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A NOTE ON UNCONDITIONAL STRUCTURES IN WEAK HILBERT SPACES

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Introduction

This note is to be considered as an addendum of [3], where an extensive study of Banach lattices, which are weak Hilbert spaces, was made. We prove that if a non-atomic separable Banach lattice is a weak Hilbert space then it is lattice isomorphic to $L_2(0,1)$. The result is a consequence of Theorem 3.11 in [3] together with an easy, short argument.

We believe that it may have some impact on the study of unconditional structures in weak Hilbert spaces.

For the convenience of the reader, in Section 1 we have given the definition of a weak Hilbert space and formulated the special case of [3], Theorem 3.11, which is needed to prove our result.

1. Notation and terminology

In this note we shall use the notation and terminology commonly used in Banach space theory as it appears in [1] and [2].

If E is a finite dimensional Banach space we denote the Banach-Mazur distance between E and the Hilbert space by $d(E)$, and if (x_j) is a sequence in a Banach space X we let $[x_j]$ denote the closed linear span of (x_j). Given a finite set A we let $|A|$ denote the cardinality of A. One of the many equivalent characterizations of a weak Hilbert space given by Pisier [4] we use as the definition.

DEFINITION 1. *A Banach space X is called a weak Hilbert space, if there exist a $\delta > 0$ and a $C \geq 1$ such that for every finite dimensional subspace $E \subseteq X$, there exist a subspace $F \subseteq E$ and a projection $P : X \to F$ with* $\dim F \geq \delta \dim E$, *$d(F) \leq C$ and $\|P\| \leq C$.*

The result below, which is a special case of one of the main theorems of [3], Theorem 3.11, is the main tool of this note.

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THEOREM 2. *Let X be a Banach lattice, which is a weak Hilbert space. There exists a constant K with the following property*:

For every 1-unconditional, normalized sequence $(x_j)_{j\in I} \subseteq X$ (finite or infinite), and every $m \in \mathbf{N}$ there exists a $J \subseteq I, |J| = m$, so that if $A \subseteq I \setminus J$ with $|A| \le m2^m$, then $(x_j)_{j\in A}$ is K-equivalent to the unit vector basis of $l_2^{|A|}$.

We refer to [2] for the definition of a Köthe function space and we let λ denote the Lebesgue measure on $[0,1]$. If $A \subseteq [0,1]$ we let 1_A denote the indicator function for A, and if f is a real-valued function on $[0,1]$ then we put $P_A f = f1_A$.

Finally, we refer to [3] for any unexplained notion on Banach lattices, which are weak Hilbert spaces.

2. The main result

Our main result is the following:

THEOREM 3. *If X is a Köthe function space on* $[0,1]$, *which is a weak Hilbert space, then X is lattice isomorphic to $L_2(0,1)$.*

Proof. Let K be the constant from Theorem 2. We wish to show that there is a constant $C \ge 1$ so that every finite sequence of positive, normalized and mutually disjoint elements of X is C-equivalent to the unit vector basis of a Hilbert space. It then follows from [2], Theorem 1.b.13, that X is lattice isomorphic to $L_2(0,1)$.

Hence let $n \in \mathbf{N}$ and $(f_j)_{j=1}^n \subseteq X$ be positive, normalized with mutually disjoint supports and put $A_j = \operatorname{supp} f_j$ for all $1 \le j \le n$.

Let $0 < \varepsilon < \frac{1}{2}$ be arbitrary. Since X is a weak Hilbert space it is order continuous and therefore there exists a $\delta > 0$ so that if $A \subseteq [0,1]$ is Lebesgue measurable then

$$\lambda(A) \le \delta \Rightarrow \|P_A f_j\| \le \varepsilon/n \quad \text{for all} \quad 1 \le j \le n. \tag{2.1}$$

Let now $m \in \mathbf{N}$ be chosen, so that $m2^{-m} \le \delta$ and $m \ge n$ and put

$$B_k = [(k-1)2^{-m}, k2^{-m}[\quad \text{for all } 1 \le k \le 2^m. \tag{2.2}$$

$$I = \{(j,k) | P_k f_j \ne 0\}, \quad \text{where } P_k = P_{B_k}, 1 \le k \le n. \tag{2.3}$$

$$g_{jk} = \|P_k f_j\|^{-1} P_k f_j \quad \text{for all } (j,k) \in I. \tag{2.4}$$

Since g_{jk} is a normalized 1-unconditional sequence and $|I| \le n2^m \le m2^m$ there is a set $J \subseteq I$ with $|I \setminus J| = m$ so that $(g_{jk})_{(j,k)\in J}$ is K-equivalent to the unit vector bases of $l_2^{|J|}$.

Put

$$A = \bigcup\{A_j \cap B_k | (j,k) \notin J\}, \qquad B = [0,1] \setminus A. \tag{2.5}$$

Clearly $\lambda(A) \leq m2^{-m} \leq \delta$ and hence

$$\|P_A f_j\| \leq \varepsilon/n \quad \text{for all } 1 \leq j \leq n. \tag{2.6}$$

If $J_j = \{k | (j,k) \in J\}$ then

$$P_B f_j = \sum_{k \in J_j} P_k f_j \quad \text{for all } 1 \leq j \leq n. \tag{2.7}$$

This shows that $(P_B f_j)_{j=1}^n$ is a block basis of (g_{jk}) and since $\|P_B f_j\| \geq 1 - \varepsilon \geq \frac{1}{2}$ for all $1 \leq j \leq n$ it is $2K$-equivalent to the unit vector basis of l_2^n. Furthermore, since

$$2\sum_{j=1}^{n} \|P_B f_j - f_j\| \leq 2\varepsilon < 1 \tag{2.8}$$

it follows that (f_j) is $2K(1+2\varepsilon)(1-2\varepsilon)^{-1}$-equivalent to the unit vector basis of l_2^n. □

As a corollary we obtain:

THEOREM 4. *Let X be a separable Banach lattice, which is not purely atomic and which is not isomorphic to a Hilbert space. If X is a weak Hilbert space then there is an infinite sequence (x_j) of mutually disjoint positive normalized vectors so that X is lattice isomorphic to $L_2(0,1) \oplus [x_j]$.*

Proof. Since every separable non-atomic Banach lattice is lattice isomorphic to a Köthe function space on [0, 1] it follows from Theorem 3 and our assumptions that X has atoms. Let $(x_n) \subseteq X$ be the sequence of atoms. We can then write

$$X = [x_j]^{\perp} \oplus [x_j]. \tag{2.9}$$

$[x_j]^{\perp}$ is a non-atomic lattice and therefore lattice isomorphic to $L_2(0,1)$ by Theorem 3. □

It is readily verified that the argument in the proof of Theorem 3 still works, if we just assume that the space X there has the following property: There is a constant $K \geq 1$ and a function $k: \mathbf{N} \to \mathbf{N}$ with $k(m)m^{-1} \to \infty$ for $m \to \infty$ so that every normalized sequence in X consisting of mutually disjoint elements has property $E(m, k(m), K)$ for all m as defined in [3].

It seems unknown whether a general Banach lattice with this property is a weak Hilbert space, if $k(m)2^{-m} \to 0$ for $m \to \infty$. Compare with the proof of Theorem 4.1 in [3].

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