$$

Hence we may assume that $p = 0$. Inequality (5.1) says that

$$12 \cdot (\mu(\Gamma)/2\pi) \geq 6r_2 + 8r_3 - 24.$$

Since $r_3 - 1 \geq d$ and $r_2 \geq a + d \geq 3$,

$$\begin{aligned} 12 \cdot (\mu(\Gamma)/2\pi) &\geq 3r_2 + 6 \cdot (r_3 - 1) + 3r_2 + 2r_3 - 18 \\ &\geq 3 \cdot (a + d) + 6d + 9 + 2 - 18 = 9d + 3a - 7. \end{aligned}$$

Finally, suppose $r_3 = 0$, so that the redundant generator removed has order two. Now we have

$$\begin{aligned} 2p &\geq d = \operatorname{rank}(A/A[2]), \\ 2p + r_2 - 1 &\geq a + d = \operatorname{rank}(A) \geq 3. \end{aligned}$$

Using inequality (5.1), we see that

$$\begin{aligned} 12 \cdot (\mu(\Gamma)/2\pi) &\geq 24p + 6r_2 - 24 \\ &= 6 \cdot (2p) + 3 \cdot (2p + r_2 - 1) + 3 \cdot (2p + r_2) - 21 \\ &\geq 6d + 3 \cdot (a + d) + 3 \cdot 4 - 21 = 9d + 3a - 9. \end{aligned}$$

A review of the calculations shows that in each case

$$\mu(\Gamma)/2\pi \geq (a + 3d - 4)/4.$$

By (2.3), $\sigma(A) \geq 1 + |A|(a + 3d - 4)/8$.

These two lemmas are summarized by the following theorem.

THEOREM 5.3. *Let the abelian group A have canonical form*

$$(Z_2)^a \times Z_{m_1} \times \cdots \times Z_{m_d}$$

where $m_1 > 2$. If $a \geq d + 2$, then

$$\sigma(A) = 1 + |A|(a + 3d - 4)/8.$$

It is interesting that this formula holds for groups with $\sigma(A) \leq 1$. We obtain a formula for the genus of an elementary abelian 2-group [7, §7] as a special case of this theorem.

COROLLARY 5.4. *The genus of the group $(Z_2)^a$ for $a \geq 2$ is given by*

$$\sigma((Z_2)^a) = 1 + 2^{a-3} \cdot (a - 4).$$

If the abelian group A has enough Z_2 factors, then the minimum genus action is attained by an NEC group with signature of Type III. Interestingly, it is easy to see that if A has a limited number of Z_2 factors, then the minimum genus is not attained by an NEC group with a Type III signature.

LEMMA 5.5. *Let the abelian group A have canonical form*

$$(Z_2)^a \times Z_{m_1} \times \cdots \times Z_{m_d}$$

where $m_1 > 2$ *and* $0 \leq a \leq d + 1$. *Suppose* $\Gamma \in \mathscr{P}_A$ *has signature*

$$(0, +, [\ \ ], \{(\ \ )^s, (2^t)\})$$

with $s \geq d$ *and* $t \geq 2$. *Then there exists an NEC group* $\Gamma' \in \mathscr{P}_A$ *with signature* $(0, +, [\ \ ], \{(\ \ )^{s+1}\})$ *satisfying* $\mu(\Gamma') < \mu(\Gamma)$.

Proof. Since $\sigma(A) \geq 2$, then $d \geq 2$. Any generating set for A must have at least d generators with order larger than 2. It follows from the canonical presentation for Γ that $s \geq d$.

Let Γ' be an NEC group with signature $(0, +, [\ \ ], \{(\ \ )^{s+1}\})$. Then Γ' is generated by the reflections $c'_1, \ldots, c'_{s+1}$ and the connecting generators $e'_1, \ldots, e'_{s+1}$ where $e'_1 \cdots e'_{s+1} = 1$. Therefore, e'_{s+1} is redundant. Since $a \leq s + 1$ and $d \leq s$, we can find a homomorphism $\phi'\colon \Gamma' \to A$ that makes $\Gamma' \in \mathscr{P}_A$. Further

$$\mu(\Gamma)/2\pi = s - 1 + t/4 > s - 1 = \mu(\Gamma')/2\pi.$$

This shows that if $a \leq d + 1$, then the genus $\sigma(A)$ is the minimum of $\sigma^\circ(A)$ and $\tau(A)$, the strong symmetric genus and the Type II genus respectively. The next lemma shows that if $1 \leq a \leq d + 1$, then $\sigma(A) = \tau(A)$.

LEMMA 5.6. *Let the abelian group A have canonical form*

$$(Z_2)^a \times Z_{m_1} \times \cdots \times Z_{m_d}$$

where $m_1 > 2$ *and* $1 \leq a \leq d + 1$. *Then all of the groups in* $\mathscr{P}_A$ *with minimal area are of Type II.*

Proof. Suppose that $\Gamma \in \mathscr{P}_A$ has minimal area. By Lemma 5.5, Γ is either of Type I or Type II.

Suppose that Γ has signature $(g, +, [\ \ ], \{\ \ \})$. It is clear that this group is not minimal if $a + d$ is odd. Assume that $a + d$ is even and $g = (a + d)/2$. In this case, $\mu(\Gamma)/2\pi = a + d - 2$. Now define Γ' to be the NEC group with signature

$$(0, +, [2^{a-1}, 2m_1, \ldots, 2m_d], \{(\ )\}).$$

It is easy to see that $\Gamma' \in \mathscr{P}_A$. Thus

$$\mu(\Gamma')/2\pi = a + d - 2 + \frac{1}{2}\left(1 - a - \sum_{i=1}^{d} \frac{1}{m_i}\right) < \mu(\Gamma)/2\pi.$$

Therefore, we may assume that Γ has signature

$$(g, +, [\lambda_1, \ldots, \lambda_r], \{\ \})$$

where $r \geq 2$. By Maclachlan's main result [5, Theorem 4], we see that $\lambda_1 = 2$ and λ_i divides λ_{i+1} for $i = 1, \ldots, r$. Now we define Γ' as the NEC group with signature

$$(g, +, [\lambda_2, \ldots, \lambda_{r-1}], \{(\ )\}).$$

Define the homomorphism $\phi'\colon \Gamma' \to A$ by

$$\phi'\colon \begin{array}{l} c_0' \to \bar{x}_1 \\ e' \to \bar{x}_r \bar{x}_1. \end{array}$$

Note $\lambda_1 = 2$, since $a \geq 1$. Hence $\Gamma' \in \mathscr{P}_A$ and it is easily calculated that

$$\mu(\Gamma') = \mu(\Gamma) - 2\pi(1/2 - 1/\lambda_r) < \mu(\Gamma).$$

Since this contradicts our assumption that Γ has minimal area, we see that Γ must be a Type II group.

All of the preceding results can be combined into one big theorem.

THEOREM 5.7. *Let A be an abelian group of even order with canonical form $(Z_2)^a \times Z_{m_1} \times \cdots \times Z_{m_d}$ where $m_1 > 2$. If the symmetric genus $\sigma(A) \geq 2$, then*

(i) $\sigma(A) = 1 + |A| \cdot (a + 3d - 4)/8$ *if* $a \geq d + 2$
(ii) $\sigma(A) = \tau(A)$ *if* $1 \leq a \leq d + 1$
(iii) $\sigma(A) = \min\{\sigma^\circ(A), \tau(A)\}$ *if* $a = 0$.

The preceding theorem is the best possible. If $a = 0$ (i.e. $\mathrm{rank}(E) \leq \mathrm{rank}(B)$), then the minimum area can occur with either groups of Type I or groups of Type II. For example, let

$$A = Z_m \times Z_{2n} \times Z_{4n}$$

where m divides n and both are odd. It is an easy computation to show that if $m = n$, then the minimum area occurs when Γ has signature

$$(0, +, [m, n, 4n], \{(\ )\})$$

of Type II. Whereas, if $m \neq n$, then the minimum area occurs when Γ has signature $(1, +, [m, m], \{\ \})$ of Type I.

Finally, we look at some special cases. Let $A = E \times B \times C$ be in alternate canonical form. If $E = 0$, then the symmetric genus and the strong symmetric genus are equal. If $B = 0$ and $E \neq 0$, then the minimum area occurs with a Type II group if $e \leq c + 1$ and with a Type III group if $e \geq c + 2$. In either case, the symmetric genus and the strong symmetric genus are not equal.

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