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THE STONE-WEIERSTRASS PROPERTY IN QUOTIENT ALGEBRAS, AND SETS OF SPECTRAL RESOLUTION

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1. Introduction

In 1960, Katznelson and Rudin, motivated by the Schwartz counterexample and Malliavin's theorem, extended the notion of the Stone-Weierstrass property to semi-simple commutative Banach algebras [9]. Since the Schwartz counterexample to the spectral synthesis can easily be modified to an example of a strongly separating self-adjoint subalgebra which is not dense in $A(R^3)$, and since Malliavin showed, in 1959, that $A(G)$ is an algebra of synthesis if and only if the LCA group G is discrete [12], [13], [14], Katznelson and Rudin were interested in investigating the Stone-Weierstrass property in $A(G)$. They concluded that $A(G)$ is a Stone-Weierstrass algebra if and only if G is totally disconnected [9], [16, Section 9.3]. Since every discrete group is totally disconnected, we can observe that every algebra $A(G)$ of synthesis is a Stone-Weierstrass algebra, or equivalently, if $A(G)$ does not have the Stone-Weierstrass property, then G contains a non-S-set. The converse is false.

In this paper we investigate the Stone-Weierstrass property in quotient algebras $A(E)$, where E is a closed subset of an LCA group. We define two classes of sets, Stone-Weierstrass sets and idempotent sets, and observe the relation between these sets and sets of spectral resolution. In this case the situations are very different from the case of an LCA group. First, the assumption "E is a set of spectral resolution" does not imply "E is discrete." A perfect Kronecker set in T (cf. [16, p. 99]) is a counterexample. If E is discrete, however, E is a set of spectral resolution (cf. [16, p. 159]). Second, even if $A(E)$ is a Stone-Weierstrass algebra, E may not be totally disconnected. Helson curves in T^n, $n \geq 2$, constructed by Kahane and McGehee serve as counterexamples [7], [15]. Third, even if E is totally disconnected, $A(E)$ may not be a Stone-Weierstrass algebra. Katznelson and Rudin constructed a counterexample [9, p. 257].

Our main results are as follows. First, every closed subset of an idempotent set and a Stone-Weierstrass set is an idempotent set and a Stone-Weierstrass

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set respectively. Note that sets of spectral resolution also share the same property: every closed subset of a set of spectral resolution is a set of spectral resolution. Second, the product of Stone-Weierstrass sets need not be a Stone-Weierstrass set although the product of idempotent sets is always an idempotent set. Compare this property with the result by Varopoulos: the product of S-sets is an S-set [20, p. 58]. Also it can be easily deduced from his theorems that the product of sets of spectral resolution need not be a set of spectral resolution. Third, the disjoint union of Stone-Weierstrass sets is a Stone-Weierstrass set, and the disjoint union of idempotent sets is an idempotent set. Fourth, we prove that, in finite-dimensional metrizable LCA groups, every set of spectral resolution is a Stone-Weierstrass set. Equivalently, if $A(E)$ does not have the Stone-Weierstrass property, then E contains a non-S-set. Fifth, we present, in T, three examples of non-S-sets which are idempotent sets, and therefore, are Stone-Weierstrass sets. Using the Herz scaffolding, these examples are easily modified to examples of proper closed subsets which are Stone-Weierstrass S-sets containing non-S-sets. Thus, we obtain the same conclusion as in the case of LCA groups: if E is a closed subset of a finite-dimensional LCA group, then every algebra $A(E)$ of synthesis is a Stone-Weierstrass algebra, and the converse is false.

As a byproduct, we obtain an upper bound for the Helson constant of a union of the type $H \cup \{H + x\}$. If the Helson constant of a compact Helson set H is α, we conclude that the Helson constant of $H \cup \{H + x\}$ is equal to or less than 2α. Since the best previously known estimate is $3^{3/2}\alpha^3$ due to Varopoulos [22], [23], [24], (cf. [3, p. 48]), and $\alpha \geq 1$ by the definition, hence, our result is better in this special case of the union problem.

In the last section, we discuss the transfer method by Varopoulos. The transfer is valid also for non-Stone-Weierstrass sets if the maximal ideal space is not totally disconnected.

2. Definitions and prerequisite theorems

In this section, we introduce basic definitions, two theorems, and a corollary which are needed in the remaining sections.

G denotes an LCA group. T^n denotes the n-dimensional torus, and Z^n is its dual group. $A(G)$ is the Fourier algebra on G, and $C(G)$ is the space of continuous functions on G. The n-dimensional Euclidian space is denoted by R^n.

Let A be a semi-simple commutative Banach algebra with maximal ideal space Δ. For a closed subset E in Δ,

$$I(E) = \{f \in A : f = 0 \text{ on } E\},$$

and $J(E)$ is the closure of the ideal

$$j(E) = \{f \in A : f = 0 \text{ on some neighborhood of } E\}.$$

If A is regular and $I(E) = J(E)$, E is said to be an S-set. E is called a set of spectral resolution if every closed subset of E is an S-set. $A(E)$ is the quotient algebra $A(G)/I(E)$ with quotient norm. If $A(E) = C(E)$, E is said to be a Helson set. The Helson constant is defined by the supremum of $\{\|f\|_{A(E)}/\|f\|_{C(E)}\}$, where f is a non-zero function in $A(E)$. If every continuous function of absolute value 1 on E can be uniformly approximated on E by characters, E is called a Kronecker set. It is a well-known fact that every Kronecker set is a Helson set. $PM(G)$ denotes the dual space of $A(G)$, the space of pseudomeasures on G. $PM(E)$ is the space of pseudomeasures supported in E, and $N(E)$ is the dual of $A(E)$, where E is a closed subset of G. We call A an algebra of synthesis if Δ does not contain any non-S-set.

For a subalgebra B of A, B is said to be strongly separating if, for any distinct elements x and y in Δ, there exists some f in B such that $f(x) = 1$ and $f(y) = 0$. If the complex conjugate function of f belongs to B whenever f is in B, we call B self-adjoint. A is said to have a Stone-Weierstrass property if every strongly separating self-adjoint subalgebra of A is dense in A with norm employed in A. In this case, A is said to be a Stone-Weierstrass algebra.

The following theorem shows the embedding of $A(K)$ into $C(K) \hat{\otimes} C(K)$, where K is a compact abelian group. This was first discussed by Varopoulos [21], and later, Herz [5] simplified the idea by introducing the maps M and P.

THEOREM 2.1 (VAROPOULOS AND HERZ). *Assume that K is a compact abelian group. Let*

$$M: C(K) \to C(K \times K) \text{ and } P: C(K \times K) \to C(K)$$

be defined by $Mf(x, y) = f(x + y)$, and $Pf(z) = \int_K f(z - x, x)\, dx$ respectively. Then:

(1) *M maps $A(K)$ isometrically into $C(K) \hat{\otimes} C(K)$.*
(2) *P maps $C(K) \hat{\otimes} C(K)$ onto $A(K)$ and the norm of this mapping is one.*

Using the Šilov idempotent theorem (cf. [1, p. 88]), Katznelson and Rudin proved the following.

THEOREM 2.2 (KATZNELSON AND RUDIN [9, P. 257]). *Let B be the closure of a strongly separating self-adjoint subalgebra of A. Then every idempotent in A belongs to B.*

COROLLARY 2.3 (KATZNELSON AND RUDIN [9, P. 257]). *If a semi-simple commutative Banach algebra A is spanned by its set of idempotents, then A has the Stone-Weierstrass property.*

3. The Stone-Weierstrass property in quotient algebras

In this section, we study the Stone-Weierstrass property in quotient algebras $A(E)$, where E is a closed subset of a finite dimensional metrizable LCA group. We call E an idempotent set or an IA-set if $A(E)$ is spanned by its set of idempotents, and a Stone-Weierstrass set or an SW-set if $A(E)$ is a Stone-Weierstrass algebra. The empty set is an IA-set and an SW-set. By Corollary 2.3 above, every idempotent set is a Stone-Weierstrass set. The following theorems, which remind us of sets of spectral resolution, are valid also for infinite dimensional LCA groups.

PROPOSITION 3.1. *Every closed subset of a Stone-Weierstrass set is a Stone-Weierstrass set.*

Proof. Suppose that E is an SW-set and that F is an arbitrary closed subset of E. Let π be the canonical quotient map from $A(E)$ onto $A(F)$. For a strongly separating self-adjoint subalgebra B of $A(F)$, we define a subalgebra H of $A(E)$ by $H = \pi^{-1}(B)$, the inverse image of B in $A(E)$. Then, H is a strongly separating self-adjoint subalgebra of $A(E)$, and therefore, the closure of H is $A(E)$. Thus, B must be dense in $A(F)$. ■

PROPOSITION 3.2. *Every closed subset of an idempotent set is an idempotent set.*

Proof. Suppose that E is an IA-set and that F is an arbitrary closed subset of E. Since E is an IA-set, there exists a set $\{f_\lambda\}_{\lambda\in\Lambda}$ in $A(E)$ such that $(f_\lambda)^2 = f_\lambda$ for all λ, and the linear combinations of elements in $\{f_\lambda\}_{\lambda\in\Lambda}$ span $A(E)$. Then, $\pi(f_\lambda)$ is an idempotent for each λ, where π is the canonical quotient map from $A(E)$ onto $A(F)$, and the linear combinations of $\{\pi(f_\lambda)\}_{\lambda\in\Lambda}$ span $A(F)$. ■

3.1. Idempotent sets and Stone-Weierstrass sets in one-dimensional metrizable LCA groups. The following theorem for T was proved by Kahane [6]. Since R is σ-compact, we can extend this result as follows:

THEOREM 3.3. *If the Haar measure of a closed subset E in R is* 0, *then E is an idempotent set.*

Note. Theorem 3.3 does not hold if R is replaced by R^n, $n \geq 2$. Take, for example, $E = \{(t,0)\colon t \in T\}$.

If E is a closed subset of a one-dimensional metrizable LCA group, by the structure theorem, we may assume that E is a closed subset of either R, T or a discrete group. Therefore, in either case, if the Haar measure of E is zero, we can conclude that E is an idempotent set.

THEOREM 3.4. *If the Haar measure of a closed subset E of a one-dimensional metrizable LCA group is* 0, *then E is an idempotent set.*

The following theorem, which indicates that sets with Haar measure zero are not the only idempotent sets, is derived from Zygmund's result [25, p. 351].

THEOREM 3.5 (ZYGMUND). *There exists a set $E \subseteq R$ such that $m(E) > 0$ and E is an idempotent set.*

3.2. Idempotent sets and Stone-Weierstrass sets in n-dimensional metrizable LCA groups, where $n \geq 2$. The following theorem is a useful tool to find some idempotent sets and thus, Stone-Weierstrass sets. The relation between the tensor norm and the L_1-norm was observed by a number of mathematicians in the fifties in France. We refer the reader, for example, to Schwartz [19, Exposé n°4, p. 4, Théorème 3] and Grothendieck [4, p. 59, Théorème 2]. Since the tensor product structure is preserved by the Fourier transform, it implies that the tensor norm and the A-norm are related as appears in Varopoulos [20, p. 58 and p. 71]:

THEOREM 3.6 (VAROPOULOS). *Let E_j be a closed subset of an LCA group G_j, $j = 1, 2, \ldots, n$. Then $A(E_1 \times \cdots \times E_n)$ is isometrically isomorphic to $A(E_1) \hat{\otimes} \cdots \hat{\otimes} A(E_n)$.*

As a corollary, the product of idempotent sets is an idempotent set:

COROLLARY 3.7. *If E_j is an idempotent set in an LCA group G_j, $j = 1, 2, \ldots, n$, then $E_1 \times \cdots \times E_n$ is an idempotent set.*

Note that, in this corollary, each G_j does not need to be finite-dimensional. Combining this corollary with Proposition 3.2, we obtain the following:

COROLLARY 3.8. *Let G_j, $j = 1, \ldots, n$, be one-dimensional metrizable LCA groups. If E is a closed subset of $G_1 \times \cdots \times G_n$, and $m(\pi_j(E)) = 0$ for $j = 1, \ldots, n$, where $\pi_j(E)$ denotes the projection of E to the j-th coordinate of $G_1 \times \cdots \times G_n$, then E is an idempotent set.*

COROLLARY 3.9. *There exists an idempotent set of positive Haar measure in R^n.*

This is a consequence of Theorem 3.5 and Corollary 3.7.

Now, we list the following two theorems in order to prove that the product of Stone-Weierstrass sets need not be a Stone-Weierstrass set.

THEOREM 3.10 (LUST [11]). *$C(T) \hat{\otimes} C(T)$ does not have the Stone-Weierstrass property.*

THEOREM 3.11 (KAHANE AND MCGEHEE [7], [15]). *For $n \geq 2$, there is a Helson set $E \subset R^n$ such that E is a continuous curve.*

By Theorem 3.11, we can observe that the class of idempotent sets and the class of Stone-Weierstrass sets are not equal in non-discrete metrizable LCA groups which are more than one-dimensional because a Helson curve is a Stone-Weierstrass set that is not an idempotent set. For one-dimensional LCA groups, the question is still open.

Now, we prove:

THEOREM 3.12. *The product of Stone-Weierstrass sets need not be a Stone-Weierstrass set.*

Proof. Let E and F be continuous Helson curves in R^n. Then they are SW-sets, but $E \times F$ is not an SW-set according to Theorem 3.6 and Theorem 3.10. ■

To compare Stone-Weierstrass sets with sets of spectral resolution, we mention the following property of sets of spectral resolution.

THEOREM 3.13. *The product of sets of spectral resolution need not be a set of spectral resolution.*

We use two results by Varopoulos to prove Theorem 3.13.

THEOREM 3.14 (VAROPOULOS [21, PP. 5167–5168]). *Let G be an LCA group, and K and L be compact subsets of G. If $K \cap L = \emptyset$ and $K \cup L$ is a Kronecker set, then $A(K + L) = C(K) \hat{\otimes} C(L)$. If, in addition, $K \cup L$ is totally disconnected, this identification is isometric.*

PROPOSITION 3.15 (VAROPOULOS [20, P. 102, THEOREM 9.2.3]). *Let G be a compact metrizable abelian group, and let K and L be perfect subsets of G. Then $K + L$ contains a closed set of non-synthesis.*

Proof of Theorem 3.13. Let K and L be non-empty closed subsets of T such that $K \cup L$ is a perfect Kronecker set and $K \cap L = \emptyset$. Then, by

Theorem 3.6 and Theorem 3.14, $A(K \times L)$ is isometrically isomorphic to $A(K + L)$. Proposition 3.15 implies that there exists a non-S-set E in $K + L$. Then, it can be verified, using the Herz maps M and P in Theorem 2.1, that the closed set F in $K \times L$ defined by $F = \{(x, y): x + y \in E\}$ is a non-S-set. ■

Finally, we prove the following:

THEOREM 3.16. *Every non-Stone-Weierstrass set E in a finite-dimensional metrizable LCA group G contains a non-S-set.*

Proof. By the structure theorem, and by the fact that every closed subset of a discrete space is an SW-set, we may assume that $G = R^n$. Let $E_j = \{\pi_j(x): x \in E\}$, where π_j is the canonical projection map from R^n on the j-th coordinate of R^n. Then Corollary 3.8 implies that one of E_j, $j = 1, \ldots, n$, must have a positive Haar measure. Assume that the Haar measure of E_k is positive. Then the function $f(x_1, \ldots, x_n)$ on E defined by $f(x_1, \ldots, x_n) = I_{E_k}(x_k)$, where I_{E_k} is the characteristic function of E_k, is a pseudofunction. Therefore, E is a set of multiplicity. Since every set of multiplicity in a non-discrete metrizable LCA group contains a non-S-set according to Saeki [17], E contains a non-S-set. ■

Thus, every set of spectral resolution is a Stone-Weierstrass set.

4. A partial converse of Theorem 3.16

The converse of Theorem 3.16 does not hold as we can observe in some counterexamples in Section 6. However, the following can be said for a regular semi-simple commutative Banach Algebra:

THEOREM 4.1. *If an S-set E in the maximal ideal space Δ of a regular semi-simple commutative Banach algebra A contains a compact non-S-set, then $A/I(E)$ contains a strongly separating subspace (not necessarily a subalgebra) that is not dense in $A/I(E)$.*

Proof. Let E_1 be a compact non-S-subset of E. Then, there exist $S \in I(E)^{\perp}$ and $h \in A/I(E)$ such that $\operatorname{supp} S \subset E_1$, $h = 0$ on E_1, and $\langle h, S \rangle = \alpha > 0$. Let $B = \{f \in A/I(E): \langle f, S \rangle = 0\}$. Now, the following three lemmas complete the proof.

LEMMA 4.2. *For all $x \in E \setminus E_1$, and for all $y \in E$ with $x \neq y$, there exists an $f \in B$ such that $f(x) = 1$ and $f(y) = 0$.*

Proof. Select neighborhoods $U(x)$, $V(y)$, and $W(E_1)$ such that $\overline{U} \cap \overline{V} = \emptyset$, $\overline{U} \cap \overline{W} = \emptyset$. By the regularity of $A/I(E)$, there exists an $f \in A/I(E)$ such that $0 \leq f \leq 1$ and

$$f = \begin{cases} 1 & \text{on some compact neighborhood } H(x) \subset U, \\ 0 & \text{on } U^c. \end{cases}$$

This f has the properties in the lemma. ■

LEMMA 4.3. *For all $x \in E_1$, and for all $y \in E_1$ with $x \neq y$, there exists an element $\varphi \in B$ such that $\varphi(x) = 1$ and $\varphi(y) = 0$.*

Proof. Choose a neighborhood U of x such that $y \notin \overline{U}$. Select an $f \in A/I(E)$ such that $0 \leq f \leq 1$ and

$$f = \begin{cases} 1 & \text{on some compact neighborhood } H(x) \subset U, \\ 0 & \text{on } U^c. \end{cases}$$

If $\langle f, S \rangle = 0$, this f will do the job in the lemma. Otherwise, let $\langle f, S \rangle = \beta \neq 0$ and $\varphi = f - (\beta/\alpha)h$. This φ has the desired properties. ■

LEMMA 4.4. *For all $x \in E_1$, and for all $y \in E \setminus E_1$, there exists an element $\varphi \in B$ such that $\varphi(x) = 1$ and $\varphi(y) = 0$.*

Proof. Choose neighborhoods $V(y)$, $W(E_1)$, compact neighborhoods $K(E_1)$, and $U(x)$ such that $U \subset K \subset W$ and $\overline{V} \cap \overline{W} = \emptyset$. Then, again by the regularity, we may select an $f \in A/I(E)$ such that

$$f = \begin{cases} 1 & \text{on some compact neighborhood } H(x) \subset U, \\ 0 & \text{on } U^c, \end{cases}$$

and $0 \leq f \leq 1$. If $\langle f, S \rangle = 0$, let $f = \varphi$, and we are done. Otherwise let $\langle f, S \rangle = \beta \neq 0$. By regularity, we may choose $g \in A/I(E)$ such that $g = 1$ on K, $g = 0$ on W^c, and $0 \leq g \leq 1$. Let $\varphi = f - (\beta/\alpha)gh$. This φ will do the job. ■

By the above lemmas, B is strongly separating. Also, it can be easily verified that B is a subspace of $A/I(E)$. Since S is a non-zero functional and S annihilates B, B is not dense in $A/I(E)$. This completes the proof of Theorem 4.1. ■

5. Idempotent sets and Stone-Weierstrass sets in infinite-dimensional metrizable LCA groups

Although many theorems we proved in Section 3 remain true for infinite-dimensional groups, we deal with this case separately for the following two reasons: First, our proof of Theorem 3.16 is valid only for finite-dimensional G. The second reason is the following theorem. If K and L are non-empty closed sets, $K \cap L = \emptyset$, and $K \cup L$ is a Kronecker set, then we call $K + L$ a disjoint Kronecker sum.

THEOREM 5.1. *If G is finite-dimensional, a disjoint Kronecker sum is an idempotent set. If G is infinite-dimensional, however, there is a disjoint Kronecker sum which is not a Stone-Weierstrass set.*

Proof. By Theorem 3.14, $A(K+L) = C(K) \hat{\otimes} C(L)$. If G is finite-dimensional, K and L are totally disconnected by Rudin [16, Theorem 5.1.4 and Theorem 5.2.9]. Thus, according to Lust [11], $C(K) \hat{\otimes} C(L)$ is spanned by idempotents. Therefore, $K + L$ is an idempotent set.

In order to prove the second part of the theorem, let $G = T^{\omega}$. Now, define

$$K = \{(2\pi t, 2\pi t^2, \dots)\ \tfrac{1}{4} \leq t \leq \tfrac{1}{3}\}, \text{ and } L = \{(2\pi s, 2\pi s^2, \dots)\ \tfrac{1}{2} \leq s \leq \tfrac{3}{4}\}.$$

Then K and L are Kronecker sets [16, Theorem 5.2.7, p. 103], so that $A(K)$ and $A(L)$ have the Stone-Weierstrass property. However, from Theorem 3.14, we have $A(K+L) \cong C(T) \hat{\otimes} C(T)$, which does not have the Stone-Weierstrass property by Theorem 3.10. ■

Because of the isometry between $A(E_1 \times \cdots \times E_n)$ and $A(E_1) \hat{\otimes} \cdots \hat{\otimes} A(E_n)$, Theorem 3.6 can be naturally extended to the countably infinite product. Thus, we have the following theorems:

THEOREM 5.2. *If E_j is an idempotent set in an LCA group G_j, $j = 1, 2, \dots$, then $E_1 \times E_2 \times \cdots$ is an idempotent set.*

THEOREM 5.3. *Let G_j, $j = 1, \dots,$ be one-dimensional metrizable LCA groups. If E is a closed subset of $G_1 \times G_2 \times \cdots$, and $m(\pi_j(E)) = 0$ for $j = 1, \dots, n$, where $\pi_j(E)$ denotes the projection of E to the j-th coordinate of $G_1 \times G_2 \times \cdots$, then E is an idempotent set.*

THEOREM 5.4. *There exists an idempotent set of positive Haar measure in T^{ω}.*

Theorem 5.4 is a consequence of Theorem 3.5 and Theorem 5.2.

6. Examples of non-S-sets which are Stone-Weierstrass sets

In this section, we present three examples of non-S-sets which are Stone-Weierstrass sets in T. These examples have Haar measure zero, and thus, they are idempotent sets. By the Herz scaffolding, these sets can be modified to Stone-Weierstrass S-sets which contain non-S-sets.

Example 6.1. By Körner [10], there exists a Helson set which is not an S-set. This set is an SW-set.

Example 6.2. Let X and Y be perfect disjoint subsets of T such that $X \cup Y$ is a Kronecker set. Then $X + Y$ is an SW-set by Theorem 5.1. From Proposition 3.15, $X + Y$ contains a closed set E that is not an S-set. By Proposition 3.1, this E is an SW-set.

In order to present the third example, we employ the following set.

DEFINITION 6.3. Let $\{t_j\}_{j=1}^{\infty}$ be a decreasing sequence of positive real numbers such that

$$\sum \left(\frac{t_{j+1}}{t_j} \right)^2 < \infty$$

and $t_j > \sum_{k=j+1}^{\infty} t_k$ for all j. Let $E = \{\sum_{j=1}^{\infty} \varepsilon_j t_j\colon \varepsilon_j = 0 \text{ or } 1 \text{ for } 1 \leq j < \infty\}$. Then E is called an ultrathin symmetric set (cf. [18]).

Example 6.4. Let E be an ultrathin symmetric set. Then, it contains a subset F that is not an S-set [8, Theorem 7]. This F has a measure zero, and thus, by Theorem 3.3 and Corollary 2.3, F is an SW-set.

7. The union problem for Stone-Weierstrass sets and idempotent sets

We prove, in this section, that the disjoint union of Stone-Weierstrass sets is a Stone-Weierstrass set and that the disjoint union of idempotent sets is an idempotent set. For the non-disjoint case, the question is still open.

THEOREM 7.1. *Assume that E_1 and E_2 are disjoint compact Stone-Weierstrass sets. Then, $E_1 \cup E_2$ is a Stone-Weierstrass set.*

Proof. Let f be an arbitrary element in $A(E_1 \cup E_2)$ and ε be any positive number. Assume that B is the closure of a strongly separating self-adjoint subalgebra of $A(E_1 \cup E_2)$. By the assumptions on B, it follows that the restriction algebras $B_1 = \{f|_{E_1}\colon f \in B\}$ and $B_2 = \{f|_{E_2}\colon f \in B\}$ are strongly separating self-adjoint subalgebras of $A(E_1)$ and $A(E_2)$ respectively.

Since E_1 and E_2 are SW-sets, there exist h_1 and h_2 in B such that $\|f|_{E_1} - h_1|_{E_1}\|_{A(E_1)} \leq \varepsilon$, and $\|f|_{E_2} - h_2|_{E_2}\|_{A(E_2)} \leq \varepsilon$. Since E_1 and E_2 are compact, and $E_1 \cap E_2 = \emptyset$, therefore B contains functions g and k such that $g = 1$ on E_1, $g = 0$ on E_2, $k = 0$ on E_1, and $k = 1$ on E_2 according to Theorem 2.2 and the Šilov Idempotent Theorem (cf. Gamelin [1, p. 88]). Since B is an algebra, we have $h_1 g + h_2 k \in B$, and

$$\|f - (h_1 g + h_2 k)\| \leq \|fg - h_1 g\| + \|fk - h_2 k\| \leq 2\varepsilon,$$

where the norm is taken in $A(E_1 \cup E_2)$. Thus, B is equal to $A(E_1 \cup E_2)$. ■

THEOREM 7.2. *If E_1 and E_2 are disjoint compact idempotent sets, $E_1 \cup E_2$ is an idempotent set.*

Proof. Let $G_\lambda \in A(G)$ such that $G_\lambda|_{E_1}$ is an idempotent in $A(E_1)$, and the family $\{G_\lambda|_{E_1}\}_\lambda$ of all such idempotents spans $A(E_1)$. Similarly, let $H_\mu \in A(G)$ such that $H_\mu|_{E_2}$ is an idempotent in $A(E_2)$, and the family $\{H_\mu|_{E_2}\}_\mu$ spans $A(E_2)$. By the regularity of the Fourier algebras [2, p. 123], there exists an $I_1 \in A(G)$ such that $I_1 = 1$ on E_1, and $I_1 = 0$ on E_2. Similarly, there exists an $I_2 \in A(G)$ such that $I_2 = 0$ on E_1, and $I_2 = 1$ on E_2. Then, $G_\lambda I_1|_{E_1 \cup E_2}$ and $H_\mu I_2|_{E_1 \cup E_2}$ are idempotents, and

$$\left\{G_\lambda I_1|_{E_1 \cup E_2}, H_\mu I_2|_{E_1 \cup E_2}\right\}_{\lambda, \mu}$$

generates $A(E_1 \cup E_2)$. ■

8. Applications

In this section, we prove that the Helson constant for $H \cup \{H + x\}$, where H is a compact Helson set with Helson constant α, is less than or equal to 2α and that $C(K + L) = C(K \times L)$ where K and L are disjoint, and $K \cup L$ is a compact Kronecker set.

LEMMA 8.1. *Let X and Y be Helson sets. If the Helson constants for X and Y are α and β respectively, we have*

$$\|f\|_{C(X)\hat{\otimes}C(Y)} \leq \|f\|_{A(X\times Y)} \leq \alpha\beta\|f\|_{C(X)\hat{\otimes}C(Y)}$$

for all $f \in A(X \times Y) = C(X) \hat{\otimes} C(Y)$. In particular, if the Helson constants for X and Y are 1, the two norms are equal.

Proof. For all $f \in C(X) \hat{\otimes} C(Y)$ and for all $\varepsilon > 0$, there exist $f_j \in C(Y)$ and $g_j \in C(Y)$ such that

$$f = \sum_{j=1}^{\infty} f_j \otimes g_j \text{ and } \|f\|_{C(X)\hat{\otimes}C(Y)} + \varepsilon/\alpha\beta \geq \sum_{j=1}^{\infty} \|f_j\|_{C(X)} \|g_j\|_{C(Y)}.$$

Thus

$$\alpha\beta \|f\|_{C(X)\hat{\otimes}C(Y)} \geq \sum_{j=1}^{\infty} \|f_j\|_{A(X)} \|g_j\|_{A(Y)} - \varepsilon$$

$$\geq \sum_{j=1}^{\infty} \|f_j \otimes g_j\|_{A(X\times Y)} - \varepsilon \geq \|f\|_{A(X\times Y)} - \varepsilon.$$

Since ε is arbitrary, we have $f \in A(X \times Y)$ and $\alpha\beta \|f\|_{C(X)\hat{\otimes}C(Y)} \geq \|f\|_{A(X\times Y)}$ for all f. ■

Since $C(X \times Y) = C(X) \hat{\otimes} C(Y)$ if and only if X or Y is a finite set, we can observe the following together with Theorem 3.6:

LEMMA 8.2. *Let X and Y be Helson sets in G. Then $X \times Y$ is a Helson set if and only if either X or Y is a finite set.*

LEMMA 8.3. *If Y consists of n points, we have*

$$\|f\|_{C(X\times Y)} \leq \|f\|_{C(X)\hat{\otimes}C(Y)} \leq n \|f\|_{C(X\times Y)}$$

for all $f \in C(X) \hat{\otimes} C(Y) = C(X \times Y)$.

Proof. The first inequality follows from the definitions of the norms. For the second inequality, let $Y = \{y_1, \ldots, y_n\}$ and $f \in C(X \times Y)$. Now, for each fixed y_j, write $f_j(x) = f(x, y_j)$, $j = 1, 2, \ldots, n$. Therefore, the function f can be expressed as $f(x, y) = f_1(x) I_{\{y_1\}}(y) + \cdots + f_n(x) I_{\{y_n\}}(y)$, where I_E denotes the characteristic function of E. Thus, $\|f\|_{C(X\times Y)} = \max_{j=1,\ldots,n}\{\|f_j\|\}$. We can complete the proof by observing that $\|f\|_{C(X)\hat{\otimes}C(Y)} \leq \|f_1\| + \cdots + \|f_n\| \leq n \|f\|_{C(X\times Y)}$. ■

THEOREM 8.4. *Let X and Y be compact Helson sets in an LCA group G. Suppose that $A(X \times Y) = C(X \times Y)$. Thus, by Lemma 8.2, we may assume that Y contains only n points. Then $A(X + Y) = C(X + Y)$, and $\alpha(X + Y) \leq \alpha(X \times Y) \leq n\alpha(X)\alpha(Y)$, where $\alpha(E)$ denotes the Helson constant for E.*

Proof. Assume that $f \in A(X \times Y)$. By Lemmas 8.1, 8.2, and 8.3, we have

$$\|f\|_{A(X\times Y)} \le \alpha(X)\alpha(Y)\|f\|_{C(X)\hat{\otimes}C(Y)} \le n\alpha(X)\alpha(Y)\|f\|_{C(X\times Y)}.$$

Therefore, the Helson constant for $X \times Y$ is less than or equal to $n\alpha(X)\alpha(Y)$.

Now, let M be the Herz map described in Theorem 2.1. Since $A(X+Y) \subseteq C(X+Y)$, we have

$$M(A(X+Y)) \subseteq M(C(X+Y)).$$

Since $M(A(X+Y))$ is a strongly separating self-adjoint subalgebra of $M(C(X+Y))$, and the $A(X \times Y)$- and $C(X \times Y)$-norms are equivalent, it follows from the Stone-Weierstrass property that

$$\overline{M(A(X+Y))}^{A(X\times Y)} = \overline{M(C(X+Y))}^{C(X\times Y)}.$$

Since $M(A(X+Y))$ is closed in $A(X \times Y)$, and $M(C(X+Y))$ is closed in $C(X \times Y)$, we have

$$M(A(X+Y)) = M(C(X+Y)).$$

Since M is one-to-one, we obtain $A(X+Y) = C(X+Y)$. The Helson constant for $X+Y$ is less than or equal to that of $X \times Y$ because the map M preserves the A-norm and the C-norm. ■

COROLLARY 8.5. *The union of a compact Helson set X with Helson constant α and a translate of X is a Helson set with Helson constant less than or equal to 2α.*

Proof. Let $Y = \{0, y\}$, $y \neq 0$, $y \in G$ in Theorem 8.4. Then we have $\alpha(X+Y) \le \alpha(X \times Y) \le 2\alpha$. ■

Thus, we obtain a new proof for a special case of the union problem for Helson sets.

If we take Y to be an independent set having n elements $\{y_j\}_{j=1}^n$, we have $\beta = 1$, and hence, by Theorem 8.4, $\alpha(X+Y) = \alpha(\{X+y_1\} \cup \cdots \cup \{X+y_n\}) \le \alpha(X \times Y) \le n\alpha$.

PROPOSITION 8.6. *If X and Y are compact disjoint subsets of G and $X \cup Y$ is a Kronecker set, then $C(X \times Y)$ is isomorphic to $C(X+Y)$.*

Proof. From Theorems 3.6 and 3.14, we have $A(X \times Y) = A(X+Y)$. Thus, we obtain $C(X \times Y) = C(X+Y)$ by taking the supremum norm. ■

9. Non-Stone-Weierstrass sets and the transfer method by Varopoulos

Unlike non-S-sets, non-Stone-Weierstrass sets for $A(G)$, where G is not totally disconnected, cannot be transferred to non-Stone-Weierstrass sets for $A(D)$ or $C(D) \hat{\otimes} C(D)$, where D is the Cantor group. This is because D is totally disconnected, and therefore, every closed subset of their maximal ideal spaces is a Stone-Weierstrass set. However, non-Stone-Weierstrass sets for $A(G)$ are transferred to non-Stone-Weierstrass sets for $C(G) \hat{\otimes} C(G)$, since G is not totally disconnected.

THEOREM 9.1. *If E is a non-Stone-Weierstrass set for $A(G)$, then the closed set F in $G \times G$ defined by*

$$F = \{(x, y): x + y \in E\}$$

is a non-Stone-Weierstrass set for $C(G) \hat{\otimes} C(G)$.

Note. If $I = \{f \in C(T) \hat{\otimes} C(T): f(x, y) = 0, (x, y) \in T \times E\}$ where E is a finite set, then

$$C(T) \hat{\otimes} C(T)/I = C(T) \hat{\otimes} C(E) = C(T \times E).$$

Thus, the quotient has the Stone-Weierstrass property. Therefore, the hypotheses in the above theorem have important roles.

Proof. Let B_1 be a proper, closed, strongly separating self-adjoint subalgebra of $A(E)$, and I be the ideal in $C(G) \hat{\otimes} C(G)$ defined by $I = \{f \in C(G) \hat{\otimes} C(G): f = 0 \text{ on } F\}$. Let B be an algebra in $C(G) \hat{\otimes} C(G)/I$ spanned by $Mf(x, y)\gamma(y) + I$, where $[f] \in B_1$ and $\gamma \in \hat{G}$. Then, B is strongly separating in $C(G) \hat{\otimes} C(G)/I$, and B is self-adjoint because B_1 is self-adjoint. B is not dense in $C(G) \hat{\otimes} C(G)/I$ because $P(C(G) \hat{\otimes} C(G)/I) = A(E)$, $P(B) = B_1$, and B_1 is not dense in $A(E)$. Here, M and P are the Herz maps as in Theorem 2.1. Thus, $C(G) \hat{\otimes} C(G)/I$ does not have the Stone-Weierstrass property. ∎

The converse of Theorem 9.1 also holds. Thus, we state as follows:

THEOREM 9.2. *For a closed subset E in G, define the closed subset F as in Theorem 9.1. Then, E is a Stone-Weierstrass set for $A(G)$ if and only if F is a Stone-Weierstrass set for $C(G) \hat{\otimes} C(G)$.*

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