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CONVERGENCE OF ERGODIC AVERAGES ON LATTICE RANDOM WALKS

J. BOURGAIN[1]

1. Introduction

This note is concerned with a question posed by H. Furstenberg and communicated to the author by Y. Katznelson.[2] For $d \geq 1$, consider the standard random walk on the lattice $\mathbf{Z}^d$, i.e., if $n(j) \in \mathbf{Z}^d$ in the j^{th} position, then $n(j+1)$ takes one of the values

$$n(j) \pm e_1, n(j) \pm e_2, \dots, n(j) \pm e_d$$

with equal probability. Here e_j stands for the j^{th} unit vector. We consider ergodic averages along the sequence $\{n(j)\}$. Thus take some probability space (Ω, μ) and d commuting, invertible measure preserving transformations $T_1, T_2, \dots, T_d$ of Ω. Given a measurable function f on Ω, define

$$A_k f = \frac{1}{k} \sum_{j=1}^{k} T_1^{n(j,1)} T_2^{n(j,2)} \dots T_d^{n(j,d)} f \tag{1.1}$$

where $n(j) = (n(j,1), \dots, n(j,d))$. We are interested in the convergence properties of the averages (1.1). In this respect, we will prove the following

THEOREM. *Almost any random walk $\{n(j)\}$ on $\mathbf{Z}^d$ has the property that given any system $(\Omega, \mu, T_1, \dots, T_d)$ of commuting transformations and $f \in L^p(\Omega, \mu)$, $p > 1$, the averages $A_k f$ given by* (1.1) *converge almost surely, along any sequence $\{k_s\}$ satisfying $k_{s+1} > k_s \log\log k_s$. In particular, there is convergence for the logarithmic averages. If moreover, one of the transformations is ergodic, the limit is $\int_\Omega f \, d\mu$..*

A comment on the restrictions made in the statement: The subsequence extraction is needed, even for $d = 1$, and a generic random walk $\{n(j)\}$ is not

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[1]IHES, Bures/Yvette; University of Illinois.
[2]Stanford University, Summer 1990.

universally summing. Also, the assumption $f \in L^1(\mu)$ does not suffice, no matter how lacunary the subsequence $\{k_s\}$ is taken. The nature of the subsequence condition relates to the law of the iterated logarithm and is thus not surprising.

Observe that for $d \geq 3$, the sequence $\{n(j)\} \subset \mathbf{Z}^d$ has zero density. The method of proof of the theorem is along the lines of [Bo 1, 2]. Thus we consider a reduction to the shift on $\mathbf{Z}^d$ (after localizing the problem) and establish then maximal and maximal variational inequalities by Fourier Analysis techniques. This approach (inspired from related problems in real variable analysis) is fairly general and flexible.

The purpose of the next section is twofold. First we recall the general scheme of the argument. Details on this matter appear in [Bo 1, 2] and particularly [Bo 1], Section 9, for the multidimensional setting. Then we state a general condition on the Fourier transforms of a sequence of convolution kernels in order to satisfy an L^p-maximal inequality. The result needed for application here is essentially speaking the most simple one (simpler than [Bo 2]). The fact that the exponential sums associated to a genuine random walk satisfy these conditions will be shown in the final section. The author is grateful to Y. Katznelson for discussions on the subject. The reader may also wish to consult [Bo 3] for related problems.

2. Preliminaries

Consider the k^{th} average

$$A_k f = \frac{1}{k} \sum_{j=1}^{k} T_1^{n(j,1)} \dots T_d^{n(j,d)} f. \tag{2.1}$$

Let $\Lambda \subset \mathbf{Z}_+$ be a set of positive integers. The first goal is to control the maximal operator

$$\mathscr{M} f = \sup_{k \in \Lambda} |A_k f| \quad \text{(pointwise)} \tag{2.2}$$

by an inequality of the form

$$|\mathscr{M} f|_p \leq c_p \|f\|_p \tag{2.3}$$

for any $p > 1$. In studying almost everywhere convergence, we are then reduced to the case of bounded measurable functions, i.e., $f \in L^\infty(\mu)$. Analogously to (2.3), we prove a maximal variational inequality of the form

$$\sum_{1 \le s \le t} \left\| \sup_{\substack{k_s \le k \le k_{s+1} \\ k \in \Lambda}} |A_k f - A_{k_s} f| \right\|_2 \le \theta(t) \cdot t \cdot \|f\|_\infty \tag{2.4}$$

provided $k_1, k_2, \dots, k_t$ increase rapidly enough. Here $\theta(t) \to 0$ for $t \to \infty$ and the statement is uniform.

Since both (2.3), (2.4) have a local character, the general transference reasoning from [Bo 1, 2] permits to replace the commuting system $(\Omega, \mu, T_1, \dots, T_d)$ by the system $(\mathbf{Z}^d; S_1, \dots, S_d)$ where $S_i x = (x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_d)$ is the i^{th} coordinate shift. Inequality (2.3) for instance may then be deduced from the statement

$$\|\mathscr{M}\phi\|_{l^p(\mathbf{Z}^d)} \le c_p \|\phi\|_{l^p(\mathbf{Z}^d)} \tag{2.5}$$

in the shift model.

One may alternatively replace the integer group $\mathbf{Z}$ by a large cyclic group $\mathbf{Z}_N = \mathbf{Z}/N\mathbf{Z}$ in order to preserve finite measure. The reader is referred to [Bo 1, 2] for more details on these matters.

The advantage of the shift model is that $A_k \phi$ is expressed by a convolution on $\mathbf{Z}^d$, *nl*

$$A_k \phi = \phi * \left[\frac{1}{k} \sum_{j=1}^{k} \delta_{n(j)} \right] \tag{2.6}$$

where δ_z stands for the Dirac measure at $z \in \mathbf{Z}^d$. The Fourier transform of the k^{th} kernel

$$K_k = \frac{1}{k} \sum_{j=1}^{k} \delta_{n(j)} \tag{2.7}$$

is a function on the d-torus Π^d, defined by

$$\hat{K}_k(\alpha) = \frac{1}{k} \sum_{j=1}^{k} e^{2\pi i \langle n(j), \alpha \rangle} \tag{2.8}$$

As in [Bo 1, 2], it is our purpose to derive (2.4), (2.5) from properties of the sequence of the Fourier multipliers $\hat{K}_k(\alpha)$, since by the Fourier inversion formula

$$(A_k\phi)(x) = \int_{\mathbf{Z}^d} \hat{K}_k(\alpha)\hat{\phi}(\alpha)e^{2\pi i\langle x,\alpha\rangle}\,d\alpha \tag{2.9}$$

holds. In proving maximal and maximal variational inequalities from Fourier transform considerations, the discrete character of the original problem will play little role in fact.

We will use the following result.

Proposition 1. *Let $\{L_r | r = 1, 2, \dots\}$ be a sequence in $l^1_+(\mathbf{Z}^d)$ verifying the following conditions for some sequence $\{N_r\}$, $N_{r+1} > 2N_r$.*

$$\int_{\mathbf{Z}^d} L_r(x)\,dx = 1, \tag{2.10}$$

$$\left|\hat{L}_r(\alpha)\right| \le (N_r\|\alpha\|)^{-c_1}, \tag{2.11}$$

$$\left|1 - \hat{L}_r(\alpha)\right| \lesssim (N_{r+1}\|\alpha\|)^{c_1}, \tag{2.12}$$

where (2.11), (2.12) *are valid for all $\alpha \in \mathbf{T}^d$, and $c_1 > 0$ is some constant. Then the sequence $\mathscr{A}_r\phi \equiv \phi * L_r$ satisfies the maximal inequality* (2.3) *for $p > 1$ and a uniform maximal variational inequality* (2.4).

Proof of Proposition 1. Let τ be a smooth bumpfunction on $\mathbf{R}^d$, $\int\tau = 1$ and define the following function on $\mathbf{Z}^d$:

$$\mathscr{X}_r(x) = N_r^{-d}\tau(N_r^{-1}x) \tag{2.13}$$

The convolution operators associated to the $\{\mathscr{X}_r\}$ of course satisfy the maximal and maximal variational inequalities (they act as conditional expectations with respect to a partitioning of $\mathbf{Z}^d$ in cubes of size N_r). Denote $\|\ \|_p$ the $l^p(\mathbf{Z}^d)$-norm.

We use the $\{\mathscr{X}_r\}$ sequence in the standard comparison argument based on a square function estimate. Thus write

$$L_r = \mathscr{X}_r + (L_r - \mathscr{X}_r) \tag{2.14}$$

$$\left\|\max_r\right\| L_r * \phi) - (\mathscr{X}_r * \phi)\| \,\|_2 \le \left\|\left(\sum_r |(L_r - \mathscr{X}_r) * \phi|^2\right)^{1/2}\right\|_2$$

$$\le \left\|\left(\sum |\hat{L}_r - \hat{\mathscr{X}}_r|^2\right)^{1/2}\right\|_\infty \|\phi\|_2 \tag{2.15}$$

by Parseval's identity. Now $\hat{L}_r$ and $\hat{\mathscr{X}}_r$ have a similar shape, in the sense that $|\hat{\mathscr{X}}_r(\alpha)| \lesssim (N_r\|\alpha\|)^{-1}$, $|1 - \hat{\mathscr{X}}_r(\alpha)| \lesssim N_{r+1}\|\alpha\|$. Hence, from (2.11), (2.12)

$$\left|\hat{L}_r(\alpha) - \hat{\mathscr{X}}_r(\alpha)\right| \lesssim \min\left(\left(N_r\|\alpha\|\right)^{-c_1}, \left(N_{r+1}\|\alpha\|\right)^{c_1}\right) \tag{2.16}$$

implying in particular a bound on $(\sum |\hat{L}_r - \hat{\mathscr{X}}_r|^2)^{1/2}$. Thus

$$(2.15) \leq c\|\phi\|_2 \tag{2.17}$$

which from the preceding establishes the maximal and maximal variational inequality for $l^2(\mathbf{Z}^d)$. It remains to prove the l^p-maximal inequality if $1 < p < 2$.

We recall that if

$$\phi = \sum_{r=0}^{\infty} \phi_r, \tag{2.18}$$

$$\operatorname{supp} \hat{\phi}_r \subset B\left(0, N_r^{-1}\right) \setminus B\left(0, N_{r+1}^{-1}\right), \tag{2.19}$$

is a Littlewood-Paley decomposition of ϕ(cf[St]), then for $1 < p < \infty$,

$$\|\phi\|_p \approx \left\|\left(\sum |\phi_r|^2\right)^{1/2}\right\|_p \tag{2.20}$$

(the discrete result may be easily derived from the corresponding theorem in $\mathbf{R}^d$). Write

$$\phi * (L_r - \mathscr{X}_r) = \sum_{s=0}^{r} \phi_{r-s} * (L_r - \mathscr{X}_r) + \sum_{s>0} \phi_{r+s} * (L_r - \mathscr{X}_r), \tag{2.21}$$

$$\sup_r |\phi * (L_r - \mathscr{X}_r)| \leq \sum_{s \geq 0} \left\{ \sup_{r \geq s} |\phi_{r-s} * (L_r - \mathscr{X}_r)| + \sup_{r \geq 0} \left|\phi_{r+s} * (L_r - \mathscr{X}_r)\right|\right\}. \tag{2.22}$$

Define M_p as an (a priori) maximal function bound in the inequality

$$\left\|\sup_r |\phi * L_r|\right\|_p \leq M_p \|\phi\|_p. \tag{2.23}$$

Standard considerations permit to assume M_p finite (by restriction to large interval) and it is our purpose to get a uniform bound on M_p from a priori inequalities.

Define similarly $A_{p,s}$(rep $B_{p,s}$), $s \geq 0$, the best constants in the inequalities

$$\left| \sup_{r\geq s} \left|\phi_{r-s} * (L_r - \mathscr{X}_r)\right| \right\|_p \leq A_{p,s}\|\phi\|_p, \tag{2.24}$$

$$\left| \sup_{r\geq 0} \left|\phi_{r+s} * (L_r - \mathscr{X}_r)\right| \right\|_p \leq B_{p,s}\|\phi\|_p. \tag{2.25}$$

Since $\|\sup_r|\phi * \mathscr{X}_r|\ \|_p \leq C_p\|\phi\|_p$, (2.22) implies

$$M_p \leq C_p + \sum_{s\geq 0} (A_{p,s} + B_{p,s}). \tag{2.26}$$

Consider $1 < p_1 < p$, define

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{2}.$$

It follows from (2.16), (2.19) that

$$\|\phi_{r\pm s} * (L_r - \mathscr{X}_r)\|_2 \leq 2^{-c_1 s}\|\phi_{r\pm s}\|_2 \text{ for } s \geq 0 \tag{2.27}$$

from Parseval. Thus

$$\begin{aligned} \left\|\left| \sup_{r\geq s} \left|\phi_{r-s} * (L_r - \mathscr{X}_r)\right| \right\|_2 &\leq \left(\sum_{r\geq s} \|\phi_{r-s} * (L_r - \mathscr{X}_r)\|_2^2 \right)^{1/2} \\ &\leq 2^{-c_1 s}\left(\sum_{r\geq s} \|\phi_{r-s}\|_2^2 \right)^{1/2} \end{aligned} \tag{2.28}$$

implying that

$$A_{2,s} \lesssim 2^{-c_1 s}, \tag{2.29}$$

and similarly

$$B_{2,s} \lesssim 2^{-c_1 s}. \tag{2.30}$$

By interpolation, one may write

$$A_{p,s} \leq 2^{-c_1 s\theta}A_{p_1,s}^{1-\theta};\ B_{p,s} \leq 2^{-c_1 s\theta}B_{p_1,s}^{1-\theta}. \tag{2.31}$$

There is also the following estimate on (2.24):

$$\left\| \sup_{r \geq s} |\phi_{r-s} * (L_r - \mathscr{K}_r)| \right\|_p \leq \left\| \left(\sum_{r \geq s} |\phi_{r-s} * L_r|^2 \right)^{1/2} \right\|_p + \left\| \left(\sum_{r \geq s} |\phi_{r-s} * \mathscr{K}_r|^2 \right)^{1/2} \right\|_p \tag{2.32}$$

The second term in (2.32) is bounded by $\|(\sum_{r \geq s}(\phi_{r-s}|^2)^{1/2}\|_p$ (appealing to an inequality of E. Stein; cf. [St]), hence by $\|\phi\|_p$.

Estimate by (2.20) and duality the first term in (2.32) by

$$\left\| \left(\sum_{r \geq s} |\psi_r * L_r|^2 \right)^{1/2} \right\|_{p'} \tag{2.33}$$

where the sequence $\{\psi_r\}$ satisfies

$$\left\| \left(\sum |\psi_r|^2 \right)^{1/2} \right\|_{p'} \leq 1 \tag{2.34}$$

Since $L_r \in l^1(\mathbf{Z})$ is positive of weight 1, one has

$$|\psi_r * L_r|^2 \leq |\psi_r|^2 * L_r \tag{2.35}$$

and thus

$$(2.33) \leq \left\| \sum_{r \geq s} \left(|\psi_r|^2 * L_r \right) \right\|_{p'/2}^{1/2} \tag{2.36}$$

Again by duality and (2.34), clearly

$$(2.36) \leq \sqrt{M_{(p'/2)'}}; \tag{2.37}$$

hence from the preceding,

$$A_{p,s} \leq c_p \left(1 + \sqrt{M_{(p'/2)'}} \right) \tag{2.38}$$

The same inequality is valid for $B_{p,s}$.

Applying these last estimates replacing p by p_1 and in conjunction with (2.31), (2.26) one finds that

$$M_p \leq c_{p_1}\left\{1 + \left(\sum_{s\geq 0} 2^{-c_1\theta s}\right)\left[\sqrt{M_{(p_1'/2)'}}\right]^{(1-\theta)/2}\right\} < c_{p,p_1}\left[M_{p_1/(2-p_1)}\right]^{1/2} \quad (2.39)$$

Since $p > 1$, p_1 may be chosen to satisfy $p = p_1/(2 - p_1)$. So (2.39) implies a bound on M_p in (2.23). This proves the proposition.

In the next section, we will show that for a random walk $\{n(j)\}$ the Fourier transforms (2.8) fulfill almost surely estimates of the form

$$\left|\hat{K}_k(\alpha)\right| < c\sqrt{\log\log k}\,(\sqrt{k}\,\|\alpha\|)^{-1/2}, \quad (2.40)$$

$$\left|1 - \hat{K}_k(\alpha)\right| < c\sqrt{\log\log k}\,\sqrt{k}\,\|\alpha\| \quad (2.41)$$

for k taken in an asymptotically dense sequence Λ_1. Of course, one has $K_{k'} \leq 2K_k$ for $k/2 < k' < k$ and also

$$|A_k f - A_{k'} f| < 2\frac{|k-k'|}{k}\|f\|_\infty.$$

From these properties, it clearly suffices to consider sequences $\{k_r\}$ contained in Λ_1, and satisfying the condition of the theorem

$$k_{r+1} > k_r \log\log k_r. \quad (2.42)$$

Define $L_r = K_{k_{3r}}$ (similarly $L_r = K_{k_{3r+1,3r+2}}$) to which Proposition 1 may be applied with $c_1 = \frac{1}{2}$, $N_r = \sqrt{k_{3r}}/\log\log k_{3r}$, taking (2.40), (2.41) into account.

Given the commuting system $T_1, \ldots, T_d$ on (Ω, μ) and $f \in L^2(\mu)$, define the spectral measure $\gamma = \gamma_f$ on Π^d by

$$\hat{\gamma}(n_1, n_2, \ldots, n_d) = \langle T_1^{n_1}, \ldots, T_d^{n_d} f, f\rangle \quad (2.43)$$

If one of the transformations is ergodic, then $0 \in \Pi^d$ is not an atom of γ_f, if we assume $\int f\,d\mu = 0$. There is the identity

$$\|A_k f\|_{L^2(\mu)} = \|\hat{K}_k\|_{L^2(\Pi^d, \gamma_f)} \quad (2.44)$$

From spectral theory. Here A_k (resp. $\hat{K}_k$) is given by (1.1) (resp. 2.8). Because of (2.40), expressing the fact taht $\hat{K}_k$ "lives" on smaller and smaller neighborhoods of 0 for $k \to \infty$, it follows from the preceding that

$$\|A_k f\|_2 \to 0 \text{ if } \int f\, d\mu = 0.$$

Thus, if one of the transformations $T_1, \ldots, T_d$ is ergodic, $A_k f \to \int f$ in measure. This justifies the second statement made in the theorem.

3. Completion of the Proof of the Theorem

It remains to check the estimates (2.13), (2.14) on $\hat{K}_k$ for a generic (in measure theoretic sense) random walk $\{n(j)\}$. The underlying measure space is of course the product

$$= \prod_{j=1}^{\infty} \{1, -1, 2, -2, \ldots, d, -d\} \tag{3.1}$$

with normalized product measure.

Consider the random walk up to time k. The main estimate is contained in following

Proposition 2. *Let $B > 1$. Then for a set of complementary measure $< e^{-cB^2}$, the polynomial* (2.8), *thus*

$$\hat{K}_k(\alpha) = \frac{1}{k} \sum_{j=1}^{k} e^{2\pi i \langle n(j), \alpha \rangle},$$

satisfies the inequalities

$$\left|\hat{K}_k(\alpha)\right| < B\left(\sqrt{k}\,\|\alpha\|\right)^{-1/2}, \tag{3.2}$$

$$\left|1 - \hat{K}_k(\alpha)\right| < B\sqrt{k}\,\|\alpha\| \tag{3.3}$$

for all $\alpha \in \Pi^d$.

Deducing from this last statement the validity of (2.40), (2.41) on an asymptotically dense sequence is then immediate. In verifying the proposition, the dimension d will play no role. We assume the reader is familiar with basic probability theory of independent random variables.

Proof of Proposition 2. We identify the product space $\prod_{j=1}^{k}\{1, -1, \ldots, d, -d\}$ with the product $\{1, -1\}^k \times \prod_{j=1}^{k}\{1, 2, \ldots, d\}$ and let $\{\varepsilon_j | j = 1, \ldots, k\}$ be independent sign choices, $\{\xi_j | j = 1, \ldots, k\}$ independent selections in the set $\{1, 2, \ldots, d\}$. Rewrite $\hat{K}_k(\alpha)$ as follows

$$\hat{K}_k(\alpha) = \frac{1}{k}\{1 + e^{2\pi i\langle n(1), \alpha\rangle} + \cdots + e^{2\pi i\langle n(k-1), \alpha\rangle}\} \frac{\sum_{s=1}^{d} \cos 2\pi i \alpha_s}{d} + D_k(\varepsilon_1, \ldots, \varepsilon_k, \omega_1, \ldots, \omega_k, \alpha) \tag{3.4}$$

where

$$D_k = \frac{1}{k}\sum_{j=1}^{k} e^{2\pi i\langle n(j-1), \alpha\rangle}\left[e^{2\pi i \varepsilon_j \alpha_{\xi_j}} - \int e^{2\pi i \varepsilon \alpha_{\xi(\omega)}}\, d\varepsilon\, d\omega\right] \tag{3.5}$$

and

$$n(0) = 0, \quad n(j) = n(j-1) + \varepsilon_j e_{\xi_j}(\omega)$$

corresponds to $(\varepsilon_1, \ldots, \varepsilon_k; \omega_1, \ldots, \omega_k)$.

The first term in (3.4) equals

$$\left[\frac{1}{k}(1 - e^{2\pi i\langle n(k), \alpha\rangle}) + \hat{K}_{k(\alpha)}\right] \frac{\sum_{s=1}^{d} \cos 2\pi i \alpha_s}{d}.$$

Hence, one gets the bound

$$|\hat{K}_k(\alpha)| < c\frac{\frac{1}{k} + |D_k|}{1 - \frac{\sum \cos 2\pi i \alpha_s}{d}} \sim \frac{\frac{1}{k} + |D_k|}{\|\alpha\|^2}. \tag{3.6}$$

Also there is the obvious estimate

$$|1 - \hat{K}_k(\alpha)| \leq \frac{1}{k}\sum_{j=1}^{k} |1 - e^{2\pi i\langle n(j), \alpha\rangle}| < \left(\frac{c}{k}\sum_{j=1}^{k} |n(j)|\right)\|\alpha\|. \tag{3.7}$$

Since $n(j) = \sum_{j'=1}^{j} \varepsilon_{j'} e_{\xi_{j'}(\omega)}$,

$$|n(j)| = \left(\sum_{s=1}^{d}\left|\sum_{\xi_j(\omega)=s} \varepsilon_j\right|^2\right)^{1/2}$$

has expected size $\sim \sqrt{j}$ and by elementary probabilistic considerations, also (fixing ω)

$$\mathbf{P}_{\varepsilon}\left[\frac{1}{k}\sum_{j=1}^{k}|n(j)| > B\sqrt{k}\right] < e^{-cB^2}. \tag{3.8}$$

From (3.7), statement (3.3) is clear. We now check (3.2), subdividing Π^d into regions

$$V_{\varepsilon} = \{\alpha \in \Pi^d | \varepsilon < \|\alpha\| < 2\varepsilon\} \tag{3.9}$$

where ε takes values $> B^2/\sqrt{k}$. Fix ε and consider a δ-net $\mathscr{V}$ in $V = V_{\varepsilon}$ $(\delta < \varepsilon)$ of size

$$|V| \sim \left(\frac{\varepsilon}{\delta}\right)^d. \tag{3.10}$$

If $\alpha \in V_{\varepsilon}$, $\alpha' \in \mathscr{V}$ satisfy $\|\alpha - \alpha'\| < \delta$, it follows that

$$\left|\hat{K}_k(\alpha) - \hat{K}(\alpha')\right| < \frac{1}{k}\sum_{1}^{k}|n(j)|\,\|\alpha - \alpha'\| < B\sqrt{k}\,\delta \tag{3.11}$$

and thus $|\hat{K}(\alpha) - \hat{K}(\alpha')| < B(\sqrt{k}\,\varepsilon)^{-1/2}$ for appropriate choice of δ, leading to a test set $\mathscr{V}$ of size

$$|\mathscr{V}| \lesssim \left(\varepsilon\sqrt{k}\right)^{(3/2)d} \tag{3.12}$$

from (3.10).

Thus from (3.6) one gets

$$\sup_{\alpha\in V}\left|\hat{K}_k(\alpha)\right| < B\left(\sqrt{k}\,\varepsilon\right)^{-1/2} + \varepsilon^{-2}\left(\frac{1}{k} + \sup_{\alpha\in\mathscr{V}}\left|D_k(\alpha)\right|\right). \tag{3.13}$$

Observe that $D_k(\varepsilon, \omega, \alpha)$ appears for a given α as a martingale difference sequence on the product space $(\{1, -1\} \times \{1, \dots, d\})^k$ with differences uniformly bounded by $\|\alpha\|/k$. Hence for given moment $q > 1$,

$$\left(\int |D_k|^q \, d\omega\, d\varepsilon\right)^{1/q} \le c\sqrt{q}\,\frac{\|\alpha\|}{\sqrt{k}} \quad (\text{cf. } [Ga]). \tag{3.14}$$

Therefore, for $\lambda > 0$, by Tchbychev's inequality,

$$\begin{aligned}\lambda^q P\Big[\max_{\alpha\in\mathscr{V}} |D_k(\cdot,\alpha)| > \lambda\Big] &\le \int \max_{\alpha\in\mathscr{V}} |D_k(\varepsilon,\omega,\alpha)|^q \, d\varepsilon\, d\omega \\ &\le \sum_{\alpha\in\mathscr{V}} \int |D_k(\varepsilon,\omega,\alpha)|^q \, d\varepsilon\, d\omega \\ &\le (\varepsilon\sqrt{k})^{3d/2}\left(c\sqrt{q}\,\frac{\varepsilon}{\sqrt{k}}\right)^q \qquad (3.15)\end{aligned}$$

by (3.12), (3.14).

Take $\lambda = B\varepsilon^2/(\varepsilon\sqrt{k})^{1/2}$. From (3.15),

$$\max_{a\in\mathscr{V}} |D_k(\cdot,\alpha)| < \frac{B\varepsilon^2}{(\varepsilon\sqrt{k})^{1/2}} \tag{3.16}$$

outside an exceptional (ε,ω)-set of measure less than

$$(\varepsilon\sqrt{k})^{3/2d}\left(\frac{q}{B^2(\varepsilon\sqrt{k})}\right)^{q/2}.$$

Take $q \sim B^2$, $q > 3(d+1)$. The measure estimate becomes

$$e^{-B^2}\cdot\frac{1}{\varepsilon\sqrt{k}} \tag{3.17}$$

where $\varepsilon\sqrt{k} > B^2$. Summation of (3.17) over the different regions V_ε again gives the e^{-B^2}-measure estimate. Substitute (3.16) in (3.13). Hence

$$\sup_{\alpha\in\mathscr{V}_\varepsilon} |\hat{K}(\alpha)| < cB(\varepsilon\sqrt{k})^{-1/2} \tag{3.18}$$

for all ε. Thus (3.2) holds, proving Proposition 2 and the theorem.

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Institut des Hautes Etudes Scientifiques
Bures-sur-Yvette, France