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SOME REMARKS ON EXTENSION THEOREMS FOR WEIGHTED SOBOLEV SPACES

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1. Introduction

Let $\mathscr{D}$ be an open set in $\mathbf{R}^n$. If α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{Z}_+^n$, we will denote $\sum_{j=1}^n \alpha_j$ by $|\alpha|$ and let

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

A locally integrable function f on $\mathscr{D}$ has a weak derivative of order α if there is a locally integrable function (denoted by $D^\alpha f$) such that

$$\int_{\mathscr{D}} f(D^\alpha \varphi)\, dx = (-1)^{|\alpha|} \int_{\mathscr{D}} (D^\alpha f) \varphi \, dx$$

for all C^∞ functions φ with compact support in $\mathscr{D}$ (we will write $\varphi \in C_0^\infty(\mathscr{D})$).

By a weight w, we mean a nonnegative locally integrable function on $\mathbf{R}^n$. By abusing notation, we will also write w for the measure induced by w. Sometimes we write dw to denote $w\,dx$. We always assume w is doubling, by which we mean $w(2Q) \le Cw(Q)$ for every cube Q, where $2Q$ denotes the cube with the same center as Q and twice its edgelength. Let μ be another weight. By $w/\mu \in A_p(\mu)$, we mean

$$\frac{1}{\mu(Q)}\left(\int_Q \frac{w}{\mu} d\mu\right)^{1/p} \left(\int_Q \left(\frac{w}{\mu}\right)^{-1/(p-1)} d\mu\right)^{(p-1)/p} \le C \text{ when } 1 < p < \infty, \text{ and}$$

$$\frac{\mu(x)}{\mu(Q)} \le C \frac{w(x)}{w(Q)} \text{ a.e. when } p = 1,$$

for all cubes Q in $\mathbf{R}^n$. If Q is a cube, let $l(Q)$ be the edgelength of Q. For $1 \le p \le \infty$, $k \in \mathbf{N}$, and any weight w, $L_{w,k}^p(\mathscr{D})$ and $E_{w,k}^p(\mathscr{D})$ are the spaces

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of functions having weak derivatives of all orders α, $|\alpha| \leq k$, and satisfying

$$\|f\|_{L^p_{w,k}(\mathscr{D})} = \sum_{0\leq|\alpha|\leq k} \|D^\alpha f\|_{L^p_w(\mathscr{D})} = \sum_{0\leq|\alpha|\leq k} \left(\int_{\mathscr{D}} |D^\alpha f|^p \, dw\right)^{1/p} < \infty \text{ if } 1 \leq p < \infty,$$

$$\|f\|_{L^\infty_{w,k}(\mathscr{D})} = \|f\|_{L^\infty_k(\mathscr{D})} = \sum_{0\leq|\alpha|\leq k} \operatorname*{esssup}_{\mathscr{D}} |D^\alpha f| < \infty$$

and

$$\|f\|_{E^p_{w,k}(\mathscr{D})} = \sum_{|\alpha|=k} \|D^\alpha f\|_{L^p_w(\mathscr{D})} < \infty$$

respectively. Moreover, in the case when $w \equiv 1$, we will denote $L^p_{w,k}(\mathscr{D})$ and $E^p_{w,k}(\mathscr{D})$ by $L^p_k(\mathscr{D})$ and $E^p_k(\mathscr{D})$ respectively.

The following theorem is by now well known.

THEOREM A. *If $\mathscr{D}$ is a Lipschitz domain and $1 \leq p \leq \infty$, then $L^p_k(\mathscr{D})$ has a bounded extension operator; i.e., there exists Λ: $L^p_k(\mathscr{D}) \to L^p_k(\mathbf{R}^n)$ such that $\Lambda f|_{\mathscr{D}} = f$ and*

$$\|\Lambda f\|_{L^p_k(\mathbf{R}^n)} \leq C\|f\|_{L^p_k(\mathscr{D})}.$$

A.P. Calderón [1] proved this theorem for the case $1 < p < \infty$ and E.M. Stein [12] extended Calderón's result (with a different extension operator) to include the endpoints $p = 1, \infty$. P. Jones [9] then extended Theorem A to connected (ε, δ) domains[1] as follows:

THEOREM B. *If $\mathscr{D}$ is a connected (ε, δ) domain and $1 \leq p \leq \infty$, then $L^p_k(\mathscr{D})$ has an extension operator. Moreover the norm of the extension operator depends only on $\varepsilon, \delta, k, p, \operatorname{rad}(\mathscr{D})$, and the dimension n.*

Furthermore he proved:

THEOREM C. *If $\mathscr{D}$ is an (ε, ∞) domain in $\mathbf{R}^n$, then $E^n_1(\mathscr{D})$ has a bounded extension operator, i.e., there exists Λ: $E^n_1(\mathscr{D}) \to E^n_1(\mathbf{R}^n)$ such that $\|\Lambda\|$ is bounded.*

Our purpose is to extend Theorem B and Theorem C to weighted Sobolev spaces, for example, when the weight satisfies Muckenhoupt's A_p condition. Indeed, we will try to extend these theorems when the weight w satisfies the following Poincaré type inequality

$$\|f - f_{Q,w}\|_{L^p_w(Q)} \leq C_0 l(Q)\|\nabla f\|_{L^p_w(Q)} \quad \forall \text{ cubes } Q \text{ in } \mathscr{D} \tag{P}$$

[1]The notations (ε, δ) domain and $\operatorname{rad}(\mathscr{D})$ will be defined in Section 2.

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$, the collection of all locally Lipschitz continuous functions (of course, one could replace $\mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$ by $\mathrm{Lip}_{\mathrm{loc}}(\overline{\mathscr{D}})$) where $f_{Q,w} = \int_Q f\,dw/w(Q)$. For example, it is well known that (P) holds when $w \in A_p$ (see [2] or [7]). Moreover, (P) holds for a class of non-A_p weights [16]. Also, note that (P) implies the following Poincaré type inequality on union of touching cubes (i.e., a face of one cube is contained in a face of the other)

$$\|f - f_{Q_1 \cup Q_2, w}\|_{L^p_w(Q_1 \cup Q_2)} \leq C \max(l(Q_1), l(Q_2)) \|\nabla f\|_{L^p_w(Q_1 \cup Q_2)}$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$ and touching cubes Q_1, Q_2 such that $1/4 \leq l(Q_1)/l(Q_2) \leq 4$. For the details, see [6].

Let $\mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n) = \{f\colon\ D^\alpha f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$ for all $|\alpha| < k\}$. Using similar technique used by P. Jones, we prove:

THEOREM 1.1. *Let $\mathscr{D}$ be an (ε, δ) domain with* $\mathrm{rad}(\mathscr{D}) > 0$ *and let k be a positive integer. If $1 \leq p < \infty$ and w is a weight such that* (P) *holds for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$ then there exists an extension operator Λ on $\mathscr{D}$ (i.e., $\Lambda f = f$ a.e. on $\mathscr{D}$) such that $\Lambda f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n)$ and*

$$\|\Lambda f\|_{L^p_{w,k}(\mathbf{R}^n)} \leq C \|f\|_{L^p_{w,k}(\mathscr{D})}$$

for all $f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n)$ where C depends only on ε, δ, k, w, p, n, C_0 and $\mathrm{rad}(\mathscr{D})$.[2]

Also, similar to Theorem C, we have:

THEOREM 1.2. *Let $\mathscr{D}$ be an (ε, ∞) domain and let k be a positive integer. If $1 \leq p < \infty$ and w is a weight such that* (P) *holds for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$, then there exists an extension operator Λ on $\mathscr{D}$ (i.e., $\Lambda f = f$ a.e. on $\mathscr{D}$) such that $\Lambda f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n)$ and*

$$\|\Lambda f\|_{E^p_{w,k}(\mathbf{R}^n)} \leq C \|f\|_{E^p_{w,k}(\mathscr{D})} \text{ for all } f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n), \tag{1.3}$$

where C is independent of $\mathrm{rad}(\mathscr{D})$.[3]

Let $\mathscr{D}$ be a bounded (ε, ∞) domain with $r = \mathrm{rad}(\mathscr{D})$ and let Ω be a bounded open set containing $\mathscr{D}$. Let W_2 be the collection of cubes in the Whitney decomposition of $(\mathscr{D}^c)^0$ and define

$$W_3 = \left\{Q \in W_2\colon l(Q) \leq \frac{\varepsilon r}{16nL}\right\},$$

where $L > 0$ is chosen so that $\Omega \subset (\cup_{Q \in W_3} Q) \cup \overline{\mathscr{D}}$.

[2] $\|\Lambda\| \to \infty$ as $\mathrm{rad}(\mathscr{D}) \to 0$ or as $\varepsilon \to 0$ or as $\delta \to 0$.

[3] Thus C depends only on ε, w, p, k, C_0 and n.

By a similar argument we also prove the next result.

THEOREM 1.4. *Let $1 \le p_i < \infty$ for $i = 0, 1, \ldots, N$. Let Ω be a bounded open set containing an (ε, ∞) domain $\mathscr{D}$ and let L and r be defined as above. Let μ be a weight and suppose that w_i are doubling weights such that*

$$\|f - f_{Q,\mu}\|_{L^{p_i}_{w_i}(Q)} \le A_i l(Q) \|\nabla f\|_{L^{p_i}_{w_i}(Q)} \quad \forall Q \text{ in } \mathscr{D}, \tag{1.5}$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$ and $i = 0, 1, \ldots, N$.

(I) There exists an extension operator Λ on $\mathscr{D}$ such that $\Lambda f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n)$ and

$$\|\nabla^k \Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i \|\nabla^k f\|_{L^{p_i}_{w_i}(\mathscr{D})}$$

for all i and all $f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n)$; in addition, if $w_i/\mu \in A_{p_i}(\mu)$ for some i, then for that value of i,

$$\|\Lambda f\|_{L^{p_i}_{w_i}(\mathbf{R}^n)} \le C_i\Big(\|f\|_{L^{p_i}_{w_i}(\mathscr{D})} + r\|\nabla f\|_{L^{p_i}_{w_i}(\mathscr{D})} + \cdots + r^{k-1}\|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}\Big)$$

$$\|\nabla \Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\Big(\|\nabla f\|_{L^{p_i}_{w_i}(\mathscr{D})} + r\|\nabla^2 f\|_{L^{p_i}_{w_i}(\mathscr{D})} + \cdots + r^{k-2}\|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}\Big)$$

$$\vdots$$

$$\|\nabla^l \Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\Big(\|\nabla^l f\|_{L^{p_i}_{w_i}(\mathscr{D})} + r\|\nabla^{l+1} f\|_{L^{p_i}_{w_i}(\mathscr{D})} + \cdots + r^{k-l-1}\|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}\Big)$$

$$\vdots$$

$$\|\nabla^{k-1} \Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i \|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}$$

for all $f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n)$.

(II) There exists another extension operator Λ' on $\mathscr{D}$ such that $\Lambda' f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n)$ and

$$\|\nabla^k \Lambda' f\|_{L^{p_i}_{w_i}(\mathbf{R}^n)} \le C_i \|\nabla^k f\|_{L^{p_i}_{w_i}(\mathscr{D})}$$

for all i and all $f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n)$; in addition, if $w_i/\mu \in A_{p_i}(\mu)$ for some i, then for that value of i,

$$\|\Lambda' f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\Big(\|f\|_{L^{p_i}_{w_i}(\mathscr{D})} + r\|\nabla f\|_{L^{p_i}_{w_i}(\mathscr{D})} + \cdots + r^{k-1}\|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}\Big)$$

$$\|\nabla \Lambda' f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\Big(\|\nabla f\|_{L^{p_i}_{w_i}(\mathscr{D})} + r\|\nabla^2 f\|_{L^{p_i}_{w_i}(\mathscr{D})} + \cdots + r^{k-2}\|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}\Big)$$

$$\vdots$$

$$\|\nabla^l \Lambda' f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\Big(\|\nabla^l f\|_{L^{p_i}_{w_i}(\mathscr{D})} + r\|\nabla^{l+1} f\|_{L^{p_i}_{w_i}(\mathscr{D})} + \cdots + r^{k-l-1}\|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}\Big)$$

$$\vdots$$

$$\|\nabla^{k-1} \Lambda' f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i \|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}$$

for all $f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n)$.

In either case, C_i depends only on w_i, μ, ε, L, p_i, A_i, k and n.[4] *(Unfortunately, L usually depends on r, but there are cases where L is independent of r and consequently C_i is independent of r.)*

Remarks. (a) It can be shown that the extension operators in Theorem 1.4 also satisfy

$$\|\nabla^l \Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\left(\|\nabla^l f\|_{L^{p_i}_{w_i}(\mathscr{D})} + r^{k-l}\|\nabla^k f\|_{L^{p_i}_{w_i}(\mathscr{D})}\right),$$
$$0 \le l \le k-2,\ 0 \le i \le N,$$

where C_i depends only on w_i, μ, ε, L, p_i, k and n. Moreover, in the first case we have

$$\|\Lambda f\|_{L^{p_i}_{w_i}(\mathbf{R}^n)} \le C_i\left(\|f\|_{L^{p_i}_{w_i}(\mathscr{D})} + r^k\|\nabla^k f\|_{L^{p_i}_{w_i}(\mathscr{D})}\right) \quad \forall i.$$

Here C_i again depends only on w_i, ε, L, p_i, k and n. Moreover, note that the assertion $w_i/\mu \in A_{p_i}(\mu)$ is not needed now.

(b) In case $k = 1$, better result could be obtained, see Corollary 4.23.

(c) Theorem 1.4 has a counterpart for infinite (ε, ∞) domains; see Theorem 4.26.

(d) These extension theorems, especially Theorem 1.4, have many applications; for example, they can be used to obtain Poincaré type inequalities (see Remark 4.15), Sobolev interpolation inequalities (see Theorem 4.34) and imbedding theorems of Sobolev spaces on (ε, ∞) domains.

2. Facts about (ε, δ) domains

DEFINITION. An open set $\mathscr{D}$ is an (ε, δ) domain if for all $x, y \in \mathscr{D}$, $|x-y| < \delta$, there exists a rectifiable curve γ connecting x, y such that γ lies in $\mathscr{D}$ and

$$l(\gamma) < \frac{|x-y|}{\varepsilon} \tag{2.1}$$

$$d(z, \partial\mathscr{D}) > \frac{\varepsilon|x-z||y-z|}{|x-y|} \quad \forall z \in \gamma. \tag{2.2}$$

Here $l(\gamma)$ is the length of γ and $d(z, \partial\mathscr{D})$ is the distance between z and the boundary of $\mathscr{D}$. Let us decompose $\mathscr{D} = \cup \mathscr{D}_\alpha$ into connected components

[4] $C_i \to \infty$ as $L \to 0$ or as $\varepsilon \to 0$.

and define

$$r = \operatorname{rad}(\mathscr{D}) = \inf_{\alpha} \inf_{x \in \mathscr{D}_\alpha} \sup_{y \in \mathscr{D}_\alpha} |x - y|.$$

We will assume $r > 0$ in most cases. Then for any $x \in \mathscr{D}$, there is a point y in the same component with $|x - y| \geq 3r/4$. Note that we always have $r > 0$ when $\mathscr{D}$ is an (ε, ∞) domain since $\mathscr{D}$ is then connected.

By a cube in $\mathbf{R}^n$, we mean a closed cube whose edges are parallel to the coordinate axes. Following the terminology used in [9], we say that two cubes touch if a face of one cube is contained in a face of the other. In particular, the union of two touching cubes of equal size is a rectangle. If Q is a cube, let $l(Q)$ denote the edgelength of Q. A collection of cubes $\{S_i\}_{i=0}^m$ is called a chain if S_i touches S_{i+1} for all i.

Also let W_1 be the cubes in the Whitney decomposition of $\mathscr{D}$ and W_2 be the cubes in the Whitney decomposition of $(\mathscr{D}^c)^0$. See [12] for the definition of the Whitney decomposition. We will write $d(Q, S) = \inf_{x \in Q,\ y \in S} |x - y|$ and $d(Q) = d(Q, \partial\mathscr{D})$.

Next let us recall some properties of the cubes in the Whitney decomposition of the open set $\mathscr{D}$ or $(\mathscr{D}^c)^0$. Since these properties are well known, we will often make use of them without explicitly mentioning them.

$$\begin{gathered}
l(Q) = 2^{-k} \quad \text{for some } k \in \mathbf{Z}, \\
Q_1^0 \cap Q_2^0 = \varnothing \quad \text{if } Q_1 \neq Q_2, \\
1/4 \leq \frac{l(Q_1)}{l(Q_2)} \leq 4 \quad \text{if } Q_1 \cap Q_2 \neq \varnothing, \\
1 \leq \frac{d(Q, \partial\mathscr{D})}{l(Q)} \leq 4\sqrt{n}.
\end{gathered}$$

Next, let us collect some facts concerning (ε, δ) domains. The reader can find the proof in [9]. Moreover, more details could be found in [4] or [5].

Let $\mathscr{D}$ be an (ε, δ) domain. Recall that W_1 and W_2 are the Whitney decompositions of $\mathscr{D}$ and $(\mathscr{D}^c)^0$ respectively. Then there exists $W_3 \subset W_2$ such that the following five properties hold.

(2.3) There exists $C > 0$ such that if $l(Q) < C$ and $Q \in W_2$ then $Q \in W_3$.

(2.4) There exists $C > 0$ such that for all $Q \in W_3$, $\exists S \in W_1$ such that $1 \leq l(S)/l(Q) \leq 4$ and $d(S, Q) \leq Cl(Q)$. We will choose such an S and write $S = Q^*$.

(2.5) There exists $C > 0$ such that for all $Q \in W_3$, and $S_1, S_2 \in W_1$ such that $S_1, S_2 = Q^*$, then $d(S_1, S_2) \leq Cl(Q)$.

(2.6) There exists $C > 0$ such that for all $S \in W_1$, there are at most C cubes $Q \in W_3$ with $Q^* = S$.

(2.7) There exists $C > 0$ such that for all $Q_1, Q_2 \in W_3$ such that $Q_1 \cap Q_2 \neq \emptyset$, we have $d(Q_1^*, Q_2^*) \leq Cl(Q_1)$.

(2.8) There exists $C > 0$ such that for all $Q_j, Q_k \in W_3$ with $Q_j \cap Q_k \neq \emptyset$, there exists a chain $F_{j,k} = \{Q_j^* = S_0, S_1, S_2, \ldots, S_m = Q_k^*\}$ of cubes in W_1 connecting Q_j^* to Q_k^* with $m \leq C$. (Then $l(S_i), l(Q_j)$ are comparable and $d(S_i, Q_j^*) \leq Cl(Q_j)$.)

Remark 2.9. Note that the constants in (2.3)–(2.8) depend only on ε, δ and n. Note also that W_3 is indeed the collection of those cubes which are sufficiently closed to $\mathscr{D}$. Moreover, when $\mathscr{D}$ is an (ε, ∞) domain, we can take $W_3 = \{Q \in W_2\colon\ l(Q) \leq \varepsilon \operatorname{rad}(\mathscr{D})/(16nL)\}$ with $L > 0$ so that properties (2.3)–(2.8) hold except that now $L \leq l(Q^*)/l(Q) \leq 4L$ for $Q \in W_3$.

3. Some preliminary results

From now on, C denotes various positive constants depending only on $\varepsilon, \delta, p, k, w$ and the dimension n, and $C(\alpha, \beta, \cdots)$ denotes such constants depending also on $\alpha, \beta, \cdots$. Again these constants may differ even in the same string of estimates. We denote by ∇ the vector

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$$

and by ∇^m the vector of all possible mth order derivatives for $m \in \mathbf{Z}_+$. By $w \in A_p$, $1 \leq p < \infty$, we mean that w satisfies the Muckenhoupt A_p condition, i.e.,

$$\frac{1}{|Q|}\left(\int_Q w(x)\,dx\right)^{1/p}\left(\int_Q w(x)^{-1/(p-1)}\,dx\right)^{(p-1)/p} \leq C$$

$$\forall \text{ cubes } Q \subset \mathbf{R}^n \text{ if } 1 < p < \infty$$

and

$$\frac{1}{|Q|}\int_Q w(x)\,dx \leq C \operatorname*{essinf}_{x \in Q} w(x) \quad \forall \text{ cubes } Q \subset \mathbf{R}^n \text{ if } p = 1.$$

Moreover, we write $W \in A_\infty$ if $w \in A_p$ for some $p \geq 1$.

Next, let us state a theorem on polynomials [5].

THEOREM 3.1. *Let E, F be unions of at most N cubes such that $E, F \subset Q$ for some cube Q and $|E|, |F| > \gamma|Q|$. If w is a doubling weight and p is a* *polynomial of degree* m, *then*

$$\|p\|_{L^q_w(E)} \le C(\gamma, m, N)\|p\|_{L^q_w(F)}.$$

This theorem is indeed the consequence of the following two lemmas.

LEMMA 3.2 [13, Chapter 3, Lemma 7]. *If* w *is a doubling measure and* m *is a positive integer, then there exists* $s_0(n, m, w)$ *such that if* $s < s_0$ *then for all cubes* Q, $\lambda > 0$ *such that*

$$w(\{x \in Q : |p(x)| > \lambda\}) \le sw(Q)$$

we have

$$\sup_{x\in Q} |p(x)| \le C\lambda,$$

where p *is any polynomial of degree* m *and* C *is a constant independent of* λ, Q *and* p.

It follows from Chebyshev's inequality and Lemma 3.2 that given m and a polynomial p of degree m,

$$\|p\|_{L^\infty(Q)} \le \frac{C}{w(Q)}\|p\|_{L^1_w(Q)}$$

with C independent of Q and p, since

$$\begin{aligned} w(\{x \in Q : |p(x)| > \lambda\}) &\le \frac{1}{\lambda}\int_{x\in Q,\, |p(x)|>\lambda} |p|\,dw \\ &\le \frac{1}{\lambda}\|p\|_{L^1_w(Q)} = sw(Q) \end{aligned}$$

by taking

$$\lambda = \frac{1}{sw(Q)}\int_Q |p|\,dw.$$

LEMMA 3.3. *Let* Q *be a cube and let* E *be a measurable set in* Q *with* $|E| > \gamma|Q|$. *If* p *is a polynomial of degree* m, *then*

$$\|p\|_{L^\infty(E)} \ge C(\gamma, m)\|p\|_{L^\infty(Q)}.$$

The reader could find the proof of this lemma in [5] or [4].

4. Main results

Let w and μ be weights such that w is doubling. Note that if $w/\mu \in A_p(\mu)$, then

$$\mu(Q)^{-1}\|f\|_{L^1_\mu(Q)} \leq Cw(Q)^{-1/p}\|f\|_{L^p_w(Q)}$$
$$\forall \text{ cubes } Q \subset \mathbf{R}^n \text{ and real functions } f.$$

Moreover it is clear that this condition is satisfied if $\mu = w$ since $1 \in A_p(w)$ for $p \geq 1$. If S is a compact set in an open set $\mathscr{D}$ and $f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n)$ we let $P_\mu(S, f)$ be the unique polynomial of degree $k-1$ such that

$$\int_S D^\alpha\big(f - P_\mu(S, f)\big)\,d\mu = 0, \quad 0 \leq |\alpha| < k.$$

First, we have the following lemma regarding these polynomials.

Lemma 4.1. *Let Q be a cube and let f and $P_\mu(Q, f)$ be as above. If $w/\mu \in A_p(\mu)$, then*

$$\begin{aligned}&\big\|D^\beta P_\mu(Q, f)\big\|_{L^p_w(Q)}\\ &\quad\leq C\Big(\big\|\nabla^{|\beta|}f\|_{L^p_w(Q)} + l(Q)\|\nabla^{|\beta|+1}f\|_{L^p_w(Q)} + \cdots + l(Q)^{k-|\beta|-1}\|\nabla^{k-1}f\big\|_{L^p_w(Q)}\Big)\end{aligned}$$

for $0 \leq |\beta| \leq k-1$.

The proof of this lemma is quite straightforward and is omitted. However, details could be found in [5].

Next, the following lemma is an essential tool in the proof of extension theorems. Note that it is similar to Lemma 2.2 in [9].

Lemma 4.2. *Let $F_{1,m} = \{S_1, S_2, \ldots, S_m\}$ be a chain of touching Whitney cubes and let R_i be the cube in $S_i \cup S_{i+1}$ such that $|R_i \cap S_i| = |R_i \cap S_{i+1}| = \frac{1}{2}\min(|S_i|, |S_{i+1}|)$ for $i = 1, 2, \ldots, m-1$. If*

$$\|f - f_{Q,\mu}\|_{L^p_w(Q)} \leq C_0 l(Q)\|\nabla f\|_{L^p_w(Q)} \quad \text{for } Q = S_i \text{ or } R_i, \tag{4.3}$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$, then

$$\begin{aligned}&\big\|D^\beta\big(P_\mu(S_m, f) - P_\mu(S_1, f)\big)\big\|_{L^p_w(S_1)}\\ &\qquad\leq C(m, C_0)l(S_1)^{k-|\beta|}\|\nabla^k f\|_{L^p_w(\cup F_{1,m})}, \quad 0 \leq |\beta| \leq k,\end{aligned}$$

for all $f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n)$.

Proof. Let us write $P_\mu(S)$ instead of $P_\mu(S,f)$. First note that we may assume $|\beta| < k$. By the triangle inequality and Theorem 3.1 we have

$$\begin{aligned}
&\left\| D^\beta\big(P_\mu(S_m) - P_\mu(S_1)\big)\right\|_{L^p_w(S_1)} \\
&\quad\le \sum_{i=1}^{m-1} \left\| D^\beta\big(P_\mu(S_{i+1}) - P_\mu(S_i)\big)\right\|_{L^p_w(S_1)} \\
&\quad\le C\sum_{i=1}^{m-1} \left\| D^\beta\big(P_\mu(S_{i+1}) - P_\mu(S_i)\big)\right\|_{L^p_w(S_i)} \\
&\quad\le C\sum \Big(\left\| D^\beta\big(P_\mu(S_i) - P_\mu(R_i)\big)\right\|_{L^p_w(S_i)} \\
&\qquad\qquad + \left\| D^\beta\big(P_\mu(S_{i+1}) - P_\mu(R_i)\big)\right\|_{L^p_w(S_i)}\Big) \\
&\quad\le C\sum \Big(\left\| D^\beta\big(P_\mu(S_i) - P_\mu(R_i)\big)\right\|_{L^p_w(R_i\cap S_i)} \\
&\qquad\qquad + \left\| D^\beta\big(P_\mu(S_{i+1}) - P_\mu(R_i)\big)\right\|_{L^p_w(R_i\cap S_{i+1})}\Big) \\
&\quad\le C\sum \Big(\left\| D^\beta\big(P_\mu(S_i) - f\big)\right\|_{L^p_w(S_i)} + \left\| D^\beta\big(P_\mu(S_{i+1}) - f\big)\right\|_{L^p_w(S_{i+1})} \\
&\qquad\qquad\qquad\qquad + \left\| D^\beta\big(P_\mu(R_i) - f\big)\right\|_{L^p_w(R_i)}\Big) \\
&\quad\le C\Big(\sum l(S_i)^{k-|\beta|}\|\nabla^k f\|_{L^p_w(S_i)} + l(S_{i+1})^{k-|\beta|}\|\nabla^k f\|_{L^p_w(S_{i+1})} \\
&\qquad\qquad\qquad\qquad + l(S_i)^{k-|\beta|}\|\nabla^k f\|_{L^p_w(R_i)}\Big)
\end{aligned}$$

by repeated applications of (4.3). Thus

$$\left\| D^\beta\big(P_\mu(S_m) - P_\mu(S_1)\big)\right\|_{L^p_w(S_1)} \le C(m) l(S_1)^{k-|\beta|}\|\nabla^k f\|_{L^p_w(\cup F_{1,m})}$$

since $l(S_1), l(S_2), \ldots, l(S_m)$ *are comparable. This completes the proof of the lemma.*

The following lemma is a consequence of Lemma 4.1 and the proof of the previous lemma.

LEMMA 4.4. *Under the assumption of Lemma* 4.2, *if we assume further that* $0 \le |\beta| \le q < k$, *then*

$$\begin{aligned}
&\| D^\beta(P_\mu(S_m, f) - P_\mu(S_1, f))\|_{L^p_w(S_1)} \\
&\quad\le C(m, C_0) l(S_1)^{q-|\beta|} \\
&\qquad\times\left(\|\nabla^q f\|_{L^p_w(\cup F_{1,m})} + \sum_{i=1}^{m} \|\nabla^q P_\mu(S_i, f)\|_{L^p_w(S_1)} + \sum_{i=1}^{m-1} \|\nabla^q P_\mu(R_i, f)\|_{L^p_w(S_1)}\right).
\end{aligned}$$

Furthermore, if $w/\mu \in A_p(\mu)$, *then*

$$\begin{aligned}\left\| D^{\beta}\big(P_{\mu}(S_m, f) - P_{\mu}(S_1, f)\big) \right\|_{L^p_w(S_1)} &\le C(m, C_0)\, l(S_1)^{q-|\beta|} \\ &\quad \times \Big(\| \nabla^q f \|_{L^p_w(\cup F_{1,m})} + l(Q) \| \nabla^{q+1} f \|_{L^p_w(\cup F_{1,m})} \\ &\quad + \cdots + l(Q)^{k-q-1} \| \nabla^{k-1} f \|_{L^p_w(\cup F_{1,m})} \Big).\end{aligned}$$

Proof. Because of the proof of the previous lemma, we only need to make the following observations. First, if $|\beta| < q$ then

$$\begin{aligned}&\left\| D^{\beta}\big(P_{\mu}(S_i, f) - f\big) \right\|_{L^p_w(S_i)} \\ &\quad \le C l(S_i)^{q-|\beta|} \left\| \nabla^q \big(P_{\mu}(S_i, f) - f\big) \right\|_{L^p_w(S_i)} \\ &\qquad\qquad \text{by repeated applications of (4.3)} \\ &\quad \le C l(S_i)^{q-|\beta|} \Big(\| \nabla^q P_{\mu}(S_i, f) \|_{L^p_w(S_i)} + \| \nabla^q f \|_{L^p_w(S_i)} \Big).\end{aligned}$$

Next, if $|\beta| = q$ we obtain the same estimate from the triangle inequality. Moreover, similar arguments can be applied to the term $\|D^{\beta}(P_{\mu}(R_i, f) - f)\|_{L^p_w(R_i)}$. Finally, our conclusion follows immediately from Theorem 3.1 and Lemma 4.1.

We can now prove Theorem 1.1. However, as it is almost exactly the same as the proof of Theorem 1 in [9] except that now we will make use of Theorem 3.1, Lemma 4.2 and the Poincaré type inequality (P) instead of Lemmas 2.1, 2.2 and 3.1 in [9], we will only give a sketch of the proof.

Sketch of the Proof. Recall that W_1 is the Whitney decomposition of $\mathscr{D}$ and W_2 is the Whitney decomposition of $(\mathscr{D}^c)^0$.

Step (1). Choose $W_3 \subset W_2$ such that properties (2.3)–(2.8) hold. Note that $l(Q) \le C$ for all $Q \in W_3$.

Sep (2). Next let us define the extension operator. For each $q_j \in W_3$, choose $0 \le \varphi_j \le \chi_{\frac{17}{16}Q_j}$, $\varphi_j \in C^{\infty}(\mathbf{R}^n)$, such that

$$\sum_{Q_j \in W_3} \varphi_j \equiv 1 \quad \text{on } \cup W_3, 0 \le \sum_{Q_j \in W_3} \varphi_j \le 1,$$

and

$$|D^\alpha \varphi_j| \le Cl(Q_j)^{-|\alpha|}, \quad 0 \le |\alpha| \le k.$$

Given $f \in \mathrm{Lip}_{\mathrm{loc}}^{k-1}$, we define $P_j(x) = P_w(Q_j^*, f)(x)$ and

$$\Lambda f(x) = \begin{cases} f(x) & \text{if } x \in \overline{\mathscr{D}} \\ \Sigma_{Q_j \in W_3} P_j(x)\varphi_j(x) & \text{if } x \in (\mathscr{D}^c)^0. \end{cases}$$

We then show that if $Q_0 \in W_3$ then

$$\| D^\alpha \Lambda f \|_{L^p_w(Q_0)} \le C \| D^\alpha f \|_{L^p_w(Q_0^*)} + Cl(Q_0)^{k-|\alpha|} \| \nabla^k f \|_{L^p_w(\cup F(Q_0))}, \quad (4.5)$$

where $0 \le |\alpha| \le k$ and $F(Q_0)$ is the collection of cubes which belong to any of the chains $F_{0,j}$ for which $Q_j \cap Q_0 \ne \varnothing$.

This inequality can be shown by repeated applications of (P), Theorem 3.1 and Lemma 4.2. The proof is quite technical but standard (see [9], [4] or [5]).

Next if $Q_0 \in W_2 \setminus W_3$, $0 \le |\alpha| \le k$, we can show

$$\| D^\alpha \Lambda f \|_{L^p_w(Q_0)} \le C(r) \sum_{Q_j \in W_3,\, Q_j \cap Q_0 \ne \varnothing} \left\{ \| \nabla^k f \|_{L^p_w(Q_j^*)} + \sum_{\beta \le \alpha} \| D^\beta f \|_{L^p_w(Q_j^*)} \right\}. \quad (4.6)$$

Again, the proof of this equality can be found in [9] or [5].

Moreover, observe that

$$\left\| \sum_{Q_j \in W_2 \setminus W_3} \; \sum_{Q_1 \in W_3,\, Q_l \cap Q_j \ne \varnothing} \chi_{Q_l^*} \right\|_{L^\infty} \le C, \quad (4.7)$$

$$\left\| \sum_{Q_j \in W_3} \chi_{\cup F(Q_j)} \right\|_{L^\infty} \le C. \quad (4.8)$$

Combining these facts with (4.5), (4.6) and using $l(Q_j) \le C$ if $Q_j \in W_3$, we have, for $0 \le |\alpha| \le k$,

$$\begin{aligned}
\|D^\alpha \Lambda f\|^p_{L^p_w((\mathscr{D}^c)^0)} &= \sum_{Q_j \in W_3} \|D^\alpha \Lambda f\|^p_{L^p_w(Q_j)} + \sum_{Q_j \in W_2 \setminus W_3} \|D^\alpha \Lambda f\|^p_{L^p_w(Q_j)} \\
&\le \sum_{Q_j \in W_3} C\left(\|D^\alpha f\|_{L^p_w(Q_j^*)} + \|\nabla^k f\|_{L^p_w(\cup F(Q_j))}\right)^p \\
&\quad + \sum_{Q_j \in W_2 \setminus W_3} \left(\sum_{Q_l \in W_3,\, Q_j \cap Q_l \ne \varnothing} C(r) \right. \\
&\qquad \left. \times \left(\|\nabla^k f\|_{L^p_w(Q_l^*)} + \sum_{\beta \le \alpha} \|D^\beta f\|_{L^p_w(Q_l^*)} \right) \right)^p \\
&\le \sum_{Q_j \in W_3} C\left(\|D^\alpha f\|^p_{L^p_w(Q_j^*)} + \|\nabla^k f\|^p_{L^p_w(\cup F(Q_j))}\right) \\
&\quad + \sum_{Q_j \in W_2 \setminus W_3} \sum_{Q_l \in W_3,\, Q_j \cap Q_l \ne \varnothing} C(r) \\
&\qquad \times \left(\|\nabla^k f\|^p_{L^p_w(Q_l^*)} + \sum_{\beta \le \alpha} \|D^\beta f\|^p_{L^p_w(Q_l^*)} \right) \\
&\le C(r) \|f\|^p_{L^p_{w,k}(\mathscr{D})}.
\end{aligned}$$

Hence

$$\|\Lambda f\|_{L^p_{w,k}(\mathscr{D}^c)^0} \le C(r) \|f\|_{L^p_{w,k}(\mathscr{D})}.$$

Step (3). We then show that $D^\alpha \Lambda f$ is locally Lipschitz for all $\alpha, 0 \le |\alpha| < k$. Observe that we have

$$\|\Lambda f\|_{L^\infty_k(Q)} \le C\left(\|\nabla^k f\|_{L^\infty(\cup F(Q))} + \sum_{Q_j \in W_3,\, Q_j \cap Q \ne \varnothing} \|f\|_{L^\infty_k(Q_j^*)} \right) \quad \forall Q \in W_2. \tag{4.9}$$

(If $Q \notin W_3$, we take $\cup F(Q) = \varnothing$). To prove (4.9), we only need to replace p by ∞ in (4.5) and (4.6) since

$$\|f - f_{Q,w}\|_{L^\infty(Q)} \le Cl(Q) \|\nabla f\|_{L^\infty(Q)} \quad \text{for all cubes } Q.$$

If Ω is a bounded set in $(\mathscr{D}^c)^0$, then $\exists G \subset W_2$ such that $\Omega \subset \cup G$ and $\cup G$

is bounded. Thus

$$\|\Lambda f\|_{L_k^\infty(\Omega)} \le \|\Lambda f\|_{L_k^\infty(\cup G)} \le C\|f\|_{L_k^\infty(K)} < \infty,$$

where K is a compact set containing $\cup F(Q)$ $\forall Q \in G$ and containing Q_j^* $\forall Q_j \in W_3$ with $Q_j \cap Q \neq \varnothing$, $Q \in G$. We now show that $D^\alpha \Lambda f$ is continuous for all α, $0 \le |\alpha| < k$. To this end, one only need to show that

$$\lim_{x \to x_0,\, x \in (\mathscr{D}^c)^0} D^\alpha \Lambda f(x) = D^\alpha f(x_0) \quad \forall x_0 \in \partial\mathscr{D}, 0 \le |\alpha| < k.$$

Nevertheless, it suffices to show that if $Q_j \in W_3$ and $d(Q_j, \partial\mathscr{D}) \to 0$ then

$$\left\| D^\alpha \Lambda f - \frac{1}{w(Q_j^*)} \int_{Q_j^*} D^\alpha f\, dw \right\|_{L^\infty(Q_j)} \to 0.$$

However, the proof is again quite standard. For the details, see [9] or [4].

Remark 4.10. (a) Let W' be the collection of all cubes S such that either $Q^* = S$ for some $Q \in W_3$ or $S \in F_{j,k}$ for some $Q_j, Q_k \in W_3$ (see property (2.8)). Indeed W' is just the collection of all cubes in W_1 near the boundary $\partial\mathscr{D}$. Also, let $W'' = \{R$ is a cube in $Q_1 \cup Q_2$ such that $|R \cap Q_1| = |R \cap Q_2| = \frac{1}{2}\min(|Q_1|, |Q_2|)$ for some touching cubes Q_1, Q_2 in $W'\}$. Then indeed one needs only to assume that (P) holds for all cubes in $W' \cup W''$ to prove Theorem 1.1.

(b) In case $\overline{\mathrm{Lip}_{\mathrm{loc}}^{k-1}(\mathbf{R}^n)}$ is dense in $L_{w,k}^p(\mathscr{D})$ and $w^{-1/p-1}$ is locally integrable on $\overline{\mathscr{D}}$ (these are true when $w \in A_p$), our extension operator Λ can be defined on $L_{w,k}^p(\mathscr{D})$ such that

$$\|\Lambda f\|_{L_{w,k}^p(\mathbf{R}^n)} \le C\|f\|_{L_{w,k}^p(\mathscr{D})}.$$

For the details, please refer to [5] or [4].

Proof of Theorem 1.2. Case (i): $\mathscr{D}$ is unbounded. Then $r = \infty$ and $W_2 = W_3$ (see Section 2).

Just as before, we will define $\Lambda f = \sum_{Q_j \in W_3} P_j \varphi_j$ on $(\mathscr{D}^c)^0$. Recall that for $Q_0 \in W_3$, we have

$$\begin{aligned}\left\| D^\alpha \sum_{Q_j \in W_3} P_j \varphi_j \right\|_{L_w^p(Q_0)} &\le \|D^\alpha P_0\|_{L_w^p(Q_0)} + \left\| D^\alpha \sum_{Q_j \in W_3} (P_j - P_0)\varphi_j \right\|_{L_w^p(Q_0)} \\ &= \left\| D^\alpha \sum_{Q_j \in W_3} (P_j - P_0)\varphi_j \right\|_{L_w^p(Q_0)}\end{aligned}$$

when $|\alpha| = k$ since P_0 is a polynomial of degree $k - 1$. Also recall that

$$\Big\| D^\alpha \sum_{Q_j \in W_3} (P_j - P_0)\varphi_j \Big\|_{L^p_w(Q_0)} \le Cl(Q_0)^{k-|\alpha|} \| \nabla^k f \|_{L^p_w(\cup F(Q_0))}$$
$$= C\| \nabla^k f \|_{L^p_w(\cup F(Q_0))}$$

since $|\alpha| = k$. By (4.8) we get as before

$$\| D^\alpha \Lambda f \|_{L^p_w(\mathscr{D}^c)^0} \le C\| \nabla^k f \|_{L^p_w(\mathscr{D})} \quad \text{if } |\alpha| = k.$$

Exactly the same as before we can show that $D^\alpha \Lambda f$ is locally Lipschitz when $|\alpha| \le k - 1$. Hence if $|\alpha| = k$,

$$\| D^\alpha \Lambda f \|_{L^p_w(\mathbf{R}^n)} \le C\| \nabla^k f \|_{L^p_w(\mathscr{D})}. \tag{4.11}$$

Case (ii): $\mathscr{D}$ is bounded. Then $\mathrm{rad}(\mathscr{D}) = r < \infty$ and $W_3 = \{Q \in W_2 \colon l(Q) \le \varepsilon r/16n\}$.

Recall that by definition, $\mathrm{rad}(\mathscr{D}) = \inf_{x \in \mathscr{D}} \sup_{y \in \mathscr{D}} |x - y|$ as now $\mathscr{D}$ is connected. Hence

$$\sup_{x, y \in \mathscr{D}} |x - y| < 3r.$$

Let $\Gamma = \cup(W_2 \setminus W_3)$ and $W_4 = \{Q \in W_3 \colon Q \cap \Gamma \ne \varnothing\}$. Note that

$$Q \in W_4 \Rightarrow l(Q) \ge \frac{1}{8}\frac{\varepsilon r}{16n}.$$

Next choose $\varphi', \varphi'_j \in C^\infty(\mathbf{R}^n)$ for $j = 1, 2, \ldots$ such that for $Q_j \in W_3$,

$$\chi_{Q_j} \le \varphi'_j \le \chi_{\frac{17}{16}Q_j}, \ |D^\alpha \varphi'_j| \le Cl(Q_j)^{-|\alpha|},$$
$$\chi_\Gamma \le \varphi' \le \chi_{\Gamma \cup \Gamma^c_\eta}, \ |D^\alpha \varphi'| \le Cr^{-|\alpha|},$$

where

$$\Gamma^c_\eta = \{x \in \Gamma^c \colon d(x, \partial\Gamma) < \eta\}, \qquad \eta = \frac{1}{32}\frac{\varepsilon r}{16n}.$$

Then $1 \le \Sigma\varphi'_j + \varphi' < C$ on $(\mathscr{D}^c)^0$. Define

$$\varphi_j = \frac{\varphi'_j}{\varphi' + \sum \varphi'_j} \quad \text{and} \quad \varphi = \frac{\varphi'}{\varphi' + \sum \varphi'_j}.$$

Since the edgelengths of cubes in W_4 are comparable to r, we have $K =$ number of cubes in $W_4 < C$ (independent of r) and

$$|D^\beta \varphi_j| \le Cl(Q_j)^{-|\beta|} \quad \text{and} \quad |D^\beta \varphi| \le Cr^{-|\beta|} \quad \text{for } 0 \le |\beta| \le k.$$

Next, define $P = (1/K)\Sigma_{Q_j \in W_4} P_j$ and

$$\Lambda f = \begin{cases} P\varphi + \Sigma_{Q_j \in W_3} P_j \varphi_j & \text{on } (\mathscr{D}^c)^0 \\ f & \text{on } \overline{\mathscr{D}}. \end{cases}$$

Now if $Q_0 \in W_3 \setminus W_4$, we can show as before that

$$\| D^\alpha \Lambda f \|_{L^p_w(Q_0)} \le C \| \nabla^k f \|_{L^p_w(\cup F(Q_0))} \quad \text{if } |\alpha| = k. \tag{4.12}$$

Next if $Q_0 \in W_4$, then since $\Sigma \varphi_j + \varphi \equiv 1$ on $\cup W_3$,

$$\begin{aligned} \| D^\alpha \Lambda f \|_{L^p_w(Q_0)} \le \| D^\alpha P_0 \|_{L^p_w(Q_0)} &+ \left\| D^\alpha \sum_{Q_j \in W_3} (P_0 - P_j)\varphi_j \right\|_{L^p_w(Q_0)} \\ &+ \| D^\alpha ((P_0 - P)\varphi) \|_{L^p_w(Q_0)}. \end{aligned}$$

Note that since $|\alpha| = k$, $D^\alpha P_0 \equiv 0$ and observe that exactly as before,

$$\begin{aligned} &\left\| D^\alpha \sum_{Q_j \in W_3} (P_0 - P_j)\varphi_j \right\|_{L^p_w(Q_0)} \\ &\quad = \left\| \sum_{Q_j \in W_3,\, Q_j \cap Q_0 \ne \varnothing} \sum_{\beta \le \alpha} C(\alpha, \beta) D^\beta (P_0 - P_j) D^{\alpha - \beta} \varphi_j \right\|_{L^p_w(Q_0)} \\ &\quad \le C \sum_{Q_j \in W_3,\, Q_j \cap Q_0 \ne \varnothing} \sum_{\beta \le \alpha} l(Q_0)^{-|\alpha - \beta|} \| D^\beta (P_0 - P_j) \|_{L^p_w(Q_0)} \\ &\quad \le Cl(Q_0)^{k - |\alpha|} \sum_{Q_j \in W_3,\, Q_j \cap Q_0 = \varnothing} \| \nabla^k f \|_{L^p_w(\cup F_{0,j})} \\ &\quad \le Cl(Q_0)^{k - |\alpha|} \| \nabla^k f \|_{L^p_w(\cup F(Q_0))} = C \| \nabla^k f \|_{L^p_w(\cup F(Q_0))} \end{aligned}$$

since $|\alpha| = k$. Also

$$\| D^\alpha ((P_0 - P)\varphi) \|_{L^p_w(Q_0)} = \left\| \sum_{\beta \le \alpha} C(\alpha, \beta) D^\beta (P_0 - P) D^{\alpha - \beta} \varphi \right\|_{L^p_w(Q_0)}.$$

Next, observe that

$$\begin{aligned}
&\| D^{\beta}(P_0 - P) D^{\alpha-\beta}\varphi \|_{L^p_w(Q_0)} \\
&\quad \leq Cr^{-|\alpha-\beta|} \| D^{\beta}(P_0 - P) \|_{L^p_w(Q_0)} \\
&\quad \leq \frac{Cr^{-|\alpha-\beta|}}{K} \sum_{Q_j \in W_4} \| D^{\beta}(P_0 - P_j) \|_{L^p_w(Q_0)} \\
&\quad \leq \frac{Cr^{-|\alpha-\beta|}}{K} \sum_{Q_j \in W_4} \sum_i \| D^{\beta}(P(S^*_{j,i}) - P(S^*_{j,i+1})) \|_{L^p_w(Q_0)} \\
&\qquad\qquad (\text{where } Q_0 = S_{j,0}, S_{j,1}, \ldots, S_{j,k_j} = Q_j \text{ is a chain in } W_4) \\
&\quad \leq \frac{Cr^{-|\alpha-\beta|}}{K} \sum_{Q_j \in W_4} \sum_i \| D^{\beta}(P(S^*_{j,i}) - P(S^*_{j,i+1})) \|_{L^p_w(S^*_{j,i})}
\end{aligned}$$

by Theorem 3.1 since $Q_0, S_{j,i} \in W_4$ and hence $Q_0, S^*_{j,i} \subset Q$ with $|Q_0|, |S^*_{j,i}| > \gamma |Q|$ for some cube Q with edgelength Cr and γ is a constant depending only on ε, k, and n. Continuing with the inequalities we have

$$\begin{aligned}
&\leq \frac{Cr^{-|\alpha-\beta|}}{K} \sum_{Q_j \in W_4} \sum_i l(S_{j,i})^{k-|\beta|} \| \nabla^k f \|_{L^p_w(\cup G_{i,j})} \quad \text{by Lemma 4.2} \\
&\qquad\qquad \text{where } G_{i,j} \text{ is a chain in } W_1 \text{ which connects } S^*_{j,i} \text{ to } S^*_{j,i+1} \\
&\leq \frac{C}{K} \sum_{Q_j \in W_4} \sum_i \| \nabla^k f \|_{L^p_w(\cup G_{i,j})} \quad \text{since } l(S_{j,i}) \leq Cr \text{ and } |\alpha| = k \\
&\leq \frac{C}{K} \sum_{Q_j \in W_4} K \| \nabla^k f \|_{L^p_w(\mathscr{D})} \quad \text{since for all } i \text{ and } j,\ \cup G_{i,j} \subset \mathscr{D} \text{ and } k_j \leq K \\
&\leq C \| \nabla^k f \|_{L^p_w(\mathscr{D})}
\end{aligned}$$

since $K < C$. Thus,

$$\| D^{\alpha}((P_0 - P)\varphi) \|_{L^p_w(Q_0)} \leq C \| \nabla^k f \|_{L^p_w(\mathscr{D})},$$

and hence if $Q_0 \in W_4$,

$$\| D^{\alpha} \Lambda f \|_{L^p_w(Q_0)} \leq C \| \nabla^k f \|_{L^p_w(\cup F(Q_0))} + C \| \nabla^k f \|_{L^p_w(\mathscr{D})} \leq C \| \nabla^k f \|_{L^p_w(\mathscr{D})}. \tag{4.13}$$

Finally, by similar methods as the preceding estimate, one can show that (details are available in [5] or [4])

$$\|D^\alpha \Lambda f\|_{L^p_w(\Gamma)} \le C\|\nabla^k f\|_{L^p_w(\mathscr{D})}. \tag{4.14}$$

It is now clear that by (4.12), (4.13) and (4.14), we have

$$\|\nabla^k \Lambda f\|_{L^p_w((\mathscr{D}^c)^0)} \le C\|\nabla^k f\|_{L^p_w(\mathscr{D})}.$$

Similarly, by checking that $D^\alpha \Lambda f$ is locally Lipschitz for all α, $|\alpha| = k - 1$, we have

$$\|\nabla^k \Lambda f\|_{L^p_w(\mathbf{R}^n)} \le C\|\nabla^k f\|_{L^p_w(\mathscr{D})}.$$

Finally, similar to case (i), we conclude the proof of case (ii).

Remark 4.15. If (P) holds for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$ and all cubes Q, then

$$\|f - f_{\mathscr{D},w}\|_{L^p_w(\mathscr{D})} \le C\,\mathrm{rad}(\mathscr{D})\|\nabla f\|_{L^p_w(\mathscr{D})}$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$ and all bounded (ε, ∞) domains $\mathscr{D}$, where C is a constant which depends only on ε, w, p and n. (In [6], the author has studied weighted inequalities of this kind over more general domains which include John domains.)

Proof. First let us choose a cube $Q \supset \mathscr{D}$ such that $l(Q)$ is comparable to $\mathrm{rad}(\mathscr{D})$. Next if $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$, we define Λf as in the proof of the previous theorem with $k = 1$. Then $\Lambda f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$ and

$$\|\nabla \Lambda f\|_{L^p_w(\mathbf{R}^n)} \le C\|\nabla f\|_{L^p_w(\mathscr{D})}.$$

Hence

$$\begin{aligned}\|f - f_{\mathscr{D},w}\|_{L^p_w(\mathscr{D})} &\le 2\|f - (\Lambda f)_{Q,w}\|_{L^p_w(\mathscr{D})}\\ &\le 2\|\Lambda f - (\Lambda f)_{Q,w}\|_{L^p_w(Q)}\\ &\le Cl(Q)\|\nabla \Lambda f\|_{L^p_w(Q)}\\ &\le C\,\mathrm{rad}(\mathscr{D})\|\nabla f\|_{L^p_w(\mathscr{D})}.\end{aligned}$$

Finally we prove Theorem 1.4. Recall that $\mathscr{D}$ is a bounded (ε, ∞) domain with $r = \mathrm{rad}(\mathscr{D})$, Ω is a bounded open set containing $\mathscr{D}$ and

$$W_3 = \left\{Q \in W_2 \colon l(Q) \le \frac{\varepsilon r}{16nL}\right\}, \quad L = 2^{-m}, m \in \mathbf{Z}_+,$$

where L is chosen so that $\Omega \subset \cup(W_3) \cup \overline{\mathscr{D}}$.

Proof of Theorem 1.4. We will only prove the theorem for the typical case $k = 3$.

By $w \in \mathrm{P}_p(\mu)$, we mean

$$\|f - f_{Q,\mu}\|_{L^p_w(Q)} \le C l(Q) \|\nabla f\|_{L^p_w(Q)} \quad \forall \text{ cubes } Q \subset \mathscr{D},\ f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n).$$

Note that $w_i \in \mathrm{P}_p(\mu)$ for all i by hypothesis. When $Q_j \in W_3$, we will write $P_j = P_\mu(Q_j^*, f)$. Moreover note that $l(Q_j^*) < Cr$ for all $Q_j \in W_3$.

Now let us prove (I). Let us define

$$\Lambda f(x) = \begin{cases} \Sigma_{Q_i \in W_3} P_i(x)\varphi_i(x) & \text{if } x \in (\mathscr{D}^c)^0 \\ f(x) & \text{if } x \in \overline{\mathscr{D}} \end{cases}$$

where the φ_i's are the same as in the proof of Theorem 1.1.

As in the proof of (4.5) we can show that if $Q_0 \in W_3$ and $|\alpha| = 3$ then

$$\|D^\alpha \Lambda f\|_{L^{p_i}_{w_i}(Q_0)} \le C(L) \|\nabla^3 f\|_{L^{p_i}_{w_i}(\cup F(Q_0))} \quad \forall i.$$

Hence when $|\alpha| = 3$, we have

$$\begin{aligned} \|D^\alpha \Lambda f\|_{L^{p_i}_{w_i}(\Omega \setminus \overline{\mathscr{D}})} &\le \|D^\alpha \Lambda f\|_{L^{p_i}_{w_i}(\cup W_3)} \\ &\le C(L) \|\nabla^3 f\|_{L^{p_i}_{w_i}(\mathscr{D})} \quad \forall i \end{aligned}$$

as in the proof of Theorem 1.1 (with the help of (4.8)).

Next we will estimate $D^\alpha \Lambda f$ for $|\alpha| \le 2$ with the help of Lemma 4.1. First observe that if $Q_o \in W_2$ and $w/\mu \in A_p(\mu)$, then

$$\begin{aligned} &\|\Lambda f\|_{L^p_w(Q_0)} \\ &\quad \le \sum_{Q_i \in W_3,\, Q_i \cap Q_0 \ne \varnothing} \|P_i\|_{LL^p_w(Q_0)} \le C(L) \sum_{Q_i \in W_3,\, Q_i \cap Q_0 \ne \varnothing} \|P_i\|_{L^p_w(Q_i^*)} \\ &\quad \le C(L) \sum_{Q_i \in W_3,\, Q_i \cap Q_0 \ne \varnothing} C\big(\|f\|_{L^p_w(Q_i^*)} + r\|\nabla f\|_{L^p_w(Q_i^*)} + r^2 \|\nabla^2 f\|_{L^p_w(Q_i^*)}\big) \end{aligned} \tag{4.16}$$

by Lemma 4.1 since $l(Q_i^*) \le Cr$ if $Q_i \in W_3$. Hence

$$
\begin{aligned}
\|\Lambda f\|^p_{L^p_w(\cup W_3)} &\le \sum_{Q_i \in W_3} \|\Lambda f\|^p_{L^p_w(Q_i)} \\
&\le \sum_{Q_i \in W_3} \Bigg(\sum_{Q_j \in W_3,\, Q_j \cap Q_i \ne \varnothing} C(L) \\
&\qquad \times \left(\|f\|_{L^p_w(Q_j^*)} + r\|\nabla f\|_{L^p_w(Q_j^*)} + r^2\|\nabla^2 f\|_{L^p_w(Q_j^*)} \right) \Bigg)^p \\
&\le \sum_{Q_i \in W_3} \sum_{Q_j \in W_3,\, Q_j \cap Q_i \ne \varnothing} C(L) \\
&\qquad \times \left(\|f\|^p_{L^p_w(Q_j^*)} + r^p\|\nabla f\|^p_{L^p_w(Q_j^*)} + r^{2p}\|\nabla^2 f\|^p_{L^p_w(Q_j^*)} \right) \\
&\le C(L) \sum_{Q_j \in W_3} \left(\|f\|^p_{L^p_w(Q_j^*)} + r^p\|\nabla f\|^p_{L^p_w(Q_j^*)} + r^{2p}\|\nabla^2 f\|^p_{L^p_w(Q_j^*)} \right) \\
&\le C(L) \left(\|f\|^p_{L^p_w(\mathscr{D})} + r^p\|\nabla f\|^p_{L^p_w(\mathscr{D})} + r^{2p}\|\nabla^2 f\|^p_{L^p_w(\mathscr{D})} \right). \qquad (4.17)
\end{aligned}
$$

Moreover,

$$
\|\Lambda f\|^p_{L^p_w(\cup(W_2 \setminus W_3))} = \Bigg\| \sum_{Q_i \in W_3} P_i \varphi_i \Bigg\|^p_{L^p_w(\cup(W_2 \setminus W_3))} = \Bigg\| \sum_{Q_i \in W_3} P_i \varphi_i \Bigg\|^p_{L^p_w(\cup W_5)}
$$

where $W_5 = \{Q \in W_2 \setminus W_3 \colon Q \cap Q_l \ne \varnothing \text{ for some } Q_l \in W_3\}$,

$$
\begin{aligned}
&\le \sum_{Q_j \in W_5} \Bigg(\sum_{Q_i \in W_3,\, Q_i \cap Q_j \ne \varnothing} \|P_i \varphi_i\|_{L^p_w(Q_j)} \Bigg)^p \\
&\le \sum_{Q_j \in W_5} \Bigg(\sum_{Q_i \in W_3,\, Q_i \cap Q_j \ne \varnothing} C(L) \|P_i\|_{L^p_w(Q_i^*)} \Bigg)^p \\
&\le \sum_{Q_j \in W_5} \sum_{Q_i \in W_3,\, Q_i \cap Q_j \ne \varnothing} C(L) \|P_i\|^p_{L^p_w(Q_i^*)} \\
&\le \sum_{Q_j \in W_5} \sum_{Q_i \in W_3,\, Q_i \cap Q_j \ne \varnothing} C(L) \\
&\qquad \times \left(\|f\|_{L^p_w(Q_i^*)} + r\|\nabla f\|_{L^p_w(Q_i^*)} + r^2\|\nabla^2 f\|_{L^p_w(Q_i^*)} \right)^p
\end{aligned}
$$

by Lemma 4.1

$$\leq \sum_{Q_j \in W_5} \sum_{Q_i \in W_3,\, Q_i \cap Q_j \neq \varnothing} C(L)$$

$$\times \left(\|f\|^p_{L^p_w(Q_i^*)} + r^p \|\nabla f\|^p_{L^p_w(Q_i^*)} + r^{2p} \|\nabla^2 f\|^p_{L^p_w(Q_i^*)} \right)$$

$$\leq C(L) \sum_{Q_j \in W_4} \left(\|f\|^p_{L^p_w(Q_j^*)} + r^p \|\nabla f\|^p_{L^p_w(Q_j^*)} + r^{2p} \|\nabla^2 f\|^p_{L^p_w(Q_j^*)} \right)$$

$$\leq C(L) \left(\|f\|^p_{L^p_w(\mathscr{D})} + r^p \|\nabla f\|^p_{L^p_w(\mathscr{D})} + r^{2p} \|\nabla^2 f\|^p_{L^p_w(\mathscr{D})} \right).$$

Thus

$$\|\Lambda f\|_{L^p_w(\cup W_2)} \leq C(L) \left(\|f\|_{L^p_w(\mathscr{D})} + r \|\nabla^2 f\|_{L^p_w(\mathscr{D})} + r^2 \|\nabla^2 f\|_{L^p_w(\mathscr{D})} \right). \quad (4.18)$$

Moreover if $|\beta| = 1$, $Q_0 \in W_3$ and $w \in P_p(\mu)$, (again, $w/\mu \in A_p(\mu)$)

$$\|D^\beta \Lambda f\|_{L^p_w(Q_0)} \leq \Big\| \sum_{Q_i \in W_3} (D^\beta P_i) \varphi_i \Big\|_{L^p_w(Q_0)} + \Big\| \sum_{Q_i \in W_3} P_i D^\beta \varphi_i \Big\|_{L^p_w(Q_0)}$$

$$\leq \sum_{Q_i \in W_3,\, Q_i \cap Q_0 \neq \varnothing} \|D^\beta P_i\|_{L^p_w(Q_0)} + \Big\| \sum_{Q_i \in W_3} (P_i - P_0) D^\beta \varphi_i \Big\|_{L^p_w(Q_0)}$$

since $\sum_{Q_i \in W_3} D^\beta \varphi_i \equiv 0$ on $\cup W_3$

$$\leq C(L) \sum_{Q_i \in W_3,\, Q_i \cap Q_0 \neq \varnothing} \|D^\beta P_i\|_{L^p_w(Q_i^*)} + C(L) l(Q_i)^{-1}$$

$$\times \sum_{Q_i \in W_3,\, Q_i \cap Q_0 \neq \varnothing} \|P_i - P_0\|_{L^p_w(Q_i^*)}$$

$$\leq C(L) \sum_{Q_i \in W_3,\, Q_i \cap Q_0 \neq \varnothing} \left(\|\nabla f\|_{L^p_w(Q_i^*)} + r \|\nabla^2 f\|_{L^p_w(Q_i^*)} + \|\nabla f\|_{L^p_w(\cup F_{0,i})} \right.$$

$$\left. + r \|\nabla^2 f\|_{L^p_w(\cup F_{0,i})} \right) \quad \text{by Lemmas 4.1 and 4.4,}$$

where $F_{0,i} = \{Q_0^* = S_{i,0}, S_{i,1}, \ldots, S_{i,m_i} = Q_i^*\}$ is a chain guaranteed by (2.8)

$$\leq C(L) \sum_{Q_i \in W_3,\, Q_i \cap Q_0 \neq \varnothing} \left(\|\nabla f\|_{L^p_w(\cup F_{0,i})} + r \|\nabla^2 f\|_{L^p_w(\cup F_{0,i})} \right). \quad (4.19)$$

Similarly, if $|\gamma| = 2$ and $Q_0 \in W_3$, we obtain (when $w/\mu \in A_p(\mu)$ and $w \in P_p(\mu)$)

$$\begin{aligned}
\| D^\gamma \Lambda f \|_{L^p_w(Q_0)} \leq & \sum_{Q_i \in W_3,\, Q_i \cap Q_0 \neq \varnothing} \| D^\gamma P_i \|_{L^p_w(Q_i^*)} \\
& + Cl(Q_i)^{-1} \sum_{Q_i \in W_3,\, Q_i \cap Q_0 \neq \varnothing} \sum_{|\beta|=1} \| D^\beta (P_i - P_0) \|_{L^p_w(Q_0)} \\
& + Cl(Q_i)^{-2} \sum_{Q_i \in W_3,\, Q_i \cap Q_0 \neq \varnothing} \| P_i - P_0 \|_{L^p_w(Q_0)} \\
\leq C(L) & \sum_{Q_i \in W_3,\, Q_i \cap Q_0 \neq \varnothing} \left(\| D^\gamma f \|_{L^p_w(Q_i^*)} + \| \nabla^2 f \|_{L^p_w(\cup F_{0,i})} \right),
\end{aligned} \tag{4.20}$$

by Lemmas 4.1 and 4.4. Thus we obtain estimates of

$$\|D^\beta \Lambda f\|_{L^p_w(Q_0)},\ \|D^\gamma \Lambda f\|_{L^p_w(Q_0)} \quad \text{for } Q_0 \in W_3.$$

Similar to the estimate of $\|\Lambda f\|_{L^p_w(\cup W_3)}$, if $|\beta| = 1$, $|\gamma| = 2$, we have, by (4.19) and (4.20),

$$\| D^\beta \Lambda f \|_{L^p_w(\cup W_3)} \leq C(L) \left(\| \nabla f \|_{L^p_w(\mathscr{D})} + r \| \nabla^2 f \|_{L^p_w(\mathscr{D})} \right), \tag{4.21}$$

$$\| D^\gamma \Lambda f \|_{L^p_w(\cup W_3)} \leq C(L) \| \nabla^2 f \|_{L^p_w(\mathscr{D})}. \tag{4.22}$$

Next as in the proof of Theorem 1.1, we can show that $D^\alpha \Lambda f$ is locally Lipschitz for $0 \leq |\alpha| \leq 2$ if $f \in \mathrm{Lip}^2_{\mathrm{loc}}(\mathbf{R}^n)$. Hence we have the desired estimates (recall that $\Omega \subset \overline{\mathscr{D}} \cup (\cup W_3)$), namely, for all i such that $w_i/\mu \in A_{p_i}(\mu)$,

$$\begin{aligned}
\| \nabla^2 \Lambda f \|_{L^{p_i}_{w_i}(\Omega)} &\leq C(L) \| \nabla^2 f \|_{L^{p_i}_{w_i}(\mathscr{D})}, \\
\| \nabla \Lambda f \|_{L^{p_i}_{w_i}(\Omega)} &\leq C(L) \left(\| \nabla f \|_{L^{p_i}_{w_i}(\mathscr{D})} + r \| \nabla^2 f \|_{L^{p_i}_{w_i}(\mathscr{D})} \right), \\
\| \Lambda f \|_{L^{p_i}_{w_i}(\mathbf{R}^n)} &\leq C(L) \left(\| f \|_{L^{p_i}_{w_i}(\mathscr{D})} + r \| \nabla f \|_{L^{p_i}_{w_i}(\mathscr{D})} + r^2 \| \nabla^2 f \|_{L^{p_i}_{w_i}(\mathscr{D})} \right),
\end{aligned}$$

and

$$\| \nabla^3 \Lambda f \|_{L^{p_i}_{w_i}(\Omega)} \leq C(L) \| \nabla^3 f \|_{L^{p_i}_{w_i}(\mathscr{D})}.$$

This proves (I).

We now consider (II). As in the proof of case (ii) of Theorem 1.2, we let $P = (1/K) \sum_{Q_j \in W_4} P_j$ (recall that K is the number of cubes in W_4 and $W_4 = \{Q \in W_3 \colon Q \cap \Gamma \neq \varnothing\}$ where $\Gamma = \cup (W_2 \setminus W_3)$). Also, similar to the proof of Theorem 1.2, we choose $\varphi', \varphi' \in C^\infty(\mathbf{R}^n)$ for $j = 1, 2, \dots$ such that for $Q_j \in W_3$,

$$\chi_{Q_j} \le \varphi_j' \le \chi_{\frac{17}{16}Q_j}, \ |D^\alpha \varphi_j'| \le Cl(Q_j)^{-|\alpha|},$$
$$\chi_\Gamma \le \varphi' \le \chi_{\Gamma \cup \Gamma_\eta^c}, \ |D^\alpha \varphi'| \le C(L) r^{-|\alpha|},$$

where

$$\Gamma_\eta^c = \{x \in \Gamma^c \colon d(x, \partial\Gamma) < \eta\}, \qquad \eta = \frac{1}{32}\frac{\varepsilon r}{16nL}.$$

then $1 \le \Sigma\varphi_j' + \varphi' < C$ on $(\mathscr{D}^c)^0$. Define

$$\varphi_j = \frac{\varphi_j'}{\varphi' + \sum \varphi_j'} \quad \text{and} \quad \varphi = \frac{\varphi'}{\varphi' + \sum \varphi_j'}.$$

Since $C_1(L)r \le$ edgelengths of cubes in $W_4 \le C_2(L)r$, we have $K =$ number of cubes in $W_4 < C(L)$ (independent of r) and

$$|D^\beta \varphi_j| \le Cl(Q_j)^{-|\beta|} \text{ and } |D^\beta \varphi| \le C(L) r^{-|\beta|} \quad \text{for } 0 \le |\beta| \le k.$$

Again we define

$$\Lambda' f = \begin{cases} P\varphi + \sum\limits_{Q_j \in W_3} P_j \varphi_j & \text{on } (\mathscr{D}^c)^0 \\ f & \text{on } \overline{\mathscr{D}}. \end{cases}$$

The rest of the proof of (II) are similar to that of (I). The reader could refer to [5] or [4] for the details.

COROLLARY 4.23. *Let $1 \le p_i < \infty$ for $i = 1, 2, \dots, N$ and let μ be a weight. Let $N_1, N \in \mathbf{Z}_+$ such that $N_1 < N$. Suppose that $w_1, \dots, w_N$ are doubling weights such that $w_i/\mu \in A_{p_i}(\mu)$ for $1 \le i \le N_1$ and*

$$\|f - f_{Q,\mu}\|_{L^{p_i}_{w_i}(Q)} \le A_i l(Q) \|\nabla f\|_{L^{p_i}_{w_i}(Q)} \quad \forall \text{ cubes } Q \text{ in } \mathscr{D},\ N_1 < i \le N,$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$. Let $\mathscr{D}$ and Ω be as in the preceding theorem. Then there exists an extension operator Λ on $\mathscr{D}$ such that $\Lambda f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$,

$$\|\Lambda f\|_{L^{p_i}_{w_i}(\mathbf{R}^n)} \le C_i \|f\|_{L^{p_i}_{w_i}(\mathscr{D})} \quad \text{for } 1 \le i \le N_1$$

and

$$\| \nabla \Lambda f \|_{L^{p_i}_{w_i}(\Omega)} \le C_i \| \nabla f \|_{L^{p_i}_{w_i}(\mathscr{D})} \quad \text{for } N_1 < i \le N,$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$ *where* C_i *depends only on* w_i, p_i, ε, A_i, L *and* n. (*Similarly, there exists another extension operator so that the above inequalities hold with* Ω *and* $\mathbf{R}^n$ *interchanged*).

Proof. One needs only to check through the preceding proof and see that the proof of (4.18) does not involved the Poincaré inequality (P) when $k = 1$.

COROLLARY 4.24. *Let* $1 \le p_i < \infty$ *for* $i = 1, 2, \ldots, N$ *and let* μ *be a weight. Let* $N_1, N \in \mathbf{Z}_+$ *such that* $N_1 < N$. *Suppose that* $w_1, \ldots, w_N$ *are doubling weights such that* $w_i/\mu \in A_{p_i}(\mu)$ *for* $1 \le i \le N_1$ *and*

$$\| f - f_{Q,\mu} \|_{L^{p_i}_{w_i}(Q)} \le A_i l(Q) \| \nabla f \|_{L^{p_i}_{w_i}(Q)} \quad \forall Q \text{ in } \mathscr{D}$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$ *and* $N_1 < i \le N$. *Let* B *be any ball in* $\mathbf{R}^n$. *If* $1 < s < \infty$, *then there exists an extension operator* Λ *on* $\mathscr{D}$ *such that* $\Lambda f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$,

$$\| \Lambda f \|_{L^{p_i}_{w_i}(sB)} \le C_i \| f \|_{L^{p_i}_{w_i}(B)} \quad \text{for } 1 \le i \le N_1,$$

and

$$\| \nabla \Lambda f \|_{L^{p_i}_{w_i}(\mathbf{R}^n)} \le C_i \| \nabla f \|_{L^{p_i}_{w_i}(B)} \quad \text{for } N_1 < i \le N,$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$ *where* C_i *depends only on* A_i, w_i, p_i, n *and* s. *Moreover,* Λf *are locally Lipschitz on* $\mathbf{R}^n$. (*Similarly, there exists another extension operator so that the above mentioned inequalities hold with* sB *and* $\mathbf{R}^n$ *interchanged*).

Proof. First note that all balls B are (ε_0, ∞) domains for some fixed $\varepsilon_0 < 1$. Next take $\mathscr{D} = B$, $\Omega = sB$ in Corollary 4.23 above. Then C_i's are independent of $\mathrm{rad}(B)$ since L depends only on s (indeed $L = C/s$).

Theorems 1.4 has another corollary:

COROLLARY 4.25. *Let* μ *be a weight and suppose that* w_i *is a doubling weight such that*

$$\| f - f_{Q,\mu} \|_{L^{p_i}_{w_i}(Q)} \le A_i l(Q) \| \nabla f \|_{L^{p_i}_{w_i}}(Q) \quad \forall \text{ cubes } Q \text{ in } \mathscr{D}$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$ *and* $i = 1, \ldots, N$. *If* $\mathscr{D}$ *is a bounded*[5] (ε, ∞) *domain,*

[5] A similar result also holds for unbounded (ε, ∞) domains.

then there exists an extension operator Λ *on* $\mathscr{D}$ *such that* $\Lambda f \in \mathrm{Lip}_{\mathrm{loc}}^{k-1}(\mathbf{R}^n)$ *and*

$$\|\nabla^k \Lambda f\|_{L^{p_i}_{w_i}(\mathbf{R}^n)} \le C_i \|\nabla^k f\|_{L^{p_i}_{w_i}(\mathscr{D})} \quad \forall i$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}^{k-1}(\mathbf{R}^n)$ *where* C_i *depends only on* ε, A_i, w_i, p_i, k *and* n.

Next, Theorems 1.4 has a counterpart for infinite (ε, ∞) domains:

THEOREM 4.26. *Let* μ *be a weight and suppose that* w_i *is a doubling weight such that*

$$\|f - f_{Q,\mu}\|_{L^{p_i}_{w_i}(Q)} \le A_i l(Q) \|\nabla f\|_{L^{p_i}_{w_i}(Q)} \quad \forall Q \text{ in } \mathscr{D}$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$ *and* $i = 1, \ldots, N$. *If* Ω *is an open set containing an infinite* (ε, ∞) *domain such that* $\sup_{x \in \Omega} d(x, \mathscr{D}) = L < \infty$, *then there exists an extension operator* Λ *on* $\mathscr{D}$ *such that* $\Lambda f \in \mathrm{Lip}_{\mathrm{loc}}^{k-1}(\mathbf{R}^n)$ *and*

$$\|\nabla^k \Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i \|\nabla^k f\|_{L^{p_i}_{w_i}(\mathscr{D})} \quad \forall i$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}^{k-1}(\mathbf{R}^n)$; *in addition, if* $w_i/\mu \in A_{p_i}(\mu)$ *for some* i, *then for that value of* i,

$$\|\Lambda f\|_{L^{p_i}_{w_i}(\mathbf{R}^n)} \le C_i\big(\|f\|_{L^{p_i}_{w_i}(\mathscr{D})} + L\|\nabla f\|_{L^{p_i}_{w_i}(\mathscr{D})} + \cdots + L^{k-1}\|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}\big)$$

$$\|\nabla \Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\big(\|\nabla f\|_{L^{p_i}_{w_i}(\mathscr{D})} + L\|\nabla^2 f\|_{L^{p_i}_{w_i}(\mathscr{D})} + \cdots + L^{k-2}\|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}\big)$$

$$\vdots$$

$$\|\nabla^l \Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\big(\|\nabla^l f\|_{L^{p_i}_{w_i}(\mathscr{D})} + L\|\nabla^{l+1} f\|_{L^{p_i}_{w_i}(\mathscr{D})} + \cdots + L^{k-l-1}\|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}\big)$$

$$\vdots$$

$$\|\nabla^{k-1} \Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i \|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}.$$

Moreover, there exists another extension operator (which we will again denote by Λ*) on* $\mathscr{D}$ *such that* $\Lambda f \in \mathrm{Lip}_{\mathrm{loc}}^{k-1}(\mathbf{R}^n)$ *and*

$$\|\nabla^k \Lambda f\|_{L^{p_i}_{w_i}(\mathbf{R}^n)} \le C_i \|\nabla^k f\|_{L^{p_i}_{w_i}(\mathscr{D})} \quad \forall i$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}^{k-1}(\mathbf{R}^n)$; *in addition, if* $w_i/\mu \in A_{p_i}(\mu)$ *for some* i, *then for that* *value of i,*

$$\|\Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\big(\|f\|_{L^{p_i}_{w_i}(\mathscr{D})} + L\|\nabla f\|_{L^{p_i}_{w_i}(\mathscr{D})} + \cdots + L^{k-1}\|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}\big)$$

$$\|\nabla\Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\big(\|\nabla f\|_{L^{p_i}_{w_i}(\mathscr{D})} + L\|\nabla^2 f\|_{L^{p_i}_{w_i}(\mathscr{D})} + \cdots + L^{k-2}\|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}\big)$$

$$\vdots$$

$$\|\nabla^l\Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\big(\|\nabla^l f\|_{L^{p_i}_{w_i}(\mathscr{D})} + L\|\nabla^{l+1} f\|_{L^{p_i}_{w_i}(\mathscr{D})} + \cdots + L^{k-l-1}\|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}\big)$$

$$\vdots$$

$$\|\nabla^{k-1}\Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathscr{D})}.$$

In either case, C_i depends only on A_i, w_i, ε, p_i, k and n.

Sketch of the proof. Let us assume that $k = 3$. First let $W_3 = \{Q \in W_2: l(Q) \le CL\}$ such that $\Omega \subset (\cup W_3) \cup \mathscr{D}$. Next, given $f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbf{R}^n)$, by Lemma 4.1, if $w/\mu \in A_p(\mu)$ and S is any cube or union of two touching Whitney cubes, then

$$\|P_\mu(S)\|_{L^p_w(S)} \le C\big(\|f\|_{L^p_w(S)} + l(S)\|\nabla f\|_{L^p_w(S)} + l(S)^2\|\nabla^2 f\|_{L^p_w(S)}\big),$$

$$\|\nabla P_\mu(S)\|_{L^p_w(S)} \le C\big(\|\nabla f\|_{L^p_w(S)} + l(S)\|\nabla^2 f\|_{L^p_w(S)}\big),$$

$$\|\nabla^2 P_\mu(S)\|_{L^p_w(S)} \le C\|\nabla^2 f\|_{L^p_w(S)}.$$

Using these estimates we can prove this theorem as we did Theorem 1.4.

Remark 4.27. The Poincaré type inequalities (P) assumed in all the above can be replaced by the Poincaré type inequalities on balls, i.e.,

$$\|f - f_{B,w}\|_{L^p_w(B)} \le C|B|^{1/n}\|\nabla f\|_{L^p_w(B)} \quad \forall \text{ balls } B \text{ in } \mathscr{D} \qquad (\mathrm{P}')$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$. To see this, it suffices to observe that one can use Whitney type decompositions of open sets in balls in those proof above. Of course, we now only have bounded overlaps instead of non-overlapping. But that is sufficient. Thus the conditions (P) and (P′) are indeed equivalent.

With the help of the extension theorems in Section 4, we are able to improve a result in [15]. First, let us prove a lemma which is essentially a consequence of the proof of Theorem 1 in [15].

LEMMA 4.28. *Let $1 \le p < \infty$ and let v, w_1, w_2 be doubling weights. Let B_0 be any ball in $\mathbf{R}^n$. Suppose that there exist constants $C_0 > 0$ and $q > p$ such that*

$$\left(\frac{1}{w_2(B)}\int_B |u - u_{B,v}|^q \, dw_2\right)^{1/q} \le C_0 |B|^{1/n} \left(\frac{1}{w_1(B)}\int_B |\nabla u|^p \, dw_1\right)^{1/p} \tag{4.29}$$

for all balls $B \subset 3B_0$, with center in B_0, $u \in \mathrm{Lip}(3\overline{B}_0)$. Suppose further that there exists $C_1 > 0$ and $1 < h \le q/p$ such that

$$\frac{w_2(\tilde{B})}{w_2(B)} \le C_1 \left[\frac{v(\tilde{B})}{v(B)}\right]^{h-1} \left(\left[\frac{|B|}{|\tilde{B}|}\right]^{p/n} \frac{w_1(\tilde{B})}{w_1(B)} + \frac{v(\tilde{B})}{v(B)}\right) \tag{4.30}$$

for $\tilde{B}, B$, such that $B \subset B_0$, $\tilde{B} \subset 3B$ and the center of $\tilde{B}$ lies in B_0. Then there exists a constant $C > 0$ (depending only on n, C_0 and C_1) such that

$$\begin{aligned} \int_B |u|^{ph} \, dw_2 \le C w_2(B) &\left[\frac{1}{v(3B)}\int_{3B} |u|^p \, dv\right]^{h-1} \\ &\times \left[\frac{|B|^{p/n}}{w_1(3B)}\int_{3B} |\nabla u|^p \, dw_1 + \frac{1}{v(B)}\int_{3B} |u|^p \, dv\right] \end{aligned} \tag{4.31}$$

for $B \subset B_0$ and $u \in \mathrm{Lip}(3\overline{B})$.

Proof. Our proof is essentially the same as the proof of equation 2.1 in [15] although it is assumed that $p = 2$ in [15]. Fix a ball B in B_0 and let $x \in B$ and B_x be a ball in $3B_0$ centered at x. By (4.29) we have

$$\begin{aligned} \int_{B_x} |u|^{ph} \, dw_2 &\le C_0 w_2(B_x) \left\{ |B_x|^{ph/n} \left[\frac{1}{w_1(B_x)}\int_{B_x} |\nabla u|^p \, dw_1\right]^h \right. \\ &\qquad \left. + \left[\frac{1}{v(B_x)}\int_{B_x} |u| dv\right]^{ph} \right\} \\ &\le C_0 w_2(B_x) \left\{ |B_x|^{ph/n} \left[\frac{1}{w_1(B_x)}\int_{B_x} |\nabla u|^p \, dw_1\right]^h \right. \\ &\qquad \left. + \left[\frac{1}{v(B_x)}\int_{B_x} |u|^p \, dv\right]^h \right\} \\ &= C_0 w_2(B_x)\{I + II\} \end{aligned} \tag{4.32}$$

by Hölder's inequality. If B_x is large, i.e., $B \subset B_x \subset 3B$, we may assume that $I \geq II$. Otherwise if $I < II$, we have

$$\int_{B_x} |u|^{ph}\, dw_2 \leq 2C_0 w_2(B_x)\left(\frac{1}{v(B_x)}\int_{B_x}|u|^p\, dv\right)^h,$$

and we have nothing to prove. Also observe that $|B_x| \to 0$ implies $I \to 0$ and $II \to |u(x)|^{ph}$. Therefore if $u(x) \neq 0$, for small balls B_x we have $I < II$. Hence given $x \in B$ with $u(x) \neq 0$ there exists $B_x \subset 3B$ such that $I = II$, i.e., such that

$$|B_x|^{p/n}\frac{1}{w_1(B_x)}\int_{B_x}|\nabla u|^p\, dw_1 = \frac{1}{v(B_x)}\int_{B_x}|u|^p\, dv. \tag{4.33}$$

Hence for this ball B_x,

$$\begin{aligned}\int_{B_x}|u|^{ph}\, dw_2 &\leq 2C_0 w_2(B_x)|B_x|^{p/n}\left[\frac{1}{w_1(B_x)}\int_{B_x}|\nabla u|^p\, dw_1\right]^h\\ &= 2C_0 w_2(B_x)\left[\frac{1}{v(B_x)}\int_{B_x}|u|^p\, dv\right]^{h-1}\\ &\quad\times|B_x|^{p/n}\left[\frac{1}{w_1(B_x)}\int_{B_x}|\nabla u|^p\, dw_1\right].\end{aligned}$$

By Besicovitch's covering lemma, there is a family $\{B_k\}_{k=1}^{\infty}$ which covers $\{x \in B\colon u(x) \neq 0\}$ and which has bounded overlaps. Since

$$\frac{1}{v(B_k)}\int_{B_k}|u|^p\, dv \leq \frac{1}{v(B_k)}\int_{3B}|u|^p\, dv,$$

we obtain

$$\begin{aligned}\int_B |u|^{ph}\, dw_2 &\leq C\sum_k w_2(B_k)\left[\frac{1}{v(B_k)}\int_{B_k}|u|^p\, dv\right]^{h-1}\left[\frac{|B_k|^{p/n}}{w_1(B_k)}\int_{B_k}|\nabla u|^p\, dw_1\right]\\ &\leq C\left(\int_{3B}|u|^p\, dv\right)^{h-1}\sum_k\frac{w_2(B_k)|B_k|^{p/n}}{v(B_k)^{h-1}w_1(B_k)}\int_{B_k}|\nabla u|^p\, dw_1.\end{aligned}$$

We now apply (4.33) and assumption (4.30) to get

$$\int_B |u|^{ph}\, dw_2 \le C\left(\int_{3B} |u|^p\, dv\right)^{h-1} \sum_k \frac{|B_k|^{p/n} w_2(B)}{v(B_k)^{h-1} w_1(B_k)} \int_{B_k} |\nabla u|^p\, dw_1$$

$$\times\left[\left(\frac{v(B_k)}{v(B)}\right)^{h-1}\left\{\left(\frac{|B|}{|B_k|}\right)^{p/n} \frac{w_1(B_k)}{w_1(B)} + \frac{v(B_k)}{v(B)}\right\}\right]$$

$$\le Cw_2(B)\left(\int_{3B} |u|^p\, dv\right)^{h-1} \sum_k \frac{|B_k|^{p/n}}{v(B)^{h-1} w_1(B_k)} \int_{B_k} |\nabla u|^p\, dw_1$$

$$\times\left\{\left[\frac{|B|}{|B_k|}\right]^{p/n} \frac{w_1(B_k)}{w_1(B)} + \frac{v(B_k)}{v(B)}\right\}$$

$$\le Cw_2(B)\left(\int_{3B} |u|^p\, dv\right)^{h-1} \sum_k \int_{B_k} |\nabla u|^p dw_1$$

$$\times\left[\frac{|B|^{p/n}}{v(B)^{h-1} w_1(B)} + \frac{|B_k|^{p/n} v(B_k)}{v(B)^h w_1(B_k)}\right]$$

$$\le Cw_2(B)\left[\frac{1}{v(B)} \int_{3B} |u|^p\, dv\right]^{h-1}$$

$$\times\left[\frac{|B|^{p/n}}{w_1(B)} \int_{3B} |\nabla u|^p\, dw_1 + \frac{1}{v(B)} \sum_k \frac{v(B_k)|B_k|^{p/n}}{w_1(B_k)} \int_{B_k} |\nabla u|^p\, dw_1\right]$$

since B_k's have bounded overlaps

$$\le Cw_2(B)\left[\frac{1}{v(3B)} \int_{3B} |u|^p\, dv\right]^{h-1}$$

$$\times\left[\frac{|B|^{p/n}}{w_1(3B)} \int_{3B} |\nabla u|^p\, dw_1 + \frac{1}{v(B)} \sum_k \int_{B_k} |u|^p\, dv\right]$$

$$\le Cw_2(B)\left[\frac{1}{v(3B)} \int_{3B} |u|^p\, dv\right]^{h-1}$$

$$\times\left[\frac{|B|^{p/n}}{w_1(3B)} \int_{3B} |\nabla u|^p\, dw_1 + \frac{1}{v(B)} \int_{3B} |u|^p\, dv\right]$$

again since B_k's have bounded overlaps. This completes the proof.

We are now ready to prove the following Sobolev interpolation inequality which is similar to Theorem 1 in [15].

THEOREM 4.34. *Let v, w_1, w_2 be doubling weights and $1 \le p < q < \infty$. Suppose that*

$$\left(\int_B |u - u_{B,v}|^p \, dw_1\right)^{1/p} \le C_0 |B|^{1/n} \left(\int_B |\nabla u|^p \, dw_1\right)^{1/p} \tag{4.35}$$

for all balls B, and $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$.

(a) *Suppose that* (4.29) *holds for all balls B and $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}^n)$. Let $1 < h \le q/p$. Then the following two inequalities are equivalent:*

$$\int_B |u|^{ph} \, dw_2 \le C w_2(B) \left[\frac{1}{v(B)} \int_B |u|^p \, dv\right]^{h-1} \times \left[\frac{|B|^{p/n}}{w_1(B)} \int_B |\nabla u|^p \, dw_1 + \frac{1}{v(B)} \int_B |u|^p \, dv\right] \tag{4.36}$$

for all $B \subset \mathbb{R}^n$ and $u \in \mathrm{Lip}(\bar{B})$;

$$\frac{w_2(\tilde{B})}{w_2(B)} \le C \left[\frac{v(\tilde{B})}{v(B)}\right]^{h-1} \left(\left[\frac{|B|}{|\tilde{B}|}\right]^{p/n} \frac{w_1(\tilde{B})}{w_1(B)} + \frac{v(\tilde{B})}{v(B)}\right) \tag{4.37}$$

for all balls $B, \tilde{B} \subset \mathbb{R}^n$ such that $\tilde{B} \subset 3B$ and the center of $\tilde{B}$ lies in B.

(b) (4.36) *holds for all B in a fixed B_0 if* (4.29) *holds for all B in $3B_0$ and* (4.37) *holds for all $B \subset B_0$ and $\tilde{B} \subset 3B$ (and the center of $\tilde{B}$ lies in B).*

Proof. First assume that (4.36) holds. Let $B, \tilde{B}$ in $\mathbb{R}^n$ such that $\tilde{B} \subset 3B$ and the center of $\tilde{B}$ lies in B. We now choose $u \in C_0^\infty$ such that $\chi_{\tilde{B}/2} \le u \le \chi_{\tilde{B}}$ and $|\nabla u| \le C/|\tilde{B}|^{1/n}$. Then

$$\int_{3B} |u|^{ph} \, dw_2 \le C w_2(3B) \left(\frac{1}{v(3B)} \int_{3B} |u|^p \, dv\right)^{h-1} \times \left[\frac{|3B|^{p/n}}{w_1(3B)} \int_{3B} |\nabla u|^p \, dw_1 + \frac{1}{v(3B)} \int_{3B} |u|^p \, dv\right].$$

Thus

$$w_2(\tilde{B}/2) \le C w_2(3B) \left(\frac{1}{v(3B)} v(\tilde{B})\right)^{h-1} \left[\frac{|3B|^{p/n}}{w_1(3B)} \frac{C}{|\tilde{B}|^{p/n}} w_1(\tilde{B}) + \frac{v(\tilde{B})}{v(3B)}\right],$$

and hence

$$\frac{w_2(\tilde{B})}{w_2(3B)} \leq C\left[\frac{v(\tilde{B})}{v(3B)}\right]^{h-1}\left[\frac{|3B|^{p/n}}{|\tilde{B}|^{p/n}}\frac{w_1(\tilde{B})}{w_1(3B)} + \frac{v(\tilde{B})}{v(3B)}\right].$$

Therefore,

$$\frac{w_2(\tilde{B})}{w_2(B)} \leq C\left[\frac{v(\tilde{B})}{v(B)}\right]^{h-1}\left[\frac{|B|^{p/n}}{|\tilde{B}|^{p/n}}\frac{w_1(\tilde{B})}{w_1(B)} + \frac{v(\tilde{B})}{v(B)}\right].$$

Next we will prove (b). First by the previous lemma, we have

$$\int_B |u|^{ph}\, dw_2 \leq C w_2(B)\left[\frac{1}{v(3B)}\int_{3B}|u|^p\, dv\right]^{h-1} \times\left[\frac{|B|^{p/n}}{w_1(3B)}\int_{3B}|\nabla u|^p\, dw_1 + \frac{1}{v(B)}\int_{3B}|u|^p\, dv\right] \quad (4.38)$$

for $u \in \operatorname{Lip}(\overline{B})$, $B \subset B_0$. Next by Corollary 4.24 and Remark 4.27, given $u \in \operatorname{Lip}(\overline{B})$ there exists an extension Λu (which is locally Lipschitz) of u such that

$$\|\Lambda u\|_{L^p_v(3B)} \leq C\|u\|_{L^p_v(B)}$$

and

$$\|\nabla\Lambda u\|_{L^p_{w_1}(3B)} \leq C\|\nabla u\|_{L^p_{w_1}(B)}.$$

Moreover, we know that $\Lambda u \in \operatorname{Lip}(3\overline{B})$, so Λu satisfies (4.38). Thus (b) holds since

$$\|u\|_{L^{ph}_{w_2}(B)} = \|\Lambda u\|_{L^{ph}_{w_2}(B)}.$$

It is now easy to see that part (a) holds.

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