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A CHARACTERIZATION OF CYLINDERLIKE SURFACES

CASSANDRA COX

Introduction

In this paper we wish to characterize the rings whose spectra are cylinderlike surfaces. We define these surfaces as follows.

DEFINITION 1. A cylinderlike surface is an affine, rational, nonsingular surface, $T = \operatorname{Spec} A$, over an algebraically closed field k of characteristic zero having the properties that $A^* = k^*$, $\operatorname{Pic} T$ is torsion, and T contains a nonempty subset U which is isomorphic to $\mathbf{A}^1 \times C$ where C is a rational curve. We will call a subset such as U a cylindrical open set.

Interest in these surfaces arose from a result of Miyanishi [M, Theorem 0], which says that if T is a cylinderlike surface and $\operatorname{Pic} T = 0$, then $T \simeq \mathbf{A}^2$. In certain applications of this result it is clear that $\operatorname{Pic} T$ is torsion and the main difficulty is to show that $\operatorname{Pic} T = 0$. This led to the consideration of cylinderlike surfaces. In general, Miyanishi's theorem fails for these surfaces. In fact, for any finite Abelian group G, there is a cylinderlike surface T with $\operatorname{Pic} T = G$. Theorem 4.1 of this paper gives an algebraic construction of such a surface. We are interested in the number of ways in which T can be chosen for any particular G.

In an attempt to answer this question, we have been considering a certain type of fibration on a cylinderlike surface.

DEFINITION 2. Suppose $T = \operatorname{Spec} A$ is a cylinderlike surface. We say that a morphism $k[x] \hookrightarrow A$ gives a cylindrical fibration of T if for each $\alpha \in k$ the fibre defined by $x - \alpha$ is irreducible and if the complement of a finite set of these fibres is a cylindrical open set.

One previously obtained result, which is due to Richard Swan and which will be proved as Proposition 1.1 of this paper, says that if $T = \operatorname{Spec} A$ is a cylinderlike surface, then there are $\alpha_i \in k$, $n_i \in \mathbf{Z}^+$, and prime ideals P_i of

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A such that

$$k[s,t] \subset A \subset k\left[s, t, \frac{1}{\prod_{i=1}^{r}(s-\alpha_i)}\right]$$

and $(s-\alpha_i) = P_i^{n_i}$. Thus,

$$T - \bigcup_{i=1}^{r} V((s-\alpha_i)) \simeq \operatorname{Spec} k\left[s, t, \frac{1}{\prod_{i=1}^{r}(s-\alpha_i)}\right]$$

is a cylindrical open set and $k[s] \hookrightarrow A$ gives a cylindrical fibration of T. We wish to determine for which cylinderlike surfaces this cylindrical fibration is unique. We hope that this will be useful in determining when two cylinderlike surfaces are isomorphic. As a step toward this goal, this paper gives an explicit description of the rings whose spectra are cylinderlike surfaces.

1. Preliminary results

This first section gives the proofs of several unpublished results of Richard Swan and David Wright which will be used later in this paper.

PROPOSITION 1.1. *Let $T = \operatorname{Spec} A$ be a cylinderlike surface and let $U \simeq \mathbf{A}^1 \times C$ be an open cylindrical subset of T. Then there are $\alpha_i \in k$, $n_i \in \mathbf{Z}^+$, and height one prime ideals P_i of A such that*

$$k[s,t] \subset A \subset k\left[s, t, \frac{1}{\prod_{i=1}^{r}(s-\alpha_i)}\right],$$

$$U \simeq \operatorname{Spec} k\left[s, t, \frac{1}{\prod_{i=1}^{r}(s-\alpha_i)}\right],$$

and

$$(s-\alpha_i) = P_i^{n_i}.$$

Proof [S]. By [I, Theorem 2.15, p. 124], T is pure of codimension 1. Thus, there are height one prime ideals P_i such that $T - U = \bigcup_{i=1}^{r} V(P_i)$. Since

Pic T is torsion, there are $a_i \in A$ such that $(a_i) = P_i^{n_i}$. Thus, for $f = \prod_{i=1}^r a_i$, we have $T - U = V((f))$ and $U = \operatorname{Spec} A_f$.

Since C is rational, we have $C \subset \mathbf{P}^1$. By making C smaller if necessary we may assume that $C \subset \mathbf{A}^1$. Then the argument above gives $C = \operatorname{Spec} B$ where $B = k[s, 1/g(s)]$. Thus, $U = \operatorname{Spec} B[t]$.

We now have $A_f = B[t]$. Let $R = A \cap B$. Since $f \in (A_f)^* = B^*$, we have $f \in R$ and $R_f = A_f \cap B_f = A_f \cap B = B$. Replace t by $f^n t$ so that $t \in A$. Since A and B are normal, R is also normal. Thus, R is an intersection of valuation rings of $k(s)$. These are $k[s^{-1}]_{(s^{-1})}$ and $k[s]_{(p)}$ where p is irreducible. Since the intersection of all of these valuation rings is k, at least one must be omitted from the intersection which forms R. Since k is algebraically closed, irreducible elements of $k[s]$ are of the form $s - \alpha$ for $\alpha \in k$. Replacing s by $1/s$ if necessary, we may assume that $k[s^{-1}]_{(s^{-1})}$ is omitted. If $k[s]_{(p)}$ is also omitted, then $p \in R^* \subset A^* = k^*$. Thus,

$$R = \bigcap_{p \text{ irreducible}} k[s]_{(p)} = k[s].$$

Since $f \in R$, we may take $f = \prod_{i=1}^r (s - \alpha_i)$ with $(s - \alpha_i) = P_i^{n_i}$. Thus,

$$k[s,t] = R[t] \subset A \subset A_f = R_f[t] = k\left[s, t, \frac{1}{\prod_{i=1}^{r}(s-\alpha_i)}\right]$$

and

$$U \simeq \operatorname{Spec} k\left[s, t, \frac{1}{\prod_{i=1}^{r}(s-\alpha_i)}\right].$$

PROPOSITION 1.2 [W]. *In Proposition* 1.1, *we have* $V((s - \alpha_i)) \simeq \mathbf{A}^1$ *and we have* $\beta_i \in k$ *such that* $t - \beta_i \in P_i$.

Proof. Since $A^* = k^*$, we have that $k[s] \subset A$ induces a surjective morphism $\pi\colon T \to \mathbf{A}^1$ which, by a lemma of Miyanishi [M, Lemma 1] together with the fact that Pic T is torsion, has the property that each fibre, when reduced, is isomorphic to $\mathbf{A}^1$. Also, $k[s,t] \subset A$ gives a birational morphism $\rho\colon T \to V = \operatorname{Spec} k[s,t]$. For $i = 1, 2, \ldots, r$, let L_i be the fibre in V defined by $s - \alpha_i$. Since $\pi\colon T \to \mathbf{A}^1$ factors through V, $\rho^{-1}(L_i)$ is a fibre of π. Thus,

$$\rho^{-1}(L_i) = V((s - \alpha_i)) \simeq \mathbf{A}^1.$$

We also have that ρ is an isomorphism on $U = T - \bigcup_{i=1}^{r} V((s - \alpha_i))$. If $\rho(V((s - \alpha_i)))$ is not a point, then ρ is an isomorphism in a neighborhood of $V((s - \alpha_i))$ and we may take

$$U = T - \bigcup_{j \neq i} V\big((s - \alpha_j)\big).$$

Thus, we may assume that $\rho(V((s - \alpha_i)))$ is a point corresponding to the maximal ideal $(s - \alpha_i, t - \beta_i)$ and that $t - \beta_i \in P_i$.

PROPOSITION 1.3. *Suppose*

$$k[s,t] \subset A \subset k\left[s, t, \frac{1}{\prod_{i=1}^{r}(s - \alpha_i)}\right]$$

with $(s - \alpha_i) = P_i^{n_i}$ *for some* $n_i \in \mathbf{Z}^+$. *Then there is* $z \in A$ *such that*

$$k[s,z] \subset A \subset k\left[s, z, \frac{1}{\prod_{n_i > 1}(s - \alpha_i)}\right].$$

Proof (adapted from [S]). Suppose $n_1 = 1$. Let $a_1, a_2, \ldots, a_N$ generate A. There are nonnegative integers m_j such that

$$(s - \alpha_1)^{m_j} a_j \in k\left[s, t, \frac{1}{\prod_{i=2}^{r}(s - \alpha_i)}\right].$$

We use induction on $\sum_{j=1}^{N} m_j$ to show that there is $z \in A$ such that

$$a_j \in k\left[s, z, \frac{1}{\prod_{i=2}^{r}(s - \alpha_i)}\right] \quad \text{for } j = 1, 2, \ldots, N.$$

Let D be the divisor corresponding to P_1. If $m_1 > 0$, then

$$(s - \alpha_1)^{m_1} a_1 = \frac{g(s,t)}{\prod_{i=2}^{r}(s - \alpha_i)}$$

with $v_D(g) > 0$. Thus,

$$g(s,t) = (s-\alpha_1)g_1(s,t) + (t-\beta)g_0(t)$$

with $v_D(t-\beta) > 0$. Since $v_D(s-\alpha_1) = n_1 = 1$, we have that $s-\alpha_1$ divides $t-\beta$ in A. Let

$$z' = \frac{t-\beta}{s-\alpha_1}.$$

Then $k[s,z'] \subset A$ and

$$(s-\alpha_1)^{m_1-1}a_1 \in k\left[s, z', \frac{1}{\prod_{i=2}^{r}(s-\alpha_i)}\right].$$

Thus, $\Sigma_{j=1}^{N} m_j$ is reduced and there is $z \in A$ with

$$k[s,z] \subset A \subset k\left[s, z, \frac{1}{\prod_{i=2}^{r}(s-\alpha_i)}\right].$$

PROPOSITION 1.4. *For* $T = \operatorname{Spec} A$ *in Proposition* 1.3, $\operatorname{Pic} T \simeq \coprod_{i=1}^{r} \mathbf{Z}/n_i\mathbf{Z}$.

Proof. Let $U = T - \bigcup_{i=1}^{r} V((s-\alpha_i))$ and let D_i be the prime divisor corresponding to P_i. By [H, Prop. 6.5, p. 133] together with the fact that $\operatorname{Pic} U = 0$, we have that $\operatorname{Pic} T$ is generated by $D_1, D_2, \dots, D_r$. Let $\rho\colon T \to V = \operatorname{Spec} k[s,t]$ be the morphism given by $k[s,t] \hookrightarrow A$. Then ρ is an isomorphism from U to $V - \bigcup_{i=1}^{r} V((s-\alpha_i))$. Thus, if $f \in k(s,t)^*$ and if the support of the divisor of f is contained in $\bigcup_{i=1}^{r} V((s-\alpha_i))$ in T, then the support of the divisor of f is also contained in $\bigcup_{i=1}^{r} V((s-\alpha_i))$ in V. Therefore, $f = \prod_{i=1}^{r}(s-\alpha_i)^{e_i}$ and the divisor of f is an element of $\langle n_1D_1, n_2D_2, \dots, n_rD_r\rangle$. Thus,

$$\operatorname{Pic} T = \frac{\langle D_1, D_2, \dots, D_r\rangle}{\langle n_1D_1, n_2D_2, \dots, n_rD_r\rangle} \simeq \coprod_{i=1}^{r} \mathbf{Z}/n_i\mathbf{Z}.$$

2. Divisibility by $s - \alpha_i$

Suppose that Spec A is a cylinderlike surface. Then we have

$$A \subset k\left[s, t, \frac{1}{\prod_{i=1}^{r}(s-\alpha_i)}\right]$$

from Proposition 1.1. We begin the task of describing A by determining the elements of $k[s,t]$ which are divisible in A by $s - \alpha_i$ for some particular i. After a linear change of variable, we may assume that $i = 1$ and that $\alpha_1 = \beta_1 = 0$ where β_1 is given by Proposition 1.2. For this section, we let P denote the height one prime ideal containing s which is given by Proposition 1.1, and we let D denote the prime divisor corresponding to P. Then by Proposition 1.2 we have $s, t \in P$ and $A/P = k[\bar{v}]$ for some $v \in A$.

We wish to define two finite sequences. The first is a sequence, $u_1 = t, u_2, \ldots, u_q$, of elements of $k[s,t]$ with the properties

$$u_{j+1} = u_j^{n_j} - \lambda_j s^{e_{0,j}} \prod_{i=1}^{j-1} u_i^{e_{i,j}}$$

and

$$v_D(u_{j+1}) > v_D(u_j^{n_j}).$$

The u_j's are used to define the second sequence, $v_1, v_2, \ldots, v_q$, of elements of A which ends with $A/P = k[\overline{v_q}]$. Then, in Proposition 2.1, we will use these two sequences to determine when an element f of $k[s,t]$ is divisible in A by s^M.

First, we let

$$d_0 = v_D(s) \tag{2.1}$$

and

$$u_1 = t. \tag{2.2}$$

Then we let $d_1 = \gcd(d_0, v_D(u_1))$, $n_1 = d_0/d_1$, $m_1 = v_D(u_1)/d_1$ and $v_1 = t^{n_1}/s^{m_1}$. Since $n_1 v_D(t) = n_1 m_1 d_1 = m_1 v_D(s)$, the divisor of v_1 is effective so that $v_1 \in A$.

Now we suppose that $l \in \mathbf{Z}^+$ and we assume that we have defined $d_j \in \mathbf{Z}^+$ and $v_j \in A$ for $j = 1, \ldots, l$ and $u_{j+1} \in k[s,t]$ for $j = 1, 2, \ldots, l-1$

as follows. Let

$$d_j = \gcd\big(d_{j-1}, v_D(u_j)\big), \tag{2.3}$$

$$n_j = \frac{d_{j-1}}{d_j}, \tag{2.4}$$

and

$$m_j = \frac{v_D(u_j)}{d_j}. \tag{2.5}$$

Let $a_{j,j} = m_j$. Then, since $\gcd(n_j, m_j) = 1$, we have that, for $i = 1, 2, \ldots, j - 1$, there are integers $a_{i,j}$ and $b_{i,j}$ with

$$0 \le b_{i,j} < n_i \tag{2.6}$$

such that

$$a_{i+1,j} = a_{i,j}n_i - b_{i,j}m_i. \tag{2.7}$$

We let $a_j = a_{1,j}$ and let

$$v_j = \frac{\left(\prod_{i=1}^{j-1} u_i^{b_{i,j}}\right) u_j^{n_j}}{s^{a_j}} \tag{2.8}$$

Then the use of induction on k together with (2.7) gives us

$$a_{k,j} = a_{1,j}n_1n_2 \ldots n_{k-1} - \sum_{i=1}^{k-1} b_{i,j}m_in_{i+1} \ldots n_{k-1} \text{ for } k = 1, 2, \ldots, j.$$

Thus, by (2.1), (2.4), and (2.5) we have

$$v_D\left(\left(\prod_{i=1}^{j-1} u_i^{b_{i,j}}\right) u_j^{n_j}\right) = v_D(s^{a_j}) \tag{2.9}$$

so that the divisor of v_j is effective and $v_j \in A$. We also assume that, for $j = 1, 2, \ldots, l - 1$, in A/P we have

$$\overline{v_j} = \gamma_j \in k^*. \tag{2.10}$$

Now let $e'_{j,j} = m_j$ for $j = 1, 2, \ldots, l$. Then, for $i = 1, 2, \ldots, j - 1$, there are unique integers $e_{i,j}$ and $e'_{i,j}$ with

$$0 \leq e_{i,j} < n_i \tag{2.11}$$

such that

$$e'_{i+1,j} = e'_{i,j} n_i + e_{i,j} m_i. \tag{2.12}$$

Let

$$e_{0,j} = e'_{1,j}. \tag{2.13}$$

Then the use of induction on k together with (2.12) gives us

$$e'_{k,j} = e_{0,j} n_1 n_2 \dots n_{k-1} + \sum_{i=1}^{k-1} e_{i,j} m_i n_{i+1} \dots n_{k-1} \text{ for } k = 1, 2, \dots, j.$$

Thus, by (2.1), (2.4), and (2.5), we have

$$v_D(u_j^{n_j}) = v_D\left(s^{e_{0,j}} \prod_{i=1}^{j-1} u_i^{e_{i,j}} \right). \tag{2.14}$$

We assume that for $j = 1, 2, \dots, l-1$ there is $\lambda_j \in k^*$ such that

$$v_D\left(u_j^{n_j} - \lambda_j s^{e_{0,j}} \prod_{i=1}^{j-1} u_i^{e_{i,j}} \right) > v_D(u_j^{n_j}) \tag{2.15}$$

and we define

$$u_{j+1} = u_j^{n_j} - \lambda_j s^{e_{0,j}} \prod_{i=1}^{j-1} u_i^{e_{i,j}} \tag{2.16}$$

Induction on j together with (2.15) and (2.16) gives us that $\sum_{i=1}^{j-1} \mu_i v_D(u_i) < v_D(u_j)$ whenever $\mu_i < n_i$. Thus, by (2.11) and (2.14),

$$e_{0,j} > 0, \tag{2.17}$$

If $A/P = k[\overline{v_l}]$, then the sequences end. Otherwise, the following lemmas will allow us to continue the sequences by showing that $\overline{v_l} = \gamma_l \in k^*$ and that there is $\lambda_l \in k^*$ such that

$$v_D\left(u_l^{n_l} - \lambda_l s^{e_{0,l}} \prod_{i=1}^{l-1} u_i^{e_{i,l}} \right) > v_D(u_l^{n_l})$$

so that we may define u_{l+1} and, hence, d_{l+1} and v_{l+1}.

LEMMA 2.1. *Let* $f \in k[s,t]$. *Then* $f = \sum_{j=1}^{n} \xi_j s^{\mu_{0,j}} \prod_{i=1}^{l} u_i^{\mu_{i,j}}$ *where* $\xi_j \in k^*$ *and where the* $\mu_{i,j}$ *are nonnegative integers with* $\mu_{i,j} < n_i$ *for* $i = 1, 2, \ldots, l-1$.

Proof. Consider $\prod_{i=1}^{l} u_i^{\sigma_i}$ where $\sigma_i \in \mathbf{Z}^{\text{nonneg}}$. Suppose $j \in \{1, 2, \ldots, l-1\}$ with $\sigma_j \geq n_j$ and $\sigma_i < n_i$ for $i = j+1, \ldots, l-1$. Let $r \in \{j+1, \ldots, l\}$ such that $\sigma_i + 1 = n_i$ for $i = j+1, \ldots, r-1$ and either $\sigma_r + 1 < n_r$ or $r = l$. Induction on $r - j$ together with (2.16) shows that

$$u_j^{n_j} u_{j+1}^{\sigma_{j+1}} \ldots u_{r-1}^{\sigma_{r-1}} = u_r + \sum_{k=j}^{r-1} \lambda_k s^{e_{0,k}} \left(\prod_{i=1}^{k-1} u_i^{e_{i,k}} \right) \left(\prod_{i=k+1}^{r-1} u_i^{\sigma_i} \right).$$

Thus,

$$\prod_{i=1}^{l} u_i^{\sigma_i} = \sum_{k=1}^{n} \zeta_k s^{\mu_{0,k}} \left(\prod_{i=1}^{l} u_i^{\mu_{i,k}} \right)$$

where, for each k, $\mu_{j,k} < \sigma_j$ and, by (2.11), $\mu_{i,k} < n_i$ for $i = j+1, \ldots, l-1$. Therefore, induction on σ_j and $l - j$ gives the result.

LEMMA 2.2. *Suppose that* N *and* $\mu_1, \mu_2, \ldots, \mu_l$ *are integers with* $v_D(\prod_{i=1}^{l} u_i^{\mu_i}) = v_D(s^N)$. *Then there are integers* $\nu_1, \nu_2, \ldots, \nu_l$ *with* $\mu_l = n_l \nu_l$ *such that*

$$\frac{\prod_{i=1}^{l} u_i^{\mu_i}}{s^N} = \prod_{i=1}^{l} v_i^{\nu_i}.$$

Proof. By (2.1), (2.4), and (2.5), we have $N n_1 n_2 \ldots n_l d_l = \sum_{i=1}^{l} \mu_i m_i n_{i+1} \ldots n_l d_l$. Thus, induction on $l - i$ together with (2.7) shows that there are integers ν_i with

$$n_i \nu_i = \mu_i - \sum_{j=i+1}^{l} b_{i,j} \nu_j$$

and

$$v_D\left(\prod_{j=i}^{l} u_j^{\mu_j} \right) = \sum_{j=i}^{l} \nu_j d_{i-1} a_{i,j}.$$

Thus, by (2.8),

$$\prod_{i=1}^{l} v_i^{\nu_i} = \frac{\prod_{i=1}^{l} u_i^{\mu_i}}{s^N}$$

and $n_l \nu_l = \mu_l$.

LEMMA 2.3. *In A/P, $\deg(\overline{v_l}) \le 1$.*

Proof. By Proposition 1.2 we have that $A/P = k[\bar{v}]$ for some $v \in A$. Since, by Proposition 1.1,

$$A \subset k\left[s, t, \frac{1}{\prod_{i=1}^{r}(s-\alpha_i)}\right],$$

we have that there is $F \in k[s, t]$ with

$$v = \frac{F}{s^M \prod_{i=2}^{r}(s-\alpha_i)^{M_i}}.$$

Since $v \notin P$, we have $v_D(F) = v_D(s^M)$. We write $\prod_{i=2}^{r}(s-\alpha_i)^{M_i} = sg(s) + \alpha$ with $g \in k[s]$ and $\alpha \in k^*$. Then

$$v_D\left(\frac{F}{s^M} - \alpha v\right) = v_D\left(\frac{sg(s)F}{s^M \prod_{i=2}^{r}(s-\alpha_i)^{M_i}}\right) > 0.$$

Thus $\overline{F}/s^M = \alpha\bar{v}$ in A/P and we may assume that $v = F/s^M$.

Let

$$F = \sum_{j=1}^{n} \xi_j s^{\mu_{0,j}} \prod_{i=1}^{l} u_i^{\mu_{i,j}}$$

as in Lemma 2.1. Let

$$m = \min_j \left\{ v_D\left(s^{\mu_{0,j}} \prod_{i=1}^{l} u_i^{\mu_{i,j}} \right) \right\},$$

let

$$J = \left\{ j \middle| v_D\left(s^{\mu_{0,j}} \prod_{i=1}^{l} u_i^{\mu_{i,j}} \right) = m \right\},$$

and let

$$F_0 = \sum_{j \in J} \xi_j s^{\mu_{0,j}} \prod_{i=1}^{l} u_i^{\mu_{i,j}} \text{ and } F_1 = \sum_{j \notin J} \xi_j s^{\mu_{0,j}} \prod_{i=1}^{l} u_i^{\mu_{i,j}}.$$

Induction on l shows that if j and k are elements of J with $\mu_{l,j} = \mu_{l,k}$, then $\mu_{i,j} = \mu_{i,k}$ for $i = 0, 1, \ldots, l-1$. Thus, for distinct elements j and k of J, we may assume $\mu_{l,j} \neq \mu_{l,k}$. Now $F = F_0 + F_1$ with $v_D(F_1) > m$ and $v_D(F) \geq m$.

If $v_D(F) = m$, then, since $v_D(F) = v_D(s^M)$, we have

$$v_D\left(\frac{F_1}{s^M}\right) > 0.$$

Thus, in A/P,

$$\bar{v} = \frac{\overline{F_0}}{s^M} = \sum_{j \in J} \xi_j \frac{\overline{\prod_{i=1}^{l} u_i^{\mu_{i,j}}}}{s^{M-\mu_{0,j}}}.$$

By Lemma 2.2, we have integers $\nu_{i,j}$ with $\bar{v} = \sum_{j \in J} \xi_j \prod_{i=1}^{l} \bar{v}_i^{\nu_{i,j}}$ and with $n_l \nu_{l,j} = \mu_{l,j}$ so that $\nu_{l,j} \geq 0$. By (2.10), $\bar{v} = \sum_{j \in J} \zeta_j \bar{v}_l^{\nu_{l,j}}$ for some $\zeta_j \in k^*$. Therefore, $\deg(\overline{v_l}) = 1$.

If $v_D(F) > m$, then $v_D(F_0) > m$. We wish to choose $\tau_i \in \mathbf{Z}^{\text{nonneg}}$ so that

$$m + \sum_{i=1}^{l} v_D(u_i^{\tau_i}) = N v_D(s)$$

for some $N \in \mathbf{Z}$. By (2.9), this can be done by fixing $k \in J$, letting $\tau_i = \sigma_i v_D(s) + b_{i,l} - \mu_{i,k}$ for $i = 1, 2, \ldots, l-1$, and letting $\tau_l = \sigma_l v_D(s) + n_l - \mu_{l,k}$ where $\sigma_i \in \mathbf{Z}$ is chosen so that $\tau_i \geq 0$. Let $g = \prod_{i=1}^{l} u_i^{\tau_i}$. Then $v_D(gF_0) > v_D(g) + m = N v_D(s)$. Therefore, in A/P, we have $0 = \overline{gF_0}/s^N$. As above,

$$\frac{\overline{gF_0}}{s^N} = \sum_{j \in J} \zeta_j \overline{v_l}^{\nu_{l,j}}$$

for some $\zeta_j \in k^*$ and some nonnegative integers $\nu_{l,j}$ with $n_l \nu_{l,j} = \mu_{l,j} + \tau_l$. Thus, for distinct elements j and k in J, since $\mu_{l,j} \neq \mu_{l,k}$, we have $\nu_{l,j} \neq \nu_{l,k}$. By (2.9) we have $\overline{v_l} \neq 0$. Therefore, $\deg(\overline{v_l}) = 0$.

LEMMA 2.4. *If* $\deg(\overline{v_l}) = 0$ *in* A/P, *then*

$$v_D\left(u_l^{n_l} - \lambda_l s^{e_{0,l}} \prod_{i=1}^{l-1} u_i^{e_{i,l}}\right) > v_D(u_l^{n_l})$$

for some $\lambda_l \in k^*$.

Proof. Let $F = F_0 + F_1$ and J be as in the proof of Lemma 2.3. Fix r and p in J so that $\mu_{l,r} \geq \mu_{l,j}$ and $\mu_{l,p} \leq \mu_{l,j}$ for all $j \in J$. Then for $j \in J$ we have

$$v_D\left(s^{\mu_{0,j}} \prod_{i=1}^{l} u_i^{\mu_{i,j}}\right) = m = v_D\left(s^{\mu_{0,r}} \prod_{i=1}^{l} u_i^{\mu_{i,r}}\right).$$

Induction on $l - k$ together with (2.1), (2.4), (2.5), (2.12), and (2.13) gives us that, for $k = 1, 2, \ldots, l$, there is an integer $\tau_{k,j}$ with

$$\begin{aligned} n_k\Bigg[&(\mu_{0,j} - \mu_{0,r})n_1 n_2 \ldots n_{k-1} \\ &+ \sum_{i=1}^{k-1} (\mu_{i,j} - \mu_{i,r}) m_i n_{i+1} \ldots n_{k-1} - \sum_{i=k+1}^{l} e'_{k,i}\tau_{i,j}\Bigg] \\ &= m_k\left[\mu_{k,r} - \mu_{k,j} + \sum_{i=k+1}^{l} e_{k,i}\tau_{i,j}\right], \end{aligned}$$

with

$$\mu_{k,j} = \mu_{k,r} + \sum_{i=k+1}^{l} e_{k,i}\tau_{i,j} - n_k\tau_{k,j},$$

and

$$\mu_{0,j} = \mu_{0,r} + \sum_{k=1}^{l} e_{0,k}\tau_{k,j}.$$

Since $\mu_{l,r} \geq \mu_{l,j}$ and $n_k > \mu_{k,j}$ for $k = 1, 2, \ldots, l-1$, the second equation gives us $\tau_{k,j} \geq 0$. Since $\mu_{l,p} \leq \mu_{l,j}$ we have $\tau_{l,p} \geq \tau_{l,j}$. Thus, by (2.15) and

(2.16), we have

$$
\begin{aligned}
F_0 &= \sum_{j\in J} \xi_j s^{\mu_{0,j}} \prod_{i=1}^{l} u_i^{\mu_{i,j}} \\
&= \sum_{j\in J} \xi_j \left(s^{e_{0,l}} \prod_{i=1}^{l-1} u_i^{e_{i,l}} \right)^{\tau_{l,j}} s^{\mu_{0,r}} \prod_{k=1}^{l-1} \left(\frac{1}{\lambda_k} (u_k^{n_k} - u_{k+1}) \right)^{\tau_{k,j}} \prod_{k=1}^{l} u_k^{\mu_{k,r} - n_k \tau_{k,j}} \\
&= \left(s^{\mu_{0,r}} \prod_{i=1}^{l-1} u_i^{\mu_{i,r}} \right) u_l^{\mu_{l,r} - n_l \tau_{l,p}} \sum_{j\in J} \zeta_j \left(s^{e_{0,l}} \prod_{i=1}^{l-1} u_i^{e_{i,l}} \right)^{\tau_{l,j}} (u_l^{n_l})^{\tau_{l,p} - \tau_{l,j}} + G(s,t)
\end{aligned}
$$

where $\zeta_j \in k^*$ and $v_D(G) > m$. Since $\deg(\overline{v_l}) = 0$, we have, from the proof of Lemma 2.3, that $v_D(F_0) > m$. Therefore,

$$
\begin{aligned}
v_D &\left(\sum_{j\in J} \zeta_j \left(s^{e_{0,l}} \prod_{i=1}^{l-1} u_i^{e_{i,l}} \right)^{\tau_{l,j}} (u_l^{n_l})^{\tau_{l,p} - \tau_{l,j}} \right) \\
&> m - v_D \left(s^{\mu_{0,r}} \prod_{i=1}^{l-1} u_i^{\mu_{i,r}} \right) u_l^{\mu_{l,r} - n_l \tau_{l,p}} \Bigg) = v_D (u_l^{n_l})^{\tau_{l,p}}.
\end{aligned}
$$

Thus, there must be $\lambda_l \in k^*$ such that $v_D(u_l^{n_l} - \lambda_l s^{e_{0,l}} \prod_{i=1}^{l-1} u_i^{e_{i,l}}) > v_D(u_l^{n_l})$.

The above lemmas allow us to form our sequences $u_1, u_2, \ldots, u_l$ and $v_1, v_2, \ldots, v_l$ as follows. Suppose that, for $j = 1, 2, \ldots, l$, we have defined u_j and v_j as in (2.2), (2.8), and (2.16). Then, by Lemma 2.3, $\deg(\overline{v_l}) \le 1$ in A/P. If $\deg(\overline{v_l}) = 1$, then $A/P = k[\overline{v_l}]$, and the sequences end. If $\deg(\overline{v_l}) = 0$, then (2.10) is satisfied for v_l and, by Lemma 2.4, (2.15) is satisfied for u_l. Thus, we can define u_{l+1} to be $u_l^{n_l} - \lambda_l s^{e_{0,l}} \prod_{i=1}^{l-1} u_i^{e_{i,l}}$ and, consequently, define d_{l+1} and v_{l+1} so that the sequences continue.

LEMMA 2.5. *There exists $q \in \mathbf{Z}^+$ such that the sequence $v_1, v_2, \ldots, v_l$ ends with $A/P = k[\overline{v_p}]$.*

Proof. Suppose that for $j = 1, \ldots, l$, we have defined u_j and v_j as in (2.2), (2.8), and (2.16). Let $F = F_0 + F_1$ and J be as in the proof of Lemma 2.3. We must have $\mu_{l,j} > 0$ for some $j \in J$, for if not, then $F_0 = \xi s^{\mu_0} \prod_{i=1}^{l-1} u_i^{\mu_i}$ and, thus, $v_D(F_0) = m < v_D(F_1)$. Therefore, $v_D(F) = m$ and, as the proof of Lemma 2.3, $\bar{v} = \xi \prod_{i=1}^{l-1} \bar{v}_i^{\nu_i}$ where the ν_i are given by Lemma 2.2. By (2.10), this contradicts the fact that $A/P = k[\bar{v}]$. Now since $\mu_{l,j} > 0$ for some j, we have that $v_D(u_l) \le m \le v_D(F)$. By (2.15), (2.16), and Lemma 2.4, the sequence $u_1, u_2, \ldots, u_l$ has the property that $v_D(u_{j+1}) > v_D(u_j^{n_j})$. Thus, the sequence must end before $v_D(u_l) > v_D(F)$. If the sequence ends with u_q, then by Lemma 2.3 and Lemma 2.4, we have $\deg(\overline{v_q}) = 1$ so $A/P = k[\overline{v_q}]$.

For the remainder of this section we will assume that $q \in \mathbf{Z}^+$ with $A/P = k[\overline{v_q}]$.

LEMMA 2.6. *Let $f \in k[s,t]$ and let $f = \sum_{j=1}^{n} \xi_j s^{\mu_{0,j}} \prod_{i=1}^{q} u_i^{\mu_{i,j}}$ as in Lemma 2.1. Then*

$$v_D(f) = \min_j \left\{ v_D\left(s^{\mu_{0,j}} \prod_{i=1}^{q} u_i^{\mu_{i,j}} \right) \right\}.$$

Proof. Let

$$m = \min_j \left\{ v_D\left(s^{\mu_{0,j}} \prod_{i=1}^{q} u_i^{\mu_{i,j}} \right) \right\},$$

and let

$$J = \left\{ j \middle| v_D\left(s^{\mu_{0,j}} \prod_{i=1}^{q} u_i^{\mu_{i,j}} \right) = m \right\}.$$

As in the proof of Lemma 2.3, we have $g \in k[s,t]$, $N \in \mathbf{Z}^+$, $\zeta_j \in k^*$, and distinct $\nu_j \in \mathbf{Z}^{\text{nonneg}}$ such that $v_D(g) + m = v_D(s^N)$ and $\overline{gf}/s^N = \sum_{j \in J} \zeta_j \overline{v_q}^{\nu_j}$. Thus, $\overline{gf}/s^N \neq 0$ and $v_D(f) = v_D(s^N) - v_D(g) = m$.

PROPOSITION 2.1. *Let $f \in k[s,t]$ and let $M \in \mathbf{Z}$. Then $f/s^M \in A$ if and only if $f/s^M \in k[I]$ where*

$$I = \left\{ \frac{\prod_{i=1}^{q} u_i^{\mu_i}}{s^N} \middle| 0 \le \mu_i < n_i \text{ for } i < q, 0 \le \mu_q, \text{ and } v_D\left(\prod_{i=1}^{q} u_i^{\mu_i} \right) \ge v_D(s^N) \right\}.$$

Proof. If z is an element of I, then the divisor of z is effective. Thus, $k[I] \subset A$. If f/s^M is an element of A, then $v_D(f) \ge v_D(s^M)$. Thus, $f/s^M \in k[I]$ by Lemma 2.1 and Lemma 2.6.

The last lemma of this section gives an additional property of the sequence $v_1, v_2, \ldots, v_q$.

LEMMA 2.7. *Let $q \in \mathbf{Z}^+$ with $A \setminus P = k[\overline{v_q}]$. Then $q > 1$ and $d_{q-1} = 1$.*

Proof. Let m be the ideal of A generated by P and v_q. Since Spec A is nonsingular and the subvariety $V(P)$ has codimension one, we have that $V(P)$ has one local equation in a neighborhood of the point corresponding to

m. Let the equation be x/y where $x, y \in A$ and $y \notin m$. Then $v_D(x) = 1$. Since $PA_m = (x/y)A_m$, there are elements z and w of A with $w \notin m$ such that $s = (x/y)(z/w)$. By Proposition 1.1,

$$x = \frac{f}{s^R \prod_{i=2}^{r} (s - \alpha_i)^{R_i}} \quad \text{and} \quad z = \frac{g}{s^Q \prod_{i=2}^{r} (s - \alpha_i)^{Q_i}}.$$

Since $xz/s = yw \notin m$, we have $v_D(fg) = v_D(s^{R+Q+1})$. As in the proof of Lemma 2.3, there is $\alpha \in k^*$ such that, in $A \setminus P$,

$$\frac{\overline{xz}}{s} = \frac{\overline{\alpha fg}}{s^{R+Q+1}}.$$

Let

$$f = \sum_{j=1}^{n} \xi_j s^{\mu_{0,j}} \prod_{i=1}^{q} u_i^{\mu_{i,j}} \quad \text{and} \quad g = \sum_{k=1}^{p} \zeta_k s^{\sigma_{0,k}} \prod_{i=1}^{q} u_i^{\sigma_{i,k}}$$

as in Lemma 2.6. Let

$$M_1 = \min_j \left\{ v_D\left(s^{\mu_{0,j}} \prod_{i=1}^{l} u_i^{\mu_{i,j}} \right) \right\}, \quad M_2 = \min_k \left\{ v_D\left(s^{\sigma_{0,k}} \prod_{i=1}^{l} u_i^{\sigma_{i,k}} \right) \right\},$$

$$J = \left\{ j \middle| v_D\left(s^{\mu_{0,j}} \prod_{i=1}^{l} u_i^{\mu_{i,j}} \right) = M_1 \right\} \quad \text{and} \quad K = \left\{ k \middle| v_D\left(s^{\sigma_{0,k}} \prod_{i=1}^{l} u_i^{\sigma_{i,k}} \right) = M_2 \right\}.$$

Then as in the proof of Lemma 2.3, for $j \in J$ and $k \in K$ we have $\nu_{j,k} \in \mathbf{Z}^{\text{nonneg}}$ with $n_l \nu_{j,k} = \mu_{l,j} + \sigma_{l,k}$ and with

$$\frac{\overline{xz}}{s} = \sum_{j \in J, k \in K} \theta_{j,k} \overline{v_q}^{\,-\nu_{j,k}}.$$

Since $xz/s \notin m$, we must have $\nu_{j,k} = 0$ for some $j \in J$ and $k \in K$. Thus, for this j, we have $\mu_{l,j} = 0$ and

$$1 = v_D(x) = v_D(f) - v_D(s^R) = v_D\left(s^{\mu_{0,j} - R} \prod_{i=1}^{q-1} u_i^{\mu_{i,j}} \right).$$

Since, by Proposition 1.3, we may assume $v_D(s) > 1$, we must have $q > 1$. Also, by (2.1), (2.4), and (2.5), we have that d_{q-1} divides $v_D(s)$ and d_{q-1} divides $v_D(u_i)$ for $i = 1, 2, \ldots, q-1$. Thus, $d_{q-1} = 1$.

3. Construction of a ring with a cylinderlike spectrum

In the previous section we began with a cylinderlike surface, Spec A, and found a subset I of $k[s, t, 1/s]$ with the property that $f/s^M \in A$ if and only if $f/s^M \in k[I]$. In this section we wish to define a finite subset J of $k[s, t, 1/s]$ with the property that Spec $k[J]$ is a cylinderlike surface. To define J, we wish to form a sequence, $u_1, u_2, \dots, u_q$, of elements of $k[s, t]$ and a sequence, $v_1, v_2, \dots, v_q$, of elements of $k[s, t, 1/s]$ which have the properties of the two sequences of section 2. We begin with an integer q with $q > 1$; $\lambda_1, \lambda_2, \dots, \lambda_{q-1} \in k^*$; and $d_0, \omega_1, \omega_2, \dots, \omega_q \in \mathbf{Z}^+$. We want to define J so that there is a prime divisor D on Spec $k[J]$ with $v_D(s) = d_0$ and $v_D(u_j) = \omega_j$ and so that $u_{j+1} = u_j^{n_j} - \lambda_j s^{e_{0,j}} \prod_{i=1}^{j-1} u_i^{e_{i,j}}$ for some n_j, $e_{i,j} \in \mathbf{Z}^{\text{noneg}}$. Thus, from Proposition 1.3, (2.15), and Lemma 2.7, we require that

$$d_0 > 1 \tag{3.1}$$

and that if, for $j = 1, 2, \dots, q$, we let

$$d_j = \gcd(d_{j-1}, \omega_j), \tag{3.2}$$

$$n_j = \frac{d_{j-1}}{d_j}, \tag{3.3}$$

and

$$m_j = \frac{\omega_j}{d_j}, \tag{3.4}$$

then

$$\omega_{j+1} > n_j \omega_j \tag{3.5}$$

and

$$d_{q-1} = 1. \tag{3.6}$$

Given these conditions, by (3.2), (3.3), and (3.4), we have, for $i = 1, 2, \dots, j-1$, that $\gcd(n_i, m_i) = 1$. Thus, we may define integers $e_{i,j}$ and $e'_{i,j}$ as in (2.11), (2.12), and (2.13). As in (2.14),

$$e_{0,j} d_0 + \sum_{i=1}^{j-1} e_{i,j} \omega_i = n_j \omega_j \tag{3.7}$$

and, as in (2.17),

$$e_{0,j} > 0. \tag{3.8}$$

We let

$$u_1 = t \tag{3.9}$$

and, for $j = 1, 2, \ldots, q-1$, we define

$$u_{j+1} = u_j^{n_j} - \lambda_j s^{e_{0,j}} \prod_{i=1}^{j-1} u_i^{e_{i,j}} \tag{3.10}$$

so that $u_j \in k[s,t]$. For $j = 1, 2, \ldots, q-1$ we let $a_{j,j} = m_j$ and for $i = 1, 2, \ldots, j-1$, we define integers $a_{i,j}$, and $b_{i,j}$ as in (2.6) and (2.7) and we let $a_j = a_{1,j}$. Then as in (2.9), we have

$$\sum_{i=1}^{j-1} b_{i,j}\omega_i + n_j\omega_j = a_j d_0. \tag{3.11}$$

Now we define

$$v_j = \frac{\left(\prod_{i=1}^{j-1} u_i^{b_{i,j}}\right) u_j^{n_j}}{s^{a_j}} \tag{3.12}$$

and

$$w_j = \frac{u_j^{d_0}}{s^{\omega_j}}. \tag{3.13}$$

Thus, v_j and w_j are elements of $k[s, t, 1/s]$. By (3.1), (3.2) and (3.6) there is an integer p with $0 < p < q$ such that $d_p = 1$ and $d_i > 1$ for $i = 0, 1, \ldots, p-1$. Let c_i and c_i' be the unique integers with

$$0 \le c_i < n_i \tag{3.14}$$

such that, if $c_{p+1}' = -1$, then $c_{i+1}' = c_i' n_i - c_i m_i$ and let $c_0 = c_1'$. Then as in (2.9)

$$\sum_{i=1}^{p} c_i \omega_i = c_0 d_0 + 1. \tag{3.15}$$

We now define

$$x = \frac{\prod_{i=1}^{p} u_i^{c_i}}{s^{c_0}} \tag{3.16}$$

so that x is also an element of $k[s, t, 1/s]$. Finally, we let

$$\Omega = \{j \in \mathbf{Z} | 1 \leq j < q \text{ and } n_j \neq 1\} \tag{3.17}$$

and we define

$$J = \{s, t, x, v_q\} \cup \{v_j, w_{j+1} | j \in \Omega\}. \tag{3.18}$$

LEMMA 3.1. *Let $M \in \mathbf{Z}$ and for $i = 1, 2, \ldots, q$, let $\mu_i \in \mathbf{Z}^{\text{nonneg}}$ with $\sum_{i=1}^{q} \mu_i \omega_i \geq M d_0$. Let $\nu_0 = \sum_{i=1}^{q} \mu_i \omega_i - M d_0$. Then there are integers ν_i with $\nu_q = \mu_q$ such that*

$$\frac{\prod_{i=1}^{q} u_i^{\mu_i}}{s^M} = x^{\nu_0} \prod_{i=1}^{q} v_i^{\nu_i}.$$

Furthermore, if $i \notin \Omega$, then $\nu_i = \mu_i$ and if $\nu_0 = 0$ and $\mu_i = 0$ for $i = j$, $j + 1, \ldots, q$, then $\nu_i = 0$ for $i = j, j + 1, \ldots, q$.

Proof. Let $p, c_i \in \mathbf{Z}$ be as in (3.15) and, for $i = p + 1, p + 2, \ldots, q$, let $c_i = 0$. Then by (3.15), we have

$$\sum_{i=1}^{q} (\mu_i - \nu_0 c_i) \omega_i = \nu_0 + M d_0 - \nu_0 (c_0 d_0 + 1) = d_0 (M - \nu_0 c_0).$$

Induction on $q - i$ together with (3.3), (3.4), (2.7), and the fact that $m_i = a_{i,i}$ shows that there are $\nu_i \in \mathbf{Z}$ such that

$$\sum_{j=1}^{i-1} (\mu_j - \nu_0 c_j) m_j n_{j+1} \ldots n_i + (\mu_i - \nu_0 c_i) m_i + \sum_{j=i+1}^{q} \nu_j (a_{i,j} n_i - b_{i,j} m_i) = n_1 n_2 \ldots n_i (M - \nu_0 c_0)$$

and

$$n_i \nu_i = (\mu_i - \nu_0 c_i) - \sum_{j=i+1}^{q} b_{i,j} \nu_j$$

and

$$\sum_{j=1}^{q} \nu_j a_{1,j} = M - \nu_0 c_0.$$

Thus, by (3.12) and (3.16),

$$x^{\nu_0}\prod_{i=1}^{q} v_i^{\nu_i} = \frac{\prod_{i=1}^{q} u_i^{\mu_i}}{s^M}.$$

Also, from above, $c_q = 0$ and by (3.3) and (3.6) we have $n_q = 1$ so that $\nu_q = \mu_q$. Similarly, if $i < q$ and $i \notin \Omega$, then by (3.17) $n_i = 1$ so by (2.6) and (3.14) we have $b_{i,j} = 0$ and $c_i = 0$ so that $\nu_i = \mu_i$. If $\nu_0 = 0$ and $\mu_i = 0$ for $i = j, j+1, \ldots, q$, then induction on $q - i$ shows that $\nu_i = 0$ for $i = j$, $j+1, \ldots, q$.

LEMMA 3.2. *Let J be given by* (3.18). *Then* $1/s \notin k[J]$.

Proof. By (3.9), by (3.11) and (3.12), by (3.13), by (3.15) and (3.16), and by (3.18) we have that the elements of J are all of the form

$$\frac{\prod_{i=1}^{q} u_i^{\mu_i}}{s^M}$$

with $M \in \mathbf{Z}$, $\mu_i \in \mathbf{Z}^{\text{nonneg}}$, and $\sum_{i=1}^{q}\mu_i\omega_i \geq Md_0$. Thus, the elements of $k[J]$ are all of the form f/s^M where

$$f = \sum_{j=1}^{R} \zeta_j s^{\mu_{0,j}} \prod_{i=1}^{q} u_i^{\mu_{i,j}}$$

with $\mu_{i,j} \in \mathbf{Z}^{\text{nonneg}}$, $\zeta_j \in k^*$ and, for each j, $\mu_{0,j} d_0 + \sum_{i=1}^{q}\mu_{i,j}\omega_i \geq Md_0$. An argument similar to the one used in Lemma 2.1 together with (3.5) shows that we may assume $\mu_{i,j} < n_i$ for $i = 1, 2, \ldots, q-1$. By (3.8), (3.9), and (3.10), we have

$$\prod_{i=1}^{q} u_i^{\mu_{i,j}} = t^{\tau_j} - sg(s,t) \quad \text{where } \tau_j = \sum_{i=1}^{q} \mu_{i,j} n_1 n_2 \ldots n_{i-1}.$$

Induction on i shows that if $\tau_j = \tau_l$, then $\mu_{i,j} = \mu_{i,l}$ for $i = 1, 2, \ldots, q$. Thus, if $f = s^N$, then $N \geq M$. Therefore, $1/s \notin k[J]$.

PROPOSITION 3.1. *Let J be given by* (3.18) *and let* $B = k[J]$. *Then* Spec B *is nonsingular and the divisor of s is $d_0 D$ for some prime divisor D. Also, if $i \in \Omega$, then v_i is not an element of any maximal ideal of B which contains s.*

Proof. Since $B_{1/s} = k[s, t, 1/s]$, we have that Spec B is nonsingular at any point of Spec $B - V(s)$. Let P be a height one prime ideal containing s. We wish to find a set of generators for P. Let l be the least element of Ω. By (2.6), (2.11), and (3.17), we have $b_{i,l} = e_{i,l} = 0$ for $i = 1, 2, \ldots, l-1$ and, thus, by (3.12), (3.9), and (3.10), we have

$$s^{a_l} v_l = u_l^{n_l} = \left(t - \sum_{i=1}^{l-1} \lambda_i s^{e_{0,i}} \right)^{n_l}.$$

Since $a_{l,l} = m_l > 0$, we have, by (2.6) and (2.7), that $a_l = a_{1,l} > 0$. Therefore, by (3.8), $t \in P$.

For $j = 1, 2, \ldots, q$, let

$$\Omega_j = \{ i \in \Omega | i < j \}. \tag{3.19}$$

If $i \notin \Omega_j$, then by (2.6), (2.11), (3.14), and (3.17) we have $e_{i,j} = b_{i,j} = c_i = 0$. Now we use induction to show that, for $j \in \Omega$, there is $\gamma_j \in k^*$ such that $v_j - \gamma_j \in P$. With l remaining as the least element of Ω, we have, by (3.5), (3.7), and (3.11), that $\omega_{l+1} > n_l \omega_l = e_{0,l} d_0 = a_l d_0$. Thus, by (3.10), (3.12), and (3.13) we have

$$(v_l - \lambda_l)^{d_0} = \left(\frac{u_l^{n_l} - \lambda_l s^{a_l}}{s^{a_l}} \right)^{d_0} = \left(\frac{u_{l+1}}{s^{a_l}} \right)^{d_0} = s^{\omega_{l+1} - a_l d_0} w_{l+1} \in P.$$

Thus, $v_l - \lambda_l \in P$ and we let $\gamma_l = \lambda_l$. For $j = 1, 2, \ldots, q$, we have, by Lemma 3.1 and by (3.7) and (3.11), that there are integers $\sigma_{i,j}$ and $\rho_{i,j}$ such that

$$\frac{\prod_{i \in \Omega_j} u_i^{b_{i,j} + e_{i,j}}}{s^{a_j - e_{0,j}}} = \prod_{i \in \Omega_j} v_i^{\sigma_{i,j}} \tag{3.20}$$

and

$$\frac{\prod_{i \in \Omega_j} u_i^{d_0 b_{i,j}}}{s^{d_0 a_j - n_j \omega_j}} = \prod_{i \in \Omega_j} v_i^{\rho_{i,j}} \tag{3.21}$$

Suppose that $j \in \Omega$ with $j > l$. Then, by (3.10), (3.12), and (3.13), we have

$$\begin{aligned}\left(v_j - \lambda_j \prod_{i \in \Omega_j} v_i^{\sigma_{i,j}} \right)^{d_0} &= \left(\frac{\left(\prod_{i \in \Omega_j} u_i^{b_{i,j}} \right) \left(u_j^{n_j} - \lambda_j s^{e_{0,j}} \prod_{i \in \Omega_j} u_i^{e_{i,j}} \right)}{s^{a_j}} \right)^{d_0} \\ &= s^{\omega_{j+1} - n_j \omega_j} \left(\prod_{i \in \Omega_j} v_i^{\rho_{i,j}} \right) w_{j+1}.\end{aligned}$$

For $i \in \Omega_j$, the induction hypothesis gives us $\gamma_i \in k^*$ with $v_i - \gamma_i \in P$. Thus, $v_i \notin P$ and so

$$\left(v_j - \lambda_j \prod_{i \in \Omega} v_i^{\sigma_{i,j}}\right)^{d_0} \in PB_P.$$

Therefore,

$$v_j - \gamma_j \in P \quad \text{where } \gamma_j = \lambda_j \prod_{i \in \Omega_j} \gamma_i^{\sigma_{i,j}} \in k^*.$$

Now suppose $j \notin \Omega$ and $j \neq q$. Let r be an integer with $r > j$ such that $n_k = 1$ for $k = j, j+1, \dots, r-1$ and either $r \in \Omega$ or $r = q$ so that $v_r \in J$ by (3.18). Then, by (3.19), $\Omega_k = \Omega_j$ for $k = j, \dots, r$ and, by (3.10),

$$u_r = u_j - \sum_{k=j}^{r-1} \lambda_k s^{e_{0,k}} \prod_{i \in \Omega_k} u_i^{e_{i,k}}.$$

By Lemma 3.1 and (3.15), there are integers τ_i such that

$$\frac{\prod_{i \in \Omega_{p+1}} u_i^{c_i d_0}}{s^{c_0 d_0 + 1}} = \prod_{i \in \Omega_{p+1}} v_i^{\tau_i}.$$

Thus, by (3.16), $x^{d_0} = s\prod_{i \in \Omega_{p+1}} v_i^{\tau_i}$ and, since $v_i \notin P$, we have $x \in P$.

Let $\sigma_{i,j}$ be given by (3.20). Then, by Lemma 3.1, and by (3.12), (3.13), and (3.7), there are $\nu_{i,j,k} \in \mathbf{Z}$ with $\nu_{0,j,k} > 0$ such that

$$\begin{aligned} v_j &= \frac{\left(\prod_{i \in \Omega_j} u_i^{b_{i,j}}\right)\left(u_r + \sum_{k=j}^{r-1} \lambda_k s^{e_{0,k}} \prod_{i \in \Omega_j} u_i^{e_{i,k}}\right)}{s^{a_j}} \\ &= x^{\nu_{0,j,r}}\left(\prod_{i \in \Omega_j} v_i^{\nu_{i,j,r}}\right) v_r^{\nu_{r,j,r}} + \sum_{k=j+1}^{r-1} \lambda_k x^{\nu_{0,j,k}} \prod_{i \in \Omega_j} v_i^{\nu_{i,j,k}} + \lambda_j \prod_{i \in \Omega_j} v_i^{\sigma_{i,j}}. \end{aligned}$$

Since, for $i \in \Omega_j$, there is no maximal ideal which contains both s and v_i, we have that v_j is regular on Spec B and, hence, $v_j \in B$. Also, since $x \in P$ and $v_i - \gamma_i \in P$ for $i \in \Omega_j$, we have $v_j - \gamma_j \in P$ where $\gamma_j = \lambda_j \prod_{i \in \Omega_j} \gamma_i^{\sigma_{i,j}} \in k^*$. Thus, for $j = 1, 2, \dots, q-1$, v_j is not in any maximal ideal containing s.

By Lemma 3.1 and (3.13), for $j = 1, 2, \dots, q$, there are integers $\mu_{i,j}$ such that

$$w_j = \prod_{i=1}^{j} v_i^{\mu_{i,j}} \tag{3.22}$$

so that w_j is regular on $\operatorname{Spec} B$ and $w_j \in B$. Also, for $j \neq q$, we have $w_j - \delta_j \in P$ where $\delta_j = \prod_{i=1}^{j} \gamma_i^{\mu_{i,j}} \in k^*$. Let $\delta_q = \prod_{i \in \Omega} \gamma_i^{\rho_{i,q}}$ where the $\rho_{i,q}$ are given by (3.21). By (3.3) and (3.6), we have $n_q = 1$ so that, by (3.12), (3.13), and (3.21), we have

$$v_q^{d_0} - \delta_q w_q = w_q\left(\prod_{i \in \Omega} v_i^{\rho_{i,q}} - \delta_q\right) \in P. \tag{3.23}$$

If $q - 1 \in \Omega$, let I be the ideal generated by

$$\left\{s, t, x, v_q^{d_0} - \delta_q w_q\right\} \cup \left\{v_j - \gamma_j | j \in \Omega\right\} \cup \left\{w_{j+1} - \delta_{j+1} | j \in \Omega_{q-1}\right\}.$$

If $q - 1 \notin \Omega$, let I be the ideal of B generated by $\{s, t, x\} \cup \{v_j - \gamma_j, w_{j+1} - \delta_{j+1} | j \in \Omega\}$. In either case, $I \subset P$ and, by (3.18), $B/I = k[\overline{v_q}]$ so that I is a height one prime ideal. Thus, $I = P$.

Now, to show that $\operatorname{Spec} B$ is nonsingular at any point of $V(s)$, we let m be a maximal ideal containing P. Since $v_i \notin m$ for $i < q$, we have, by Lemma 3.1 and (3.9), that s and t are elements of $(x)B_m$. By (3.10), (3.12), and (3.20), we have, for $j = 1, 2, \ldots, q - 1$, that

$$v_j - \lambda_j \prod_{i \in \Omega_j} v_i^{\sigma_{i,j}} = \frac{\left(\prod_{i \in \Omega_j} u_i^{b_{i,j}}\right) u_{j+1}}{s^{a_j}}$$

which is in $(x)B_m$ by (3.5), (3.11), and Lemma 3.1. Using induction, we have $v_j - \gamma_j \in (x)B_m$. Thus, by (3.22), $w_j - \delta_j \in (x)B_m$ and, by (3.23), $v_q^{d_0} - \delta_q w_q \in (x)B_m$. Therefore, $PB_m = (x)B_m$.

Since $B/P = k[\overline{v_q}]$, and $PB_m = (x)B_m$, we have $mB_m = (x, v_q - \zeta)B_m$ for some $\zeta \in k$. Let $\hat{m} = mB_m$. Then $\dim(\hat{m}/\hat{m}^2) \leq 2 = \dim B_m$. Therefore, $\operatorname{Spec} B$ is nonsingular at the point corresponding to m. Since $PB_P = (x)B_P$ and since, by Lemma 3.1, there are integers θ_i such that $s = x^{d_0} \prod_{i \in \Omega} v_i^{\theta_i}$ with $v_i \notin P$, we have that the divisor of s is $d_0 D$ where D is the prime divisor corresponding to P.

4. A characterization of cylinderlike surfaces

In this section we combine the results of the previous two sections to give an explicit description of the rings whose spectra are cylinderlike surfaces.

THEOREM 4.1. *Let k be an algebraically closed field of characteristic zero.* $\operatorname{Spec} A$ *is a cylinderlike surface over k if and only if $A = k[J]$ where J is a subset of* $\operatorname{frac}(k[s, t])$ *defined as follows. Let $r \in \mathbf{Z}^+$ and for $i = 1, 2, \ldots, r$, let $\alpha_i, \beta_i \in k$ with $\alpha_i \neq \alpha_j$ for $i \neq j$; let $q_i \in \mathbf{Z}$ with $q_i > 1$; let $\lambda_{1,i}$,*

$\lambda_{2,i}, \ldots, \lambda_{q_i,i} \in k^*$; and let $d_{0,i}, \omega_{1,i}, \omega_{2,i}, \ldots, \omega_{q_i,i} \in \mathbf{Z}^+$ such that, for each i, and for $j = 1, 2, \ldots, q_i$, we have that $\lambda_{j,i}$, $d_{0,i}$, and $\omega_{j,i}$ satisfy (3.1), (3.5), *and* (3.6). *For each i, let $u_{1,i} = t - \beta_i$ and for $j = 1, 2, \ldots, q_i$, define $u_{j,i}, v_{j,i}, w_{j,i}, x_i$ and J_i as in* (3.10), (3.12), (3.13), (3.16), *and* (3.18) *using $s - \alpha_i$ instead of s. Define J to be $\bigcup_{i=1}^r J_i$. In this case* $\operatorname{Pic}\operatorname{Spec} A \simeq \coprod_{i=1}^r \mathbf{Z}/d_{0,i}\mathbf{Z}$.

Proof. Suppose that J is defined as above and that $A = k[J]$. Let $T = \operatorname{Spec} A$. Then T is an affine, rational surface over k. Note that $J_i \subset k[s, t, 1/s - \alpha_i]$. Let $f = \prod_{i=1}^r (s - \alpha_i)$. Since $A \subset k[s, t, 1/f]$, we have that T contains the open cylindrical subset

$$U = T - V((f)) \simeq \operatorname{Spec} k[s, t, 1/f].$$

Let $A_i = k[J_i]$ and let $f_i = \prod_{j \neq i} (s - \alpha_j)$. By Lemma 3.2, $1/(s - \alpha_i) \notin A_i$. Thus, $1/(s - \alpha_i) \notin (A_i) f_i = A f_i$ and so $1/(s - \alpha_i) \notin A$. Therefore, $A^* = k^*$.

To show that T is nonsingular, let $f = \prod_{i=1}^r (s - \alpha_i)$ as before. Since $A_f = k[s, t, 1/f]$, we have that T is nonsingular on $T - \bigcup_{i=1}^r V((s - \alpha_i))$. Again, let

$$A_i = k[J_i] \quad \text{and} \quad f_i = \prod_{j \neq i} (s - \alpha_j).$$

Let $T_i = \operatorname{Spec} A_i$. Since $A_i \subset A \subset A_{f_i} = (A_i)_{f_i}$, we have a morphism ρ_i: $T \to T_i$ which is an isomorphism from $T - V((f_i))$ to $T_i - V((f_i))$. By Proposition 3.1, T_i is nonsingular and the divisor of $s - \alpha_i$ on T_i is $d_{0,i} C_i$ for some prime divisor C_i. On T,

$$V((s - \alpha_i)) \cap V((f_i)) = \emptyset,$$

so T is nonsingular at any point of $V((s - \alpha_i))$ and the divisor of $s - \alpha_i$ on T is $d_{0,i} D_i$ for some prime divisor D_i on T. Thus, by Proposition 1.4, $\operatorname{Pic} T \simeq \coprod_{i=1}^r \mathbf{Z}/d_{0,i}\mathbf{Z}$. Therefore, T is a cylinderlike surface.

Conversely, suppose that $\operatorname{Spec} A$ is a cylinderlike surface. Let $r \in \mathbf{Z}^+$ be given by Proposition 1.1 and, for $i = 1, 2, \ldots, r$, let $\alpha_i, \beta_i \in k$ be given by Proposition 1.1 and Proposition 1.2, respectively. Let P_i be the prime ideal given by Proposition 1.1 and let D_i be the prime divisor corresponding to P_i. For $i = 1, 2, \ldots, r$, let $d_{0,i} = v_{D_i}(s - \alpha_i)$ and let $\omega_{1,i} = v_{D_i}(t - \beta_i)$. For each i, we have $q_i \in \mathbf{Z}$ with $q_i > 1$ given by Lemma 2.5; we have a sequence, $u_{1,i} = t - \beta_i, u_{2,i}, \ldots, u_{q_i,i}$, of elements of $k[s, t]$ given by (2.16) and the discussion following Lemma 2.4; and we have a sequence, $v_{1,i}, v_{2,i}, \ldots, v_{q_i,i}$, of elements of A given by (2.8) and such that $A/P = k[\overline{v_{q_i,i}}]$ by Lemma 2.5. Let $\lambda_{1,i}, \lambda_{2,i}, \ldots, \lambda_{q_i,i}$ be the elements of k^* given by (2.15) and Lemma 2.4

and, for $j = 1, 2, \ldots, q_i$, let $\omega_{j,i} = v_{D_i}(u_{j,i}) \in \mathbf{Z}^+$. By Proposition 1.3, we may assume that the $d_{0,i}$ satisfy (3.1). For each i, we have, by (2.15), (2.16), and Lemma 2.4, that the $\omega_{j,i}$ satisfy (3.5) and we have, by Lemma 2.7, that the $d_{q-1,i}$ satisfy (3.6). Thus, we may define J as in the statement of the theorem.

Let

$$I_i = \left\{ \frac{\prod_{j=1}^{q_i} u_{j,i}^{\mu_j}}{(s-\alpha_i)^N} \,\middle|\, 0 \le \mu_j < n_{j,i} \text{ for } j < q_i,\ 0 \le \mu_{q_i}, \text{ and } v_{D_i}\left(\prod_{j=1}^{q_i} u_{j,i}^{\mu_j}\right) \ge v_{D_i}\left((s-\alpha_i)^N\right) \right\}$$

for $i = 1, 2, \ldots, r$. By Proposition 2.1, $f(s,t)/(s-\alpha_i)^M \in A$ if and only if $f(s,t)/(s-\alpha_i)^M \in k[I_i]$. By (3.9), by (3.11) and (3.12), by (3.13), by (3.15) and (3.16), and by (3.18), $J_i \subset I_i$. Thus, $k[J] \subset A$.

Let $I = \bigcup_{i=1}^r I_i$. By Proposition 1.1, an element of A has the form

$$\frac{f(s,t)}{\prod_{i=1}^r (s-\alpha_i)^{M_i}}.$$

We have

$$\frac{f(s,t)}{\prod_{i=1}^r (s-\alpha_i)^{M_i}} = \frac{1}{\alpha_r - \alpha_1}\left(\frac{f(s,t)}{(s-\alpha_1)^{M_1-1}\prod_{i=2}^r (s-\alpha_i)^{M_i}} - \frac{f(s,t)}{(s-\alpha_r)^{M_r-1}\prod_{i=1}^{r-1}(s-\alpha_i)^{M_i}}\right)$$

for $r > 1$. Thus, induction on r and on M_1 and M_r shows that $A \subset k[I]$. Let

$$z = \frac{\prod_{j=1}^{q_i} u_{j,i}^{\mu_j}}{(s-\alpha_i)^N}$$

be an element of I_i. By Lemma 3.1, there are integers τ_j with $\tau_0, \tau_{q_i} \geq 0$ such that

$$z = x_i^{\tau_0}\left(\prod_{j \in \Omega_i} v_{j,i}^{\tau_j}\right)$$

where, for each i, Ω_i is given by (3.17). By Proposition 3.1, there are no maximal ideals containing both $s - \alpha_i$ and $v_{j,i}$. Thus, z is regular on Spec $k[J]$ and, hence, $z \in k[J]$. Therefore, $A = k[J]$.

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