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REGULAR ORBITS OF NILPOTENT SUBGROUPS OF SOLVABLE GROUPS

WALTER CARLIP[1]

1. Introduction

Suppose that G is a solvable group, $\mathbf{F}$ a field, V a faithful $\mathbf{F}G$-module, and $A \subseteq G$ a nilpotent subgroup of G. The action of G permutes the vectors of V, and it is natural to ask under what conditions on G, A, $\mathbf{F}$, and V, the subgroup A is guaranteed to have a regular orbit on V. This question has been studied extensively and, if $\mathbf{F}$ has characteristic relatively prime to the order of A, definitive results have been obtained by T. R. Berger [2–8], B. Hargraves [16] and others. (See [9] for an overview of known results.)

When char($\mathbf{F}$) divides $|A|$, the picture is much less clear. P. Hall and G. Higman obtained the first related result in their renowned paper *On the p-length of p-soluble groups and reduction theorems for Burnside's problem* [15]. They show there that if A is a cyclic p-group, R an extraspecial r-group for some prime $r \neq p$, $\mathbf{F}$ a field of characteristic p that is also a splitting field for R, and G a group of the form $G = AR$, with A acting irreducibly and faithfully on $R/Z(R)$, then $V|_A$ *always* has a regular orbit. Although it appears to be very special, the Hall-Higman configuration is extremely important because it turns up regularly in minimal structures associated with many theorems.

The noncoprime configuration has also been studied by A. Espuelas. In [11] he studied the case in which A is a p-group and p = char $\mathbf{F}$. He proves the following theorem:

THEOREM 1.1 (Espuelas [11], p. 4). *Let G be a solvable group with $\mathcal{O}_p(G) = 1$ and let A be a p-subgroup of G, p a prime. Suppose that V is a faithful* $\mathbf{F}G$*-module with* char($\mathbf{F}$) $= p$. *If $p = 2$, assume that A is $Z_2 \wr Z_2$-free. Then V contains a regular A-orbit.*

In this paper, Espuelas asks whether his theorem can be extended to A nilpotent. It is the main purpose of this paper to extend Theorem 1.1 to allow

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A to be nilpotent under the additional hypothesis that G has odd order. In particular, we prove:

THEOREM 1.2. *Suppose that G is a solvable finite group and that p is a prime such that $\mathcal{O}_p(G) = 1$. Let A be a nilpotent subgroup of G and V a faithful* **F***G-module over a field* **F** *of characteristic p. Assume that $|G|$ and p are both odd.*

Then A has at least two regular orbits on V.

In view of the work of Hargraves and Espuelas, it seems likely that this theorem can be generalized to avoid restrictions on the order of the solvable group or the characteristic of the field. The following conjecture seems natural:

CONJECTURE 1.3. Suppose that G, A, **F** and V are, respectively, a finite group, a subgroup of G, a field of characteristic $p > 0$, and a faithful **F**G-module. Assume that (G, A, V) satisfies the following conditions:

(a) G is solvable;

(b) $\mathcal{O}_p(G) = 1$;

(c) A is nilpotent; and

(d) A involves no wreath product $Z_r \wr Z_r$ for $r = 2$ or r a Mersenne prime.

Then A has a regular orbit on V.

Remark 1.4. The involvement of a wreath product often creates difficulties in regular orbit theorems (see, e.g., Theorem 1.1 where the dihedral group $Z_2 \wr Z_2$ is explicitly forbidden). Thus, in particular, if G is permitted to be of even order Theorem 1.2 fails if wreath products are not banned. Consider, for example, that group $G = \mathbf{GL}(2, 3)$. It is easy to see that G contains a subgroup isomorphic to D_8 that has no regular orbits.

The methods used here are somewhat different than those used in the articles of Berger, Hargraves and Espuelas mentioned above, though the first step is similar: One argues inductively that the result follows from the quasiprimitive case. The reduction to the quasiprimitive case appears in §2, below.

The main argument in the quasiprimitive case is a counting argument, requiring a careful analysis of the size of the centralizers $C_V(a)$ of elements a of prime order in A and an investigation of the representations of a minimal counterexample. The proof of the main theorem begins in §5.

2. Preliminary reductions

To prove Theorem 1.2 we begin with a minimal counterexample. To that end we formulate the following hypothesis.

HYPOTHESIS 2.1. *Suppose that G, A, $\mathbf{F}$ and V are, respectively, a finite group, a subgroup of G, a field of odd characteristic $p > 0$, and a faithful $\mathbf{F}G$-module. Assume that (G, A, V) satisfies the following conditions:*

(a) *G is solvable of odd order;*
(b) *$\mathscr{O}_p(G) = 1$;*
(c) *A is nilpotent; and*
(d) *A has no regular orbit on V.*

By a *minimal triple* satisfying Hypothesis 2.1, we will mean a triple (G, A, V) that satisfies the hypothesis and for which $|G| + \dim_{\mathbf{F}}(V)$ is minimal and, among such triples, one for which $|A|$ is as small as possible.

We begin with two lemmas that constrain the possible complexity of the subgroup A and of the group G.

LEMMA 2.2. *Suppose that (G, A, V) is a minimal triple satisfying Hypothesis 2.1. Then A is generated by its elements of prime order.*

Proof. Suppose otherwise. Let $B \subseteq A$ be the subgroup of A that is generated by the elements of prime order in A. By hypothesis, $B \neq A$. Clearly (G, B, V) satisfies all of the hypotheses of Hypothesis 2.1 except possibly (d). Hence, by minimality of (G, A, V), the subgroup B has a regular orbit on V. Let v generate such an orbit. If $C_A(v) \neq 1$, then, by Cauchy's Theorem, $C_A(v)$ has an element a of prime order. But then $a \in B$, which contradicts the fact that v generates a regular B-orbit. □

LEMMA 2.3. *Suppose that (G, A, V) is a minimal triple satisfying Hypothesis 2.1. Then $G = AF(G)$ and $p \notin \pi(F(G))$.*

Proof. By minimality of the triple (G, A, V), it will suffice to prove that the triple $(AF(G), A, V)$ satisfies Hypothesis 2.1 and this in turn will be obvious once we show that $\mathscr{O}_p(AF(G)) = 1$.

Since $\mathscr{O}_p(G) = 1$, it follows that $F(G)$ is a p'-subgroup of G. (Otherwise, a Sylow p-subgroup of $F(G)$, being characteristic in $F(G)$, would be a normal p-subgroup of G.) Now, since G is solvable, $C_G(F(G)) \subseteq F(G)$. On the other hand,

$$[\mathscr{O}_p(AF(G)), F(G)] \subseteq \mathscr{O}_p(AF(G)) \cap F(G) = 1.$$

Consequently, $\mathscr{O}_p(AF(G))$ centralizes $F(G)$ and

$$\mathscr{O}_p(AF(G)) \subseteq \mathscr{O}_p(AF(G)) \cap F(G) = 1.$$

It is clear now that $(AF(G), A, V)$ satisfies Hypothesis 2.1 and, by minimality, $G = AF(G)$, as desired. □

Next, we turn to the structure of the representation of G on V and some consequences that restrict even more the possible structure of the group G. The argument given here is similar to that in [11].

LEMMA 2.4. *Suppose that (G, A, V) is a minimal triple satisfying Hypothesis 2.1. Then V is an irreducible* $\mathbf{F}G$*-module.*

Proof. First, we show for each $a \in A^{\#}$ that there exists an irreducible $\mathbf{F}G$-submodule $V(a)$ of V that satisfies two conditions:

(a) if $a \in A \cap F(G)$, then $\langle a \rangle$ acts nontrivially on $V(a)$; and

(b) if $a \notin F(G)$ then $[a, F(G)]$ acts nontrivially on $V(a)$.

Take $a \in A^{\#}$. If $a \in A \cap F(G)$, let $Q = \langle a \rangle$ and if $a \notin F(G)$, let $Q = [a, F(G)]$. Since G is solvable, $C_G(F(G)) \subseteq F(G)$ and if $a \notin F(G)$, then $Q = [a, F(G)] \neq 1$. Thus, in both cases $Q \neq 1$. Since $(|F(G)|, \mathrm{char}(\mathbf{F})) = 1$, Maschke's Theorem allows us to decompose $V|_{F(G)}$ as the direct sum of homogeneous components:

$$V|_{F(G)} = V_1 \oplus V_2 \oplus \cdots \oplus V_n.$$

For each i, let U_i be an irreducible $\mathbf{F}F(G)$-submodule of V_i. Then V_i contains every irreducible $\mathbf{F}F(G)$-submodule of V that is isomorphic to U_i. Since G acts faithfully on V, it follows (in both cases) that Q does not centralize all of the homogeneous components. Without loss of generality, we can assume that Q acts nontrivially on V_1. It follows from the homogeneity of V_1 that Q acts nontrivially on U_1. Let

$$X = \sum_{x \in A} V_1^x.$$

Since $G = AF(G)$, X is a $\mathbf{F}G$-submodule of V. Let W be an irreducible $\mathbf{F}G$-submodule of X. Since W is $F(G)$-invariant and every irreducible $\mathbf{F}F(G)$-submodule of W is conjugate to an irreducible $\mathbf{F}F(G)$-submodule of V_1, it follows that W contains an irreducible $\mathbf{F}F(G)$-submodule that is isomorphic to U_1. Consequently, Q acts nontrivially on W. We can now let $V(a) = W$.

Consider the $\mathbf{F}G$-module

$$U = \sum_{a \in A^{\#}} V(a).$$

Since U is the sum of irreducible $\mathbf{F}G$-modules, U is completely reducible.

Moreover, U is completely reducible as a module for $\overline{G} = G/C_G(U)$. Therefore $\mathcal{O}_p(\overline{G})$ centralizes each direct summand and thus centralizes U. This proves that $\mathcal{O}_p(\overline{G}) = 1$.

Suppose that $1 \neq a \in A \cap C_G(U)$. Then both $\langle a \rangle$ and $[a, F(G)]$ centralize U, contrary to our construction of U. Thus, $A \cap C_G(U) = 1$.

Now suppose that G is not faithful on U and denote by $\overline{A}$ the image of A in $\overline{G}$. Then $|\overline{G}| < |G|$ and we know from the minimality of (G, A, V), that $(\overline{G}, \overline{A}, V)$ cannot satisfy all of the conditions of Hypothesis 2.1. Clearly, the only possibility is that $\overline{A}$ has a regular orbit on U and hence A has a regular orbit on V, a contradiction. Thus, G is faithful on U and, again, by minimality of (G, A, V), we can conclude that $U = V$.

Since V is completely reducible, we can choose $\{a_i \mid a_i \in A\}$ so that

$$V = V(a_1) \oplus V(a_2) \oplus \cdots \oplus V(a_s).$$

For each $a \in A^{\#}$, let $K(a) = C_G(V(a))$.

If $s > 1$, then, for each i, $\dim(V(a_i)) < \dim(V)$ and $V(a_i)$ is a faithful, irreducible $G/K(a_i)$-module. Thus, $\mathcal{O}_p(G/K(a_i)) = 1$ and $(G/K(a_i), AK(a_i)/K(a_i), V(a_i))$ satisfies each condition of Hypothesis 2.1 except possibly (d). Hence, by minimality, $AK(a_i)/K(a_i)$ has a regular orbit on $V(a_i)$. Take $v_i \in V(a_i)$ such that $C_{AK(a_i)/K(a_i)}(v_i) = 1$. Then $C_A(v_1 + v_2 + \cdots + v_s) \subseteq K(a_1) \cap \cdots \cap K(a_s) = 1$, since G is faithful on V. Thus, $v_1 + \cdots + v_s$ generates a regular A-orbit, contrary to our choice of G and V. Therefore, $s = 1$, $V = V(a_1)$ and V is irreducible, as desired. □

Recall the following definition.

DEFINITION 2.5. An irreducible $\mathbf{F}G$-module V is said to be *quasiprimitive* if $V|_N$ is homogeneous for every $N \lhd G$.

In Lemma 2.12 we reduce our study to quasiprimitive modules. We will need the following well-known theorems about regular orbits under the "induced" action of a group (similar to representation of a group on an induced module). Proofs of these theorems can be found in Appendix A of [9]. First a definition:

DEFINITION 2.6. Suppose that A acts (on the right) on a set Ω with kernel $K = C_A(\Omega)$. Let

$$\Omega_A^G = \left\{f\colon G \to \Omega \mid f(ag) = f(g)^{a^{-1}} \text{ for each } a \in A \text{ and } g \in G\right\}.$$

The *induced action* on G on Ω_A^G is defined by

$$f^g(x) = f(xg^{-1}).$$

Remark 2.7. Notice that for each $k \in K$, we have $f(kg) = f(g)^{k^{-1}} = f(g)$ so the functions in Ω_A^G are constant on cosets $K \backslash G$. Moreover, each function $f \in \Omega_A^G$ is determined by its values on a (right) transversal to A in G, that is, a choice of representatives of the cosets $A \backslash G$.

THEOREM 2.8. *Suppose that $A \subseteq G$ and that A acts on a set Ω with kernel $K = C_G(\Omega)$. Assume that $|\Omega| > 1$. Let G act on Ω_A^G via the induced action. Then*

$$C_G(\Omega_A^G) = \bigcap_{x \in G} K^x = \operatorname{core}(K).$$

THEOREM 2.9. *Suppose that $A \subseteq G$ and that A acts on a set Ω with kernel $K = C_G(\Omega)$. Assume that $|\Omega| > 1$. Let G act on Ω_A^G via the induced action. Let $\{1 = x_0, x_1, \dots, x_{n-1}\}$ be a fixed right transversal of A in G.*

(a) *The map $f \mapsto (f(1), f(x_1), \dots, f(x_{n-1}))$ is a bijection from Ω_A^G to Ω^n.*

(b) *Let $\Theta\colon G \to S_{\Omega^n}$ be the action on Ω^n corresponding to the induced action of G on Ω_A^G and let*

$$\Phi\colon S_n \wr K \backslash A \to S_{\Omega^n}$$

be the natural action of $S_n \wr K \backslash A$ arising from the definition of the wreath product. Then there is a homomorphism $\Psi\colon G \to S_n \wr K \backslash A$ such that $\Phi \circ \Psi = \Theta$.

(c) $\ker(\Psi) = \operatorname{core}(K)$.

THEOREM 2.10. *Suppose that G is a solvable group and let $A \triangleleft G$ be a subgroup of G. Suppose W is an $\mathbf{F}A$-module and $V = \operatorname{ind}_A^G(W) \cong W_A^G$ the module described above. Let $K = C_A(W)$ and $C = \bigcap_{x \in G} K^x$. Suppose that $K \backslash A$ has a regular orbit on W. Then either*

(a) *$C \backslash G$ has a regular orbit on V, or*

(b) *G involves a wreath product $Z_r \wr Z_s$ for primes r and s.*

Furthermore, in case (b), *$|\mathbf{F}| - 1$ divides $|K \backslash A|$ and the primes r and s can be chosen so that r divides $|A \backslash G|$ and s is a prime divisor of $|\mathbf{F}| - 1$.*

Remark 2.11. Some cases of the theorem above are of special interest. If both char($\mathbf{F}$) and $|K \backslash A|$ are odd, then conclusion (a) always holds, as $|\mathbf{F}| - 1$ does not divide $|K \backslash A|$. If G is nilpotent, then G can never involve a wreath product $Z_r \wr Z_s$ for $r \neq s$ so one need only check for $Z_r \wr Z_r$.

LEMMA 2.12. *Suppose that (G, A, V) is a minimal triple satisfying Hypothesis* 2.1. *Then V is a quasiprimitive $\mathbf{F}G$-module.*

Proof. Let $N \lhd G$ and suppose $V|_N = V_1 \oplus \cdots \oplus V_n$ where the V_i are homogeneous components of V with respect to N. Assume, for a contradiction, that $n > 1$.

For each i, let $N_i = N_G(V_i)$. Then it follows from one form of Clifford's Theorem ([21], Theorem 8.1.3, p. 210) that V_i is an irreducible N_i-module and $V = V_i^G$. Moreover, each V_i is a faithful, irreducible $N_i/C_{N_i}(V_i)$ module, so $\mathcal{O}_p(N_i/C_{N_i}(V_i)) = 1$. Since $\dim(V_i) < \dim(V)$, the minimality of (G, A, V) tells us that

$$\left(N_i/C_{N_i}(V_i), (A \cap N_i)N_i/C_{N_i}(V_i), V_i\right)$$

cannot satisfy all the requirements of Hypothesis 2.1, and again only (d) of Hypothesis 2.1 can fail. Hence the image of $A \cap N_i$ in $N_i/C_{N_i}(V_i)$ has a regular orbit on V_i. That is,

(2.1) $N_A(V_i)C_{N_i}(V_i)/C_{N_i}(V_i) = N_A(V_i)/C_A(V_i)$ has a regular orbit on V_i.

The set $\Omega = \{V_1, \dots, V_n\}$ is permuted by A. Renumbering this set if necessary, let $\{V_1, \dots, V_t\}$ be representatives of the A-orbits on Ω and, for each $i = 1, \dots, t$, let

$$W_i = \sum_{a \in A} V_i^a,$$

i.e., the sum of the modules in the orbit of V_i. Then each W_i is an $\mathbf{F}N_G(W_i)$-module, $A \subseteq N_G(W_i)$, and $V = W_1 \oplus \cdots \oplus W_t$.

Suppose, for each $i \in \{1, 2, \dots, t\}$, that W_i has a regular $A/C_A(W_i)$-orbit and let $v_i \in W_i$ generate a regular orbit. Since A acts faithfully on V, the element $(v_1, v_2, \dots, v_t)$ generates a regular A-orbit, a contradiction. Therefore, for some index i, W_i has no regular $A/C_A(W_i)$-orbit. Without loss of generality, we can assume that W_1 has no regular $A/C_A(W_1)$-orbit.

If $A \subseteq N_1 = N_G(V_1)$ then $W_1 = V_1$ and, by (2.1), $A/C_A(W_1)$ has a regular orbit on W_1. Therefore, $A \not\subseteq N_G(V_1)$.

Let $A_0 = N_A(V_1)$. Since A is nilpotent, we can choose subgroups A_i such that the series below is part of a composition series of A:

$$N_A(V_1) = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{s-1} \subseteq A_s = A.$$

Thus, for each i, $A_{i-1} \lhd A_i$ and, since A is nilpotent, $[A_i : A_{i-1}] = p_i$ for certain primes $p_i \in \pi(A)$. For each i define the module U_i by

$$U_i = \sum_{a \in A_i} V_1^a,$$

so, for example,

$$U_s = \sum_{a \in A} V_1^a = W_1.$$

We will show by induction on i that, for each i, $A_i/C_{A_i}(U_i)$ has a regular orbit on U_i. For $i = s$ this says that $A_s/C_{A_s}(U_s) = A/C_A(W_1)$ has a regular orbit on W_1, contrary to our choice of W_1.

For $i = 0$, $U_0 = V_1$ and our conclusion says $N_A(V_1)/C_A(V_1)$ has a regular orbit on V_1, which is precisely equation (2.1). This starts the induction.

Assume that $A_i/C_{A_i}(U_i)$ has a regular orbit on U_i and choose a transversal $\{1 = x_0, x_1, x_2, \dots, x_{p_i-1}\}$ to A_i in A_{i+1}. We claim that

$$U_{i+1} \cong U_i^{A_{i+1}} \cong (U_i \otimes 1) \oplus (U_i \otimes x_1) \oplus \cdots \oplus (U_i \otimes x_{p_i-1}) = U_i \bigotimes_{\mathbf{F}A_i} A_{i+1}.$$

It suffices to check that the subspaces $U_i^{x_j}$ are pairwise disjoint, and this follows since G permutes the homogeneous components V_i and $N_A(V_1) \subseteq A_i$.

We now apply Theorem 2.10. Clearly, A_{i+1} is nilpotent and $A_i \lhd A_{i+1}$. Moreover, by induction, $A_i/C_{A_i}(U_i)$ has a regular orbit on U_i. Also, both $|A|$ and char($\mathbf{F}$) are odd. Hence (a) holds: $A_{i+1}/C_{A_{i+1}}(U_{i+1})$ has a regular orbit on U_{i+1}. We need only check that $C_{A_{i+1}}(U_{i+1}) = \bigcap_{x \in A_{i+1}} C_{A_i}(U_i)^x$, and this follows from Theorem 2.8 and Theorem 2.9.

This proves that $A/C_A(W_1)$ has a regular orbit on W_1, a contradiction. □

LEMMA 2.13. *Suppose that (G, A, V) is a minimal triple satisfying Hypothesis* 2.1. *Then every normal abelian subgroup of G is cyclic.*

Proof. This is an immediate consequence of Lemma 2.12. □

Remark 2.14. Lemma 2.13 requires only that V be faithful and quasiprimitive.

LEMMA 2.15. *Suppose that (G, A, V) is a minimal triple satisfying Hypothesis* 2.1. *Let $F = F(G)$, and, for each prime $r \in \pi(F)$, let F_r be the Sylow r-subgroup of F. Then F is the direct product of its Sylow subgroups. Moreover, for each $r \in \pi(F)$, $F_r \cong E_r \circ C_r$, where E_r is either an extraspecial r-group of exponent r or the identity group and C_r is cyclic. If $E_r \neq 1$, then the central product is taken by identifying $Z(E_r)$ with $\Omega_1(C_r)$.*

Proof. Clearly, F_r char G. Hence, each characteristic abelian subgroup of F_r is normal in G and, by Lemma 2.13, cyclic. Thus, F_r is of symplectic type and, by P. Hall's Theorem [14], Theorem 2.9, the result follows. □

Remark 2.16. Lemma 2.15 follows immediately from Philip Hall's Theorem and the fact that V is quasiprimitive: no other properties of the minimal triple are required.

3. Representation theory

By the reductions in §2, the Fitting subgroup of a minimal counterexample to Theorem 1.2 is a product of groups each of which is either cyclic, or the product of a cyclic group and an extraspecial group. The representation theory of extraspecial groups and of critical groups (defined below) are therefore essential to our analysis.

DEFINITION 3.1. A group G is called *critical* if it satisfies the following conditions:

(a) G has a normal, extraspecial r-subgroup R for some prime r;

(b) G has a cyclic, nonidentity r'-subgroup A such that $G = AR$ and $A \cap R = 1$;

(c) $[A, Z(R)] = 1$ and each nonidentity element $x \in A$ induces a fixed-point free automorphism on $R/Z(R)$.

The representation theory of critical groups is described in [17], Satz V.17.13, p. 574 and [18], Theorem IX.2.6, p. 422. We reproduce these theorems for reference, translating the first into the language of modules.

THEOREM 3.2 (Dade [17], Satz V.17.13, p. 574). *Suppose that $G = AR$ is a critical group, R is a normal, extraspecial r-group of order r^{2e+1} and $A = \langle a \rangle$ is a cyclic group. Let $\mathbf{K}$ be an algebraically closed field of characteristic p such that $(|G|, p) = 1$ and let V be an irreducible $\mathbf{K}G$-module on which R is represented faithfully. Then V is an irreducible $\mathbf{K}R$-module and $\dim_{\mathbf{K}}(V) = r^e$. Furthermore, either $|A|$ divides $r^e - 1$ and*

$$V|_A \cong s(\mathbf{K}[A]) \oplus W,$$

where $s = (r^e - 1)/|A|$ and W is an irreducible $\mathbf{K}A$-module, or $|A|$ divides $r^e + 1$ and

$$V|_A \cong s(\mathbf{K}[A]) \oplus (\mathbf{K}[A]/W),$$

where $s = ((r^e + 1)/|A|) - 1$ and W is an irreducible $\mathbf{K}A$-module.

THEOREM 3.3 (Hall-Higman [18], Theorem IX.2.6, p. 422). *Suppose that $G = AR$ is a critical group, R is a normal, extraspecial r-group of order r^{2e+1} and $A = \langle a \rangle$ is a cyclic group of order p^l. Let $\mathbf{K}$ be an algebraically closed field of characteristic p and let V be an irreducible $\mathbf{K}G$-module on which R is* *represented faithfully. Then V is an irreducible* $\mathbf{K}R$*-module and* $\dim_{\mathbf{K}}(V) = r^e$. *Furthermore,*

$$V|_A \cong s(\mathbf{K}[A]) \oplus P,$$

where P is indecomposable and either

(1) $s = (r^e - 1)/|A|$ *and* $\dim_{\mathbf{K}}(P) = 1$, *or*

(2) $s = ((r^e + 1)/|A|) - 1$ *and* $\dim_{\mathbf{K}}(P) = p^l - 1$.

The following fact is required in the proof of Lemma 5.3 below.

THEOREM 3.4. *Suppose that H is isomorphic to the direct product of Sylow subgroups, each of which is either cyclic or the central product of a cyclic group and an extraspecial group. Let W be a faithful, irreducible* $\mathbf{F}H$*-module. Let U be an irreducible $Z(H)$-submodule of W and* $f = \dim_{\mathbf{F}}(U)$ *and* $e^2 = |H/Z(H)|$. *Then*

(a) $\dim_{\mathbf{F}}(\mathrm{Hom}_{\mathbf{F}H}(W, W)) = f$, *and*

(b) $\dim_{\mathbf{F}}(W) = ef$.

Proof. See [9], Lemma 4.3.6. Slightly different versions of this theorem with varying amounts of proof can be found in [11] and [20], Lemma 4.3.7, p. 353. □

4. Quoted results

In this section we list several results from the literature that pertain to regular orbits. The first two theorems provide the primary motivation for the technique used in the proof of Theorem 1.2.

If A is a finite group and V is an $\mathbf{F}A$-module, then an element $v \in V$ generates a regular A-orbit precisely when v is *moved* by every element of A; thus, v generates a regular A orbit if v lies outside the centralizer of every element $a \in A$. Clearly, if v is fixed by an element $a \in A$, it is fixed by all powers of a, and, in particular, by an element of prime order in A. Thus, v generates a regular orbit if and only if it lies outside of the union of the centralizers in V of the elements of prime order in A. Of course, $C_V(a)$ is a subspace of V. Theorem 4.1 exploits this argument in a straight-forward way: Since an infinite vector space cannot be written as the union of proper subspaces, the union of the centralizers of elements of the finite group A cannot cover V. Theorem 4.2 uses the same idea, requiring a careful accounting of vectors centralized by elements of prime order in A. These two theorems appear with proof as Theorem 3.1.2 and Theorem 3.1.4 of [9].

THEOREM 4.1. *Suppose that A is any finite group and V is a faithful* $\mathbf{F}A$*-module with no regular A-orbits. Then* $\mathbf{F}$ *is finite.*

THEOREM 4.2. *Suppose that A is any finite group and V is a faithful $\mathbf{F}A$-module with no regular A-orbits. Assume that $|\mathbf{F}| = q = p^s$ and let m be the number of minimal subgroups of A. Then $m > q$ and $|A| > q$.*

The next result is a new theorem of A. Espuelas, the culmination of a long study of actions of groups on symplectic modules. If R is an extraspecial p-group, then $R/Z(R)$ has a natural structure as a symplectic space, with symplectic inner product given by the commutator operation in R (see, e.g., [1], (23.10), p. 109). If $R \lhd G$ and G centralizes $Z(R)$, then the action of G on $R/Z(R)$ induced by conjugation preserves the symplectic form on $R/Z(R)$, and $R/Z(R)$ becomes a symplectic module for G.

Espuelas' Theorem is crucial in the proof of Theorem 5.2, below. Since, by Theorem 2.15, the Fitting subgroup of a minimal counterexample to Theorem 1.2 is (usually) built up from extraspecial groups. Espuelas' Theorem provides a mechanism by which we are able to compare $|F(G)|$ and $|A|$.

THEOREM 4.3 (Espuelas [12], Theorem 1, p. 1). *Let G be a group of odd order that acts on an extraspecial r-group R, for r an odd prime. Suppose that $R/Z(R)$ is a faithful and completely reducible G-module. Then $R/Z(R)$ contains at least two regular G-orbits.*

If the Fitting subgroup of a minimal counterexample to Theorem 1.2 is abelian, we need an alternate route to the conclusion. The two theorems below, which appeared in the dissertation of A. Turull, [22], are used in Lemma 5.4 to analyze this situation. We begin with some required notation.

Notation 4.4. Let $\mathbf{K} = \mathbf{GF}(p^n)$ and $\mathbf{F} \subseteq \mathbf{K}$ the prime field of $\mathbf{K}$. Set $G = \mathrm{Gal}(\mathbf{K}/\mathbf{F}) \ltimes \mathbf{K}^{\#}$. Suppose that q is a prime that divides $|\mathrm{Gal}(\mathbf{K}/\mathbf{F})|$ and let Q be the subgroup of $|\mathrm{Gal}(\mathbf{K}/\mathbf{F})|$ of order q. Define

$$N = \left\{x \in \mathbf{K}^{\#} \mid \prod_{\sigma \in Q} x^{\sigma} = 1\right\} \text{ and } GN(1, p^n, q) = Q \ltimes N,$$

so $GN(1, p^n, q) \subseteq G$. We can view $\mathbf{K}$ as a one-dimensional vector space over the field $\mathbf{K}$. There is a natural action of G on $\mathbf{K}$, with the normal subgroup $\mathbf{K}^{\#}$ acting by field multiplication and $\mathrm{Gal}(\mathbf{K}/\mathbf{F})$ acting via its natural Galois automorphisms.

THEOREM 4.5 (Turull [22], Proposition 1.4, p. 52). *Let G be given above with its natural action on $\mathbf{K}$ and suppose that $A \subseteq G$. Then A has a regular orbit on $\mathbf{K}^{\#}$ if and only if for every prime q which divides* $|\mathrm{Gal}(\mathbf{K}/\mathbf{F})|$, *the group $GN(1, p^n, q)$ is not conjugate in G to a subgroup of A.*

THEOREM 4.6 (Turull [22], Proposition 1.2(6), p. 49). *If the group* $GN(1, p^n, q)$ *is nilpotent, then either* (a) $p = 2$ *or* (b) $q = 2$ *and* $GN(1, p^n, q)$ *has even order.*

5. The main result

In this final section we prove Theorem 1.2. The proof requires a careful counting argument reminiscent of the proofs of Theorem 4.1 and Theorem 4.2. In essence, these theorems show that if a finite group A acts faithfully on a vector space V and the vector space V is *large*, then it must contain a regular orbit. The following observation is crucial: A vector $v \in V$ generates a regular A-orbit if v is not centralized by any element of prime order in A. In Theorem 4.1 it is shown that a vector space over an infinite field cannot be written as the union of proper subspaces; hence the subspaces of fixed points of elements of prime order in A cannot cover V, thereby leaving a vector v that generates a regular orbit. The argument is sharpened in the proof of Theorem 4.2 by crudely estimating the number of elements centralized by the elements of prime order in A; if V is large enough, some vector is left over and generates a regular A-orbit.

We begin with a careful examination of the fixed points of elements of prime order in a nilpotent subgroup A of a group G satisfying the hypotheses of Theorem 1.2 and the properties of a minimal counterexample enumerated in §2. By comparing our count of such fixed points to $|V|$, we are able to show (in most cases) that at least one vector v remains and, therefore, that V has a regular A-orbit. Of course, if v generates a regular A-orbit, then $-v$ also generates a regular A-orbit and, since A has odd order, v and $-v$ lie in different orbits. Thus A has two regular orbits.

In our analysis we use a new theorem of A. Espuelas to produce regular orbits of sections of A on sections of $F(G)$ and thereby bound $|A|$ as a function of $|V|$. This effectively forces V to be *large enough* to have regular A-orbits. In one final configuration we apply two theorems of A. Turull that examine the natural action on a extension field $\mathbf{K}$ of the semidirect product of the Galois group, $\mathrm{Gal}(\mathbf{K}/\mathbf{F})$, with the multiplicative group, $\mathbf{K}^{\#}$.

Naturally, the proof of Theorem 1.2 requires several lemmas. To begin, we bound the sizes of the centralizers of elements of prime order in A.

LEMMA 5.1. *Suppose that G is a finite solvable group of odd order and V is a faithful, quasiprimitive* $\mathbf{F}G$ *module. Suppose that A is a nilpotent subgroup of G and $a \in A$ an element of prime order q. Then*

(a) $\dim(C_V(a)) \le (3/7)\dim(V)$ *and* $|C_V(a)| \le |V|^{3/7}$.

Moreover, if $a \in A \cap F(G)$, *then*

(b) $\dim(C_V(a)) \le (1/3)\dim(V)$ *and* $|C_V(a)| \le |\mathrm{V}|^{1/3}$.

Proof. Let $\mathbf{K} \supseteq \mathbf{F}$ be a finite extension of $\mathbf{F}$ to a splitting field for G and let $V_{\mathbf{K}} = V \otimes_{\mathbf{F}} \mathbf{K}$. Then

$$\dim_{\mathbf{K}}\left(C_{V_{\mathbf{K}}}(a)\right) = \dim_{\mathbf{F}}\left(C_V(a)\right). \tag{5.1}$$

Now, $V_{\mathbf{K}} = M_1 \oplus M_2 \oplus \cdots \oplus M_t$ for some Galois conjugate irreducible $\mathbf{K}G$-modules M_i (see, e.g., [18], Theorem VII.1.20, p. 23 and [19], Theorem 9.21, p. 154). Since the M_i are Galois conjugate, $\dim(C_{M_1}(a)) = \dim(C_{M_i}(a))$ for each i and hence it suffices to prove (a) and (b) with M_1 in place of V. Moreover, the Galois conjugacy implies that G is faithful on each summand M_i. Thus, for the purposes of this lemma, we may assume that $\mathbf{F}$ is a splitting field for G and $V = V_{\mathbf{K}}$.

We examine the two cases $a \notin F(G)$ and $a \in F(G)$ separately.

Case 1. a $\notin$ F(G)

Step 1. *There exists an a-invariant r-subgroup $Q \subseteq F(G)$, for some prime $r \neq q$, such that*

(a) *Q is either cyclic of prime order or extraspecial*;

(b) $[Q, \langle a\rangle] = Q$;

(c) $[\Phi(Q), \langle a\rangle] = 1$;

(d) $Z(Q) \lhd G$; *and*

(e) *the action of a on $Q/Z(Q)$ is fixed-point free.*

Proof. Suppose that $a \in C_A(\mathcal{O}_{q'}(F(G))$ and let A_q be the Sylow q-subgroup of A. Then $a \in C_{A_q}(\mathcal{O}_{q'}(F(G))$. Since $\mathcal{O}_q(F(G)) \lhd G$ and $G = AF(G), C_{A_q}(\mathcal{O}_{q'}(F(G)))\mathcal{O}_q(F(G)) \lhd G$. But $C_{A_q}(\mathcal{O}_{q'}(F(G)))\mathcal{O}_q(F(G))$ is a q-group, so $a \in C_{A_q}(\mathcal{O}_{q'}(F(G)))\mathcal{O}_q(F(G)) \subseteq F(G)$, a contradiction.

Thus, $[a, \mathcal{O}_r(F(G))] \neq 1$ for some prime $r \neq q$. Let $R = \mathcal{O}_r(F(G))$. By Remark 2.16 and Lemma 2.15, R has the form $R = E \circ C$, where either $E = 1$ or E is extraspecial and C is cyclic.

If $E = 1$, then $R = C$ and is cyclic. In this case take $Q = \Omega_1(C)$. By [14], Theorem 5.3.10, if a centralizes Q, then a centralizes C, contrary to our choice of R. Properties (a), (b), (c), (d), and, trivially, (e) follow immediately.

Next, assume that E is extraspecial. If $[\langle a\rangle, Z(R)] \neq 1$, then let $Q = Z(E) = \Omega_1(Z(R))$. Again, (a), (b), (c), (d), and (e) are immediate.

Finally suppose that E is extraspecial and $[\langle a\rangle, Z(E)] = 1$. Now, $E = \Omega_1(R)$, so $E \operatorname{char} R$. Let $Q = [\langle a\rangle, E]$. Since $Z(E) = \Omega_1(C)$, once again a centralizes C. Since $C = Z(R)$, we have $[\langle a\rangle, R] = [\langle a\rangle, E] = Q$. By [14], Theorem 5.3.6, $Q = [R, \langle a\rangle] = [[R, \langle a\rangle], \langle a\rangle] = [Q, \langle a\rangle]$ and Q satisfies (b).

Let $\bar{E} = E/Z(E)$. By [14], Theorem 5.3.15, $C_{\bar{E}}(\langle a\rangle) = C_E(\langle a\rangle)/Z(E)$. Since $[E, \langle a\rangle] \lhd E$, and $|Z(E)| = r$, it follows that $Z(E) \subseteq [E, \langle a\rangle]$. Thus,

$[\bar{E}, \langle a \rangle] = [E, \langle a \rangle]/Z(E)$ and [14], Theorem 5.3.5 and Theorem 5.2.3, imply that

$$(5.2)\qquad E = [E, \langle a \rangle] C_E(\langle a \rangle) \quad \text{and} \quad \bar{E} = \left[\bar{E}, \langle a \rangle\right] \times C_{\bar{E}}(\langle a \rangle).$$

It follows that $[C_E(\langle a \rangle), Q] \subseteq Z(E)$, and $[[C_E(\langle a \rangle), Q], \langle a \rangle] \subseteq [Z(E), \langle a \rangle] = 1$. Moreover, $[[\langle a \rangle, C_E(\langle a \rangle)], Q] = 1$, so by the Three-Subgroup Lemma,

$$(5.3)\qquad \left[Q, C_E(\langle a \rangle)\right] = \left[[Q, \langle a \rangle], C_E(\langle a \rangle)\right] = 1.$$

We claim that Q is extraspecial. If Q is abelian, then Q centralizes $[Q, \langle a \rangle] = Q$ and (5.2) and (5.3) imply that Q centralizes E. But then $Q \subseteq Z(E)$ and $[R, \langle a \rangle] = [Q, \langle a \rangle] \subseteq [Z(E), \langle a \rangle] = 1$, contrary to our choice of R. Thus, Q is not abelian. Since $Q/Z(E)$ is elementary abelian, $1 \subset Q' \subseteq \Phi(Q) \subseteq Z(E) \subseteq Z(Q)$. On the other hand, by (5.2) and (5.3), $Z(Q) \subseteq Z(E)$, and so $Q' = \Phi(Q) = Z(Q) = Z(E)$. It follows that Q is extraspecial. Since $Z(Q) = Z(E) = \Omega_1(Z(R))$, (*c*) and (*d*) follow. Finally, since $[Q, \langle a \rangle] = Q$, [14], Theorem 5.2.3, implies that $C_{Q/Z(Q)}(\langle a \rangle) = 1$. Thus $\langle a \rangle$ is fixed-point free on $Q/Z(Q)$, as desired. □

Step 2. $\dim(C_V(a)) \leq (3/7)\dim(V)$.

Proof. Take Q as in Step 1 and consider $V|_{\langle a \rangle Q}$. Let

$$(5.4)\qquad 0 \subset V_1 \subset \cdots \subset V_l = V$$

be an $\langle a \rangle Q$-composition series for V with quotients $\bar{V}_i = V_i/V_{i-1}$. Thus, each $\bar{V}_i$ is an irreducible $\langle a \rangle Q$-module.

By our choice of Q we know that $Z(Q) \lhd G$. Since V was obtained by tensoring a quasiprimitive module up to a splitting field, $V|_{Z(Q)}$ is the direct sum of Galois conjugate irreducible modules. Since G is faithful on V, $Z(G)$ is faithful on every irreducible summand of $V|_{Z(Q)}$. By the Jordan-Holder Theorem, these are the only irreducibles that can occur in $\bar{V}_i|_{Z(Q)}$, so

$$(5.5)\qquad Z(Q) \text{ is faithful on } \bar{V}_i \text{ for each } i.$$

Suppose now that Q is extraspecial of order r^{2e+1}. Since $\langle a \rangle$ is fixed-point free on Q and centralizes $Z(Q)$, the semidirect product $\langle a \rangle Q$ is a critical group. The faithful, irreducible representations of critical groups are described in Theorem 3.2 and Theorem 3.3. There are four cases depending upon whether q divides $r^e + 1$ or $r^e - 1$ and whether $q = p$ or $q \neq p$. In particular, if $q | r^e + 1$ and $q = p$, then

$$(5.6)\qquad \bar{V}_i|_{\langle a \rangle} = s(\mathbf{F}[\langle a \rangle]) \oplus P,$$

where the module P is the unique indecomposable $\mathbf{F}\langle a\rangle$-module of dimension $q-1$ and $s=((r^e+1)/q)-1$. If $q|r^e+1$ and $q\neq p$, then

$$\bar{V}_i|_{\langle a\rangle}=s(\mathbf{F}[\langle a\rangle])\oplus(\mathbf{F}[\langle a\rangle]/W), \tag{5.7}$$

where W is a 1-dimensional irreducible $\mathbf{F}\langle a\rangle$-module and $s=((r^e+1/q)-1$. And, finally, if $q|r^e-1$, then

$$\bar{V}_i|_{\langle a\rangle}=t(\mathbf{F}[\langle a\rangle])\oplus W, \tag{5.8}$$

where W is a 1-dimensional, irreducible module and $t=(r^e-1)/q$. Clearly, $\dim(C_{\mathbf{F}[\langle a\rangle]}(a))=1$, $\dim(C_P(a))=1$, $\dim(C_{\mathbf{F}[\langle a\rangle]/W}(a))\leq 1$, and $\dim(C_W(a))\leq 1$.

If q divides r^e+1, by (5.6) and (5.7), $\dim(C_{\bar{V}_i}(a))\leq s+1$, while $\dim(\bar{V}_i)=sq+q-1$. Thus,

$$\dim\left(C_{\bar{V}_i}(a)\right)\leq\frac{s+1}{sq+q-1}\dim\left(\bar{V}_i\right).$$

If $q\geq 5$, then

$$\dim\left(C_{\bar{V}_i}(\langle a\rangle)\right)\leq\frac{1}{q-1}\dim\left(\bar{V}_i\right)\leq\frac{1}{4}\dim\left(\bar{V}_i\right).$$

On the other hand, if $q=3$, we know that $s=(r^e+1)/3-1\neq 0$, for otherwise $r^e=2$, contrary to our hypothesis that $|G|$ is odd. Thus, $s\geq 1$ and

$$\dim\left(C_{\bar{V}_i}(\langle a\rangle)\right)\leq\frac{2}{5}\dim\left(\bar{V}_i\right).$$

Consequently, in both cases,

$$\dim\left(C_{\bar{V}_i}(\langle a\rangle)\right)\leq\frac{2}{5}\dim\left(\bar{V}_i\right)<\frac{3}{7}\dim\left(\bar{V}_i\right).$$

If q divides r^e-1, by (5.8), $\dim(C_{\bar{V}_i}(a))\leq t+1$, while $\dim(\bar{V}_i)=tq+1$. Thus,

$$\dim\left(C_{\bar{V}_i}(a)\right)\leq\frac{t+1}{tq+1}\dim\left(\bar{V}_i\right)=\frac{1}{q}\left(1+\frac{q-1}{tq+1}\right)\dim\left(\bar{V}_i\right).$$

Since $t\geq 1$, it follows that $tq+1>q-1$. If $q\geq 5$, then

$$\dim\left(C_{\bar{V}_i}(a)\right)\leq\frac{2}{5}\dim\left(\bar{V}_i\right).$$

On the other hand, if $q = 3$, since q and r are both odd and $q | r^e - 1$, we know that $t = (r^e - 1)/q \geq 2$. Thus,

$$\dim\left(C_{\overline{V}_i}(a)\right) \leq \frac{1}{3}\left(1 + \frac{2}{7}\right)\dim\left(\overline{V}_i\right) = \frac{3}{7}\dim\left(\overline{V}_i\right).$$

Again, in both cases,

$$\dim\left(C_{\overline{V}_i}(\langle a \rangle)\right) \leq \frac{3}{7}\dim\left(\overline{V}_i\right).$$

Now suppose that $U \subseteq V$ is an $\langle a \rangle$-submodule of V. Then

$$\begin{aligned} C_V(\langle a \rangle)/C_U(\langle a \rangle) &\cong C_V(\langle a \rangle)/U \cap C_V(\langle a \rangle) \\ &\cong C_V(\langle a \rangle) \oplus U/U \subseteq C_{V/U}(\langle a \rangle), \end{aligned}$$

and hence

$$\begin{aligned} \dim(C_V(\langle a \rangle)) &= \dim(C_V(\langle a \rangle)/C_U(\langle a \rangle)) + \dim(C_U(\langle a \rangle)) \\ &\leq \dim\left(C_{V/U}(\langle a \rangle)\right) + \dim(C_U(\langle a \rangle)). \end{aligned}$$

Clearly, we can extend this argument by induction to conclude that

$$(5.9)\quad \dim(C_V(\langle a \rangle)) \leq \sum_{i=1}^{l} \dim\left(C_{\overline{V}_i}(\langle a \rangle)\right) \leq \frac{3}{7}\sum_{i=1}^{l}\dim\left(\overline{V}_i\right) = \frac{3}{7}\dim(V).$$

Now consider the case that Q is cyclic. By Clifford's Theorem,

$$\overline{V}_i|_Q = W_1 \oplus W_2 \oplus \cdots \oplus W_s,$$

where the W_i are homogeneous components that are permuted transitively by $\langle a \rangle$. But, a has prime order so there are two possibilities: $s = q = \circ(a)$ or $s = 1$. Since $\mathbf{F}$ is a splitting field for G, if $s = 1$, then Q acts on $\overline{V}_i$ by scalars. Thus, $Q = [\langle a \rangle, Q] \subseteq C_{\langle a \rangle Q}(\overline{V}_i)$, which contradicts (5.5).

Suppose, on the other hand, that $s = q$. Then $v \in C_{\overline{V}_i}(\langle a \rangle)$ if and only if $v = w + w^a + w^{a^2} + \cdots + w^{a^{q-1}}$. Thus, $C_{\overline{V}_i}(\langle a \rangle) \cong W_1$, and, clearly, $\dim(C_{\overline{V}_i}(\langle a \rangle)) = (1/q)\dim(\overline{V}_i) \leq (1/3)\dim(\overline{V}_i)$. Now the same argument as above shows that

$$(5.10)\qquad \dim(C_V(\langle a \rangle)) \leq (1/q)\dim(V) \leq (1/3)\dim(V).$$

Since $1/3 < 3/7$, combining (5.9) and (5.10) yields the desired result. □

Case 2. $a \in F(G)$.

Let Q be the Sylow q-subgroup of $F(G)$. Again, by Remark 2.16 and Lemma 2.15 can write $Q = E \circ C$ where either $E = 1$ or E is extraspecial and C is cyclic. Moreover, $V|_Q$ is the direct sum of Galois conjugate irreducible $\mathbf{F}Q$-submodules. Let W be one such irreducible $\mathbf{F}Q$-summand of $V|_Q$.

First, suppose that $a \in Z(Q) = C$. We claim that a is fixed-point free on W. Otherwise, $C_W(a)$ is a nontrivial $\mathbf{F}Q$-subspace of W, and, since W is irreducible, $W = C_W(a)$. But, if a centralizes W, then it also centralizes each Galois conjugate of W, and hence a centralizes V. But A is faithful on V, a contradiction. Now, since V is the direct sum of Galois conjugates of W, it follows that $C_V(a) = 0$. Obviously, this satisfies the lemma.

Now, suppose that $a \notin Z(Q)$. Then $E \neq 1$ and Q is the central product of an extraspecial group and a cyclic group. We claim that in this situation, $W|_{\langle a \rangle} \cong s\mathbf{F}[\langle a \rangle]$, i.e, a sum of copies of the regular $\mathbf{F}\langle a \rangle$-module. Since $(\mathrm{char}(\mathbf{F}), |Q|) = 1$, this is an easy exercise in ordinary character theory. By [10], Theorem 31.5, p. 181, the character of a faithful, irreducible, representation of Q vanishes outside of $Z(Q)$. Consequently, if χ is the character of $W|_{\langle a \rangle}$ and ρ the character of the regular $\mathbf{F}\langle a \rangle$-module, then

$$\chi = \frac{\chi(1)}{\rho(1)}\rho = \frac{\chi(1)}{|\langle a \rangle|}\rho.$$

Hence, our claim holds with $s = \chi(1)/|\langle a \rangle|$.

It now follows that

$$\dim(C_W(\langle a \rangle)) = (1/q)\dim(W),$$

and, since W was an arbitrary summand of V,

$$\dim(C_V(\langle a \rangle)) = (1/q)\dim(V).$$

Finally, since $q \geq 3$, it follows that

$$\dim(C_V(\langle a \rangle)) = (1/3)\dim(V) \leq (3/7)\dim(V).$$

This completes the proof of the lemma. □

In the next two lemmas, we will complete the analysis in the case that $F(G)$ is not abelian. We have already bounded the size of the centralizer of an element of prime order in A by a function of $|V|$. In the next two lemmas, under the additional hypothesis that $F(G)$ is not abelian, we bound $|A|$ as a function of $|V|$. The first of these, Lemma 5.2, expresses a relationship between $|A|$ and $|F(G)|$. The second, Lemma 5.3 then describes a relationship between $|V|$ and $|F(G)|$ and completes the proof of Theorem 1.2 in the case that $F(G)$ is not abelian.

Lemma 5.2 makes critical use of a new result of A. Espuelas [12], Theorem 1 (reproduced here as Theorem 4.3) to find regular orbits of A on certain sections of the Fitting subgroup of G. His result is a generalization of an earlier result, [13], Theorem A, in which he examines the action of solvable subgroups of symplectic groups on their natural symplectic spaces.

LEMMA 5.2. *Suppose that G is a finite solvable group of odd order and V is a faithful, quasiprimitive* $\mathbf{F}G$ *module. Suppose that A is a nilpotent subgroup of G such that $G = AF(G)$ and such that $F(G)$ is not abelian. Let $e^2 = |F(G)/Z(F(G))|$.*

Then:

(a) *if $A \subseteq F(G)$, then $|A| \leq e^2|Z(F(G))|$;*
(b) *if $A \nsubseteq F(G)$ and $Z(F(G))$ has no cyclic Sylow subgroups, then $|A| \leq (1/6)e^4|Z(F(G))|$; and*
(c) *if $A \nsubseteq F(G)$ and $Z(F(G))$ has a cyclic Sylow subgroup, then $|A| \leq (1/36)e^4|Z(F(G))|^2$.*

Proof. For each prime $r \in \pi(F(G))$, let F_r be the Sylow r-subgroup of $F(G)$. By Remark 2.16, we can apply Lemma 2.15. Thus, for each $r, F_r = E_r \circ Z(F_r)$, where $Z(F_r)$ is cyclic and either $E_r = 1$ or E_r is an extraspecial r-group of exponent r. Let H be the product of the nonabelian Sylow subgroups of $F(G)$ and C the product of the cyclic Sylow subgroups of $F(G)$, so $F(G) = H \times C$. Since by hypothesis $F(G)$ is not abelian, $H \neq 1$. Let $e_r^2 = |E_r/Z(E_r)| = |F_r/Z(F_r)|$. Since $F(G)$ is nilpotent, $F(G)$ is the direct product of the Sylow r-subgroups F_r, and hence

$$e^2 = \prod_{r \in \pi(F(G))} e_r^2 = \prod_{r \in \pi(H)} e_r^2.$$

Fix a prime $r \in \pi(H)$ and let $\overline{E}_r = E_r/Z(E_r)$. Let $A_{r'} = \mathscr{O}_{r'}(A)$ and consider the action of $A_{r'}$ on $\overline{E}_r$. Clearly $A_{r'}/C_{A_{r'}}(\overline{E}_r)$ is faithful on $\overline{E}_r$ and since $(r', r) = 1$, Maschke's Theorem implies that $\overline{E}_r$ is completely reducible. By Theorem 4.3, $\overline{E}_r$ contains at least two regular $A_{r'}/C_{A_{r'}}(\overline{E}_r)$-orbits. Consequently,

$$\left|A_{r'}/C_{A_{r'}}(\overline{E}_r)\right| \leq \frac{1}{2}|\overline{E}_r| = \frac{1}{2}e_r^2. \tag{5.11}$$

Since A is nilpotent, it is the direct product of its Sylow subgroups and, for each $r \in \pi(H)$, we define the projections π_r: $A \to A_{r'}$. We also define the quotient maps $\theta_r\colon A_{r'} \to A_{r'}/C_{A_{r'}}(\bar{E}_r)$. Finally, let $\psi_0\colon A \to A/C_A(C)$. Combining these homomorphisms, we produce a map

$$\Psi\colon A \to A/C_A(C) \times \prod_{r\in\pi(H)} A_{r'}/C_{A'_r}(\bar{E}_r) \tag{5.12}$$

given by $\Psi = \psi_0 \times \prod_{r\in\pi(H)}\psi_r$, where $\psi_r = \theta_r \circ \pi_r$.

Let $K = \mathrm{Ker}(\Psi)$ and suppose that $a \in K$. For each prime $r \in \pi(K)$, let K_r be the Sylow r-subgroup of K. We claim that $a \in C_A(C) \cap F(G)$. Clearly, it is sufficient to prove this for elements of prime-power order, so we assume that $a \in K_r$ for some prime r. First note that since $a \in \ker(\Psi)$, we know $a \in \ker(\psi_0)$, and therefore $a \in C_A(C)$. Moreover, $a \in \ker(\psi_q)$ for each $q \in \pi(H)$. If $r \neq q$, this implies that $a \in C_{A_{q'}}(\bar{E}_q)$. But since $(r, q) = 1$, [14], Theorem 5.1.4, Burnside's theorem, implies that $a \in C_A(E_q)$. Since $q \in \pi(H)$, E_q is extraspecial and therefore a centralizes $Z(E_q) = \Omega_1(Z(F_q))$. By [14], Theorem 5.3.10, a centralizes $Z(F_q)$. Thus, $a \in C_A(F_q)$. This proves that a centralizes F_q for each $q \neq r$.

Now consider the group $K_rF(G)$. Since a was an arbitrary element of K_r, it follows from the previous paragraph that K_r centralizes each Sylow q-subgroup of $F(G)$ for $q \neq r$. Hence, $K_rF(G)$ is nilpotent. Moreover, since K_r char $K \lhd A$, it follows that $K_rF(G) \lhd AF(G) = G$. Thus, $K_rF(G) \subseteq F(G)$, and therefore, $a \in K_r \subseteq C_A(C) \cap F(G)$. This proves the claim.

Conversely, it is clear that $C_A(C) \cap F(G) \subseteq \ker(\Psi)$, and therefore,

$$\ker(\Psi) = C_A(C) \cap F(G).$$

It now follows from (5.11), (5.12) and the fact that $F(G)$ is not abelian that

$$|A/(C_A(C) \cap F(G))| \leq \prod_{r\in\pi(H)} \frac{1}{2}e_r^2 \frac{|A|}{|C_A(C)|} \leq \frac{1}{2}e^2\frac{|A|}{|C_A(C)|}. \tag{5.13}$$

We are now in a position to prove (a), (b), and (c). Suppose first that $A \subseteq F(G)$. Then

$$|A| \leq |F(G)| = |F(G)/Z(F(G))|\,|Z(F(G))| = e^2|Z(F(G))|.$$

This proves (a).

Next, suppose $A \not\subseteq F(G)$ and $C = 1$. Then

$$A = C_A(C) \quad\text{and}\quad A/(C_A(C) \cap F(G)) = A/(A \cap F(G))$$

. Moreover, $A \cap F(G) \neq F(G)$, as otherwise $F(G) \subseteq A$, so $G = AF(G) = A$ and hence $A \subseteq F(G)$, contrary to assumption. Thus $A \cap F(G)$ is a proper subgroup of $F(G)$ and, since $F(G)$ has odd order, $[F(G): A \cap F(G)] \geq 3$. Now, (5.13) yields

$$|A| = \frac{|A|}{|A \cap F(G)|}|A \cap F(G)| \leq \frac{1}{2}e^2|A \cap F(G)|$$
$$\leq \frac{1}{6}e^2|F(G)| \leq \frac{1}{6}e^4|Z(F(G))|.$$

This proves (b).

Finally, assume that $A \not\subseteq F(G)$ and $|C| > 1$. Again, $A \cap F(G)$ is a proper subgroup of $F(G)$, so certainly $C_A(C) \cap F(G)$ is a proper subgroup of $F(G)$. Consequently,

$$|C_A(C) \cap F(G)| \leq |A \cap F(G)| \leq (1/3)|F(G)| = (1/3)e^2|Z(F(G))|.$$

Now, by (5.13),

$$|A|\frac{|A|}{|C_A(C) \cap F(G)|}|C_A(C) \cap F(G)| \leq \frac{1}{6}e^4\frac{|A|}{|C_A(C)|}|Z(F(G))|.$$

Now, $A/C_A(C)$ is isomorphic to a subgroup of $\mathrm{Aut}(C)$ of odd order. Since C is nontrivial, $\mathrm{Aut}(C)$ has even order, and hence $|A/C_A(C)| \leq (1/2)|\mathrm{Aut}(C)| < (1/2)|C|$. Finally, since $F(G)$ has odd order and is not abelian, $|C| \leq (1/3)|Z(F(G))|$. Combining these results yields

$$|A| < \frac{1}{36}e^4|Z(F(G))|^2.$$

This proves (c) and completes the proof of the lemma. □

LEMMA 5.3. *Suppose that G is a finite solvable group of odd order and V is a faithful, quasiprimitive* $\mathbf{F}G$ *module. Suppose that A is a nilpotent subgroup of G such that $G = AF(G)$ and $F(G)$ is not abelian. Then A has a regular orbit on V.*

Proof. Let W be a faithful, irreducible $F(G)$-submodule of V and U an irreducible $Z(F(G))$-submodule of W. Set $f = \dim_{\mathbf{F}}(U)$ and $e^2 = |F(G)/Z(F(G))|$. We claim that

(5.14) $$|Z(F(G))| \text{ divides } (|U| - 1).$$

We give a quick proof of this fact. First, since V is quasiprimitive, $Z(F(G))$ is faithful on U. Let $\mathbf{K} = \mathrm{Hom}_{\mathbf{F}Z(F(G))}(U)$. By Schur's Lemma and Wedderburn's Theorem on finite division rings we know that $\mathbf{K}$ is a field, and we can view U

as a $\mathbf{K}Z(F(G))$-module. As such, U is irreducible and faithful. Since $\mathbf{K}$ is a splitting field for $Z(F(G))$, it follows that $\dim_{\mathbf{K}}(U) = 1$, and hence $Z(F(G))$ is isomorphic to a subgroup of $\mathbf{K}^{\#}$. Clearly, $|U| = |\mathbf{K}|$, and therefore, $|Z(F(G))|$ divides $|U| - 1$, as desired.

It follows from (5.14) that if $|Z(F(G))| \geq 5$, then $|U| - 1 \geq 5$, and, since $|U|$ is a power of the characteristic of $\mathbf{F}$, $|U| \geq 7$. Similarly, if $|Z(F(G))| = 3$, then it follows from (5.14) that $|U| \geq 4$ and $|U| - 1$ is divisible by 3. Again, since $|U|$ must be a power of an odd prime, $|U| \geq 7$. Since $|Z(F(G))| \geq 3$ in any case, we have

$$|U| \geq 7. \tag{5.15}$$

Moreover, by Lemma 3.4,

$$\dim_{\mathbf{F}}(W) = ef = e \dim_{\mathbf{F}}(U) \quad \text{and} \quad |W| = |U|^e. \tag{5.16}$$

Finally, since $|Z(F(G))|$ and U are both odd, (5.14) implies that

$$|Z(F(G))| < \frac{|U| - 1}{2} < \frac{|U|}{2}. \tag{5.17}$$

We now turn to the counting technique described in §4. Recall that every element of V that is centralized by no minimal subgroup of A must generate a regular A-orbit. Thus, to show that A has a regular orbit on V, it is sufficient to show that

$$|V| - \left| \bigcup_{a \in \tilde{A}} C_V(a) \right| > 0, \tag{5.18}$$

where $\tilde{A}$ contains one generating element from every minimal subgroup of A. In particular, every element $a \in \tilde{A}$ has prime order.

In order to show (5.18), we break the analysis into three cases corresponding to the three parts of Lemma 5.2.

Case 1. $A \subseteq F(G)$.

If $A \subseteq F(G)$, then by (5.17) and Lemma 5.2(a)

$$|A| \leq e^2 |Z(F(G))| \leq \frac{1}{2} e^2 |U|.$$

By Lemma 5.1, since $A \subseteq F(G)$, we know $|C_V(a)| \leq |V|^{1/3}$ for every $a \in A$.

Since A has odd order, we know $|\tilde{A}| < |A|/2$ and, by (5.16), $|V| \geq |W| = |U|^e$. Thus,

$$|V| - \left| \bigcup_{a \in \tilde{A}} C_V(a) \right| \geq |V| = \frac{1}{2}|A|\,|V|^{1/3} \geq |V| - \frac{1}{4}e^2|U|\,|V|^{1/3},$$

and to prove (5.18) it is sufficient to show that

$$|U|^{(2e/3)-1} - \frac{1}{4}e^2 > 0. \tag{5.19}$$

Now, by (5.15), $|U| \geq 7$, and so $|U|^{(2e/3)-1} \geq 7^{(2e/3)-1}$. Moreover, since $F(G)$ has odd order and is not abelian, $e \geq 3$. It is an easy exercise in differential calculus to prove that $f(x) = 7^{(2x/3)-1} - (1/4)x^2 > 0$, for all $x \geq 3$. This verifies (5.19), and hence (5.18) in this case.

Case 2. *$A \not\subseteq F(G)$ and $Z(F(G))$ has no cyclic Sylow subgroups.*
If $A \not\subseteq F(G)$ and $Z(F(G))$ has no cyclic Sylow subgroups, then by (5.17) and Lemma 5.2(b)

$$|A| \leq \frac{1}{6}e^4|(F(G))| \leq \frac{1}{12}e^4|U|.$$

This time, Lemma 5.1 tells us that $|C_V(a)| \leq |V|^{3/7}$ for every $a \in A$. By the argument presented in Case 1, to prove (5.18) it is sufficient to show that

$$|V| - \frac{1}{2}|A|\,|V|^{3/7} > 0.$$

Using our bound on $|A|$ and the fact that $|V| \geq |U|^e$, it is sufficient to show that

$$|U|^{4e/7-1} - \frac{1}{24}e^4 > 0.$$

Since $|U| \geq 7$, this reduces to showing $7^{4e/7-1} - (1/24)e^4 > 0$. In this case, a brief detour through differential calculus shows that $f(x) = 7^{4x/7-1} - (1/24)x^4 > 0$ whenever $x \geq 3$. Since $F(G)$ is not abelian, $e > 1$, and is an odd integer, this inequality completes the proof of Case 2.

Case 3. *$A \not\subseteq F(G)$ and $Z(F(G))$ has a cyclic Sylow subgroup.*
If $A \not\subseteq F(G)$ and $Z(F(G))$ has a cyclic Sylow subgroup, then by (5.17) and Lemma 5.2(c),

$$|A| \leq \frac{1}{36}e^4|Z(F(G))|^2 \leq \frac{1}{144}e^4|U|^2.$$

Again, Lemma 5.1 yields $|C_V(a)| \leq |V|^{3/7}$ for every $a \in A$. As in the previous cases, it is sufficient to show that

$$|V| - \tfrac{1}{2}|A|\,|V|^{3/7} > 0,$$

and this time, using our bound on $|A|$, it is enough to show that

$$|U|^{4e/7-2} - \frac{1}{288}e^4 > 0.$$

Since $|U| \geq 7$, this reduces to showing $7^{4e/7-2} - (1/288)e^4 > 0$. As before, we rely on differential calculus to verify that $f(x) = 7^{4x/7-2} - (1/288)x^4 > 0$ whenever $x \geq 3$. Since $F(G)$ is not abelian, $e > 1$, and is an odd integer, this inequality completes the proof of Case 3 and of the lemma. □

If the Fitting subgroup of a minimal counterexample to Theorem 1.2 is abelian, then by Theorem 2.15 it is cyclic. In the next lemma we show in this case that $V|_{F(G)}$ is irreducible. It follows that the $\mathbf{F}G$-module V can be viewed as a $\mathbf{K}G$-module, where $\mathbf{K} = C_{\mathrm{End}_{\mathbf{F}}(V)}(F(G)) = \mathrm{Hom}_{\mathbf{F}F(G)}(V)$. Since $\mathbf{K}$ is a splitting field for $F(G)$, $V_{\mathbf{K}}$ (the module V viewed as a $\mathbf{K}G$-module) is one-dimensional and its structure precisely determined. This one-dimensional configuration was examined by A. Turull in his dissertation [22], and he gives there explicit criteria for the existence of a regular orbit. The two relevant theorems are reproduced in §4 as Theorem 4.5 and Theorem 4.6.

LEMMA 5.4. *Suppose that G is a finite solvable group of odd order and V is a faithful, quasiprimitive* $\mathbf{F}G$ *module. Suppose that A is a nilpotent subgroup of G such that $G = AF(G)$ and $F(G)$ is abelian. Then A has a regular orbit on V.*

Proof. Let $H = F(G)$. By hypothesis, $H = Z(H)$ and by Lemma 2.13 and Remark 2.14, H is cyclic. As in Lemma 5.3, let W be a faithful, irreducible $\mathbf{F}H$-submodule of V. We claim that in this situation $V = W$, i.e., $V|_H$ is irreducible.

As in Lemma 5.1, let $\mathbf{K} \supset \mathbf{F}$ be a finite extension of $\mathbf{F}$ to a splitting field for G and let $V_{\mathbf{K}} = V \otimes_{\mathbf{F}} \mathbf{K}$. By Theorem 4.1, we may assume that $\mathbf{F}$ is finite, and hence, by [19], Theorem 9.21, p. 154, the Schur index is one. Thus, [18], Theorem VII.1.18, p. 21 implies that $V_{\mathbf{K}} \cong W_1 \oplus W_2 \oplus \cdots \oplus W_t$, where the W_i are nonisomorphic, irreducible, Galois conjugate $\mathbf{K}G$-modules. Since H is faithful on $V_{\mathbf{K}}$ and the modules W_i are Galois conjugates, H is faithful on each W_i.

Let U be a Clifford component of $W_1|_H$. By Clifford's Theorem, W_1 is the direct sum of the Clifford components, which are permuted by G. Since H is cyclic, $C_H(U) \operatorname{char} H \lhd G$, and hence, H is faithful on U. Since $\mathbf{K}$ is a splitting field for G and U is homogeneous, H acts on U by scalar multiplication. Now suppose that $g \in N_G(U)$. Then, for every $h \in H$, the commutator $[g,h]$ acts trivially on U. But $H \lhd G$, so $[g,h] \in H$ and, since H is faithful on U, we discover that $[g,h] = 1$. But then, $g \in C_G(H)$. Now, since $H = F(G)$ and G is solvable, [14], Theorem 6.1.3, implies that $g \in H$. Thus, $N_G(U) = H$. Consequently, [21] Theorem 8.1.3(iv) tells us that U is an irreducible $\mathbf{K}H$-module and hence one-dimensional. Moreover, it is immediate from the Clifford decomposition that $W_1 = U^G$. Since the modules W_i are all nonisomorphic, U is not a summand of W_i for $i \neq 1$. We can now conclude that $V_{\mathbf{K}}|_H$ is the direct sum of nonisomorphic, irreducible, faithful, one-dimensional $\mathbf{K}H$-modules.

On the other hand, if $V|_H$ is not irreducible, then by Maschke's theorem $V|_H$ can be decomposed into a nontrivial direct sum of irreducible $\mathbf{F}H$-submodules. Since $V|_H$ is homogeneous, these summands are isomorphic, and we can write $V|_H = sW$, where W is an irreducible $\mathbf{F}H$-module and $s > 1$. But then $V_{\mathbf{K}} = sW_{\mathbf{K}}$, where $W_{\mathbf{K}} = \mathbf{K} \otimes_{\mathbf{F}} W$. Clearly, an irreducible $\mathbf{K}H$-submodule of $sW_{\mathbf{K}}$ has multiplicity $s > 1$, a contradiction. Thus, $V|_H$ is irreducible.

Now, by hypothesis $G = AH$ and, by Remark 2.16, G satisfies the conclusion of Lemma 2.15. We can now follow the procedure described in §4.3 of [9] to produce a faithful $\mathbf{K}H$-module M and a homomorphism $\sigma\colon G \to \mathrm{Gal}(\mathbf{K}/\mathbf{F})$ having kernel $C_G(Z(H)) = C_G(H)$ such that the action of H on M extends to a semilinear action of G on M (with respect to σ), $M \cong V$ as G-sets and $M_{\mathbf{F}} \cong V$. Since $(M|_{\mathbf{H}})_{\mathbf{F}} = (M_{\mathbf{F}})|_H$, it follows that $(M|_H)_{\mathbf{F}}$ is irreducible and therefore $M_{\mathbf{H}}$ is irreducible. Moreover, $\mathbf{K}$ is a splitting field for $Z(H) = H$ and so M is a faithful, one-dimensional $\mathbf{K}H$-module. Therefore, H is isomorphic to a subgroup of $\mathbf{K}^{\#}$ and $M \cong \mathbf{K}$ with the action of H on M given by field multiplication. Furthermore, since $C_G(H) = H$, it follows that G/M is isomorphic to a subgroup of $\mathrm{Gal}(\mathbf{K}/\mathbf{F})$. (See also [3], (2.1), p. 515 and [22], Proposition 2.1, p. 55.)

We can summarize the previous paragraph by saying that G is isomorphic to a subgroup of $\mathrm{Gal}(\mathbf{K}/\mathbf{F}) \ltimes \mathbf{K}^{\#}$ and M is isomorphic to $\mathbf{K}$, and the action of G on M is the natural action on the one-dimensional vector space $\mathbf{K}$, i.e., with the normal subgroup $\mathbf{K}^{\#}$ acting by field multiplication and $\mathrm{Gal}(\mathbf{K}/\mathbf{F})$ acting as Galois automorphisms. The group $\mathrm{Gal}(\mathbf{K}/\mathbf{F}) \ltimes \mathbf{K}^{\#}$ has been studied by A. Turull in [22].

By Theorem 4.5, A has a regular orbit on $\mathbf{K}$ unless there is a prime q such that A contains a subgroup conjugate in $\mathrm{Gal}(\mathbf{K}/\mathbf{F}) \ltimes \mathbf{K}^{\#}$ to $GN(1, p^n, q)$. (See §4 for the definition of the group $GN(1, p^n, q)$.) But, by Theorem 4.6, $GN(1, p^n, q)$ is not nilpotent unless either $\mathrm{char}(\mathbf{F}) = p = 2$ or $q = 2$, in which case $|GN(1, p^n, q)|$ is even. Thus, it follows from Turull's analysis that in our situation A always has a regular orbit on $\mathbf{K}$ and hence A has a regular orbit on V. □

Proof of Theorem 1.2. We argue first that A has at least one regular orbit on V. To this end, assume that (G, A, V) is a minimal triple satisfying

Hypothesis 2.1. Clearly, $F(G)$ is either abelian or not abelian. By the reductions stated in §2, G, A, and V satisfy the hypotheses of Lemma 5.3 and Lemma 5.4. Thus if $F(G)$ is not abelian, by Lemma 5.3, A has a regular orbit on V, a contradiction. On the other hand, if $F(G)$ is abelian, Lemma 5.4 provides a contradiction.

It follows that under the hypotheses of the theorem, A has at least one regular orbit. Now suppose that $v \in V$ generates a regular A-orbit. Since char(**F**) is odd, $1 \neq -1 \in \mathbf{F}$ and, clearly, the vector $-v$ also generates a regular orbit. Moreover, since $|A|$ is also odd, -1 is not an eigenvalue for any element of A. Thus, v and $-v$ generate disjoint orbits and A has at least two regular orbits on V. □

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Ohio University
Athens, Ohio