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KNOTS AND SHELLABLE CELL PARTITIONINGS OF S^3

STEVE ARMENTROUT

A *cell partitioning* of S^3 is a finite covering H of S^3 by 3-cells such that if m is any positive integer and exactly m 3-cells of H intersect, their common part is a cell of dimension $4 - m$, where cells of negative dimension are empty. The 3-cells of a cell partitioning of S^3 fit together in a staggered, brick-like pattern.

A cell partitioning H of S^3 is *shellable* if and only if there is a counting $\langle h_1, h_2, \cdots, h_n \rangle$ of H such that if i is an integer and $1 \leqq i < n$, then $h_1 \cup h_2 \cup \cdots \cup h_i$ is a 3-cell. Such a counting is a *shelling* of H.

In this paper, we shall study a connection between knots in S^3 and shellability of cell partitionings of S^3. We shall use these results to construct nonshellable cell partitionings of S^3.

Our results involve the use of the bridge number of a knot in S^3. In Section 1 of this paper, we shall review some results concerning knots in S^3 and bridge numbers of knots in S^3. In Section 2, we shall establish the main result of the paper. In Section 3, we shall establish a variant of the main result that is useful in some situations. In Section 4, we shall use the results of this paper to construct a nonshellable cell partitioning of S^3 and, as a variation on that construction, a nest of nonshellable cell partitionings of S^3.

Throughout this paper, we shall assume that S^3 has its standard piecewise linear structure.

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1. Knots in S^3

A *knot* in S^3 is a polygonal simple closed curve in S^3. Two knots k and l in S^3 are of the *same knot type* in S^3 if and only if there is an orientation preserving PL homeomorphism $f: S^3 \to S^3$ such that $f(k) = l$. A knot in S^3 is *trivial* if and only if it has the same knot type as the boundary of a 2-simplex in S^3.

Suppose C is a 3-cell. Then α is a *spanning arc* of C if and only if α is an arc in C such that $\operatorname{Bd} \alpha \subset \operatorname{Bd} C$ and $\operatorname{Int} \alpha \subset \operatorname{Int} C$. D is a *semispanning disc* of C if and only if D is a disc in C such that $\operatorname{Int} D \subset \operatorname{Int} C$ and $D \cap \operatorname{Bd} C$ is an arc on $\operatorname{Bd} C$. The statement that β is a *straight* spanning arc of C means that β is a spanning arc of C and there is a semispanning disc D of C such that $\beta \subset \operatorname{Bd} D$. Recall that if β is a polyhedral straight spanning arc of a

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polyhedral 3-cell C and α is any polyhedral arc on Bd C with Bd β = Bd α, then there is a polyhedral semispanning disc Δ in C with Bd $\Delta = \alpha \cup \beta$.

The statement that $\alpha_1, \alpha_2, \ldots,$ and α_n are *simultaneously straight* in C means that $\alpha_1, \alpha_2, \ldots,$ and α_n are mutually disjoint spanning arcs of C and there exist mutually disjoint semispanning discs $D_1, D_2, \ldots,$ and D_n of C such that for each i, $\alpha_i \subset$ Bd D_i.

Suppose l is a knot in S^3, C is a polyhedral 3-cell in S^3, and m is a positive integer. Then l is in m-*bridge position on C* if and only if there exist mutually disjoint arcs $\alpha_1, \alpha_2, \ldots,$ and α_m on Bd C and mutually disjoint arcs $\beta_1, \beta_2, \ldots,$ and β_m simultaneously straight in C, such that $l = (\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_m) \cup (\beta_1 \cup \beta_2 \cup \cdots \cup \beta_m)$.

If k is a knot in S^3, then the *bridge number* of k, denoted by br k, is defined to be the least positive integer m such that there exist a knot l in S^3 and a polyhedral 3-cell C in S^3 such that (1) l and k have the same knot type, and (2) l is in m-bridge position on C.

For basic results concerning the bridge number of a knot, see [8]. It is clear that bridge number is an invariant of knot type. A knot in S^3 is trivial if and only if the knot has bridge number 1. It is easily seen, for example, that the trefoil knot in S^3 has bridge number 2.

2. The main result

In this section we shall establish a relationship between the bridge number of a knot in S^3 and the nonshellability of a cell partitioning of S^3 related to the knot in a special way. First we shall introduce some terminology.

Suppose H is a cell partitioning of S^3. If h and k are distinct intersecting 3-cells of H, then $h \cap k$ is a disc. By a *face* of H is meant such a disc. The 2-*skeleton* of H, denoted by 2-skel H, is the union of all the faces of H. The 1-*skeleton* of H, 1-skel H, is the set of all points common to three or more sets of H. The 0-*skeleton* of H, 0-skel H, is the set of all points common to four sets of H.

Suppose that H is a polyhedral cell partitioning of S^3, and k is a knot in S^3. Then k is *compatible with H* if and only if (1) k and 2-skel H are in relative general position in S^3, and (2) if $h \in H$ and k intersects h, then $h \cap k$ is a single straight spanning arc. Suppose k is compatible with H. Then the *partitioning of k induced by H* is $\pi(k, H) = \{k \cap h\colon h \in H \text{ and } h \cap k \neq \phi\}$. Let $|\pi(k, H)|$ denote the number of arcs in the partitioning of k induced by H.

We are now prepared to prove the main result of this paper. It was suggested by examples due to Bing (pp. 110–111 of [3]). In this connection, see also [4], [5], [6], and [7].

THEOREM 1. *Suppose H is a polyhedral cell partitioning of S^3, k is a knot in S^3, and k is compatible with H. If $|\pi(k, H)| < 2$ br k, then H is not shellable.*

Proof. Suppose that H is shellable. Then there is a shelling $\langle h_1, h_2, \ldots, h_n \rangle$ of H. If $1 \leqq i < n$, let C_i denote $h_1 \cup h_2 \cup \cdots \cup h_i$; C_i is a 3-cell.

Now we shall give a brief outline of the proof. By simple geometric moves, we shall construct a knot l in S^3 such that (1) l and k are of the same knot type and (2) for some positive integer r such that $2r \leqq |\pi(k, H)|$ and some polyhedral 3-cell C in S^3, l is in r-bridge position on C. Thus $\text{br}\, l \leqq r$ and since k and l are of the same knot type, then $\text{br}\, k \leqq r$. Since by hypothesis, $|\pi(k, H)| < 2\, \text{br}\, k$, then $2r \leqq |\pi(k, H)| < 2\, \text{br}\, k$, and thus $\text{br}\, k \leqq r < \text{br}\, k$. This is a contradiction.

We shall obtain l as follows. Let $k_0 = k$. Let m be the largest positive integer j such that k intersects h_j. If $1 \leqq i < m$, we shall construct a knot k_i, of the same knot type as k, and obtained from k_{i-1} by adjusting the part of k_{i-1} in C_i. It is to be true that $k_i - C_i = k - C_i$. Further, there are integers p_i and q_i such that (1) $k_i \cap C_i$ is the union of q_i simultaneously straight spanning arcs of C_i and p_i mutually disjoint arcs on $\text{Bd}\, C_i$, and (2) $p_i + q_i$ is at most the number of cells among $h_1, h_2, \ldots,$ and h_i that k intersects. We obtain l by an analogous adjustment of k_{m-1}, and l has the properties that $l \subset C$, and l is the union of q_m simultaneously straight spanning arcs of C and p_m mutually disjoint arcs on $\text{Bd}\, C$ where $p_m + q_m$ is at most the number of cells among $h_1, h_2, \ldots,$ and h_m that l intersects. Thus $p_m + q_m \leqq |\pi(k, H)|$. Since $l \subset C$, then $p_m = q_m$. If $r = p_m = q_m$, then l is in r-bridge position on C.

Now we shall give the details concerning the construction of the knots $k_1, k_2, \ldots,$ and k_{m-1}. Recall that m is the largest positive integer j such that k intersects h_j. Let k_0 denote k. Let t be the least positive integer i such that k intersects h_i. If $1 < i < n$, let D_i denote $C_{i-1} \cap h_i$; by Lemma 5, D_i is a disc. If $1 < i < n$, let E_i denote $(\text{Bd}\, h_i) - (\text{Int}\, D_i)$. Let $E_1 = \text{Bd}\, h_1$.

Let $k_1 = k_2 = \cdots = k_{t-1} = k$. Let β_{t1} denote $k \cap h_t$. Then β_{t1} is a straight spanning arc of h_t with $\text{Bd}\, \beta_{t1} \subset \text{Int}\, E_t$. Hence there exist a polygonal arc λ_{t1} in $\text{Int}\, E_t$ with $\text{Bd}\, \beta_{t1} = \text{Bd}\, \lambda_{t1}$ and a polyhedral semispanning disc Δ_{t1} in h_t such that $\text{Bd}\, \Delta_{t1} = \beta_{t1} \cup \lambda_{t1}$. Also, we require that λ_{t1} and the boundaries of the faces of H on $(\text{Bd}\, h_t)$ be in relative general position on $\text{Bd}\, h_t$.

For each positive integer i such that $t \leqq i < m$, let S_i denote the following statement.

S_i: There exist
(1) a knot k_i in S^3 of the same knot type as k_{i-1},
(2) nonnegative integers p_i and q_i such that $p_i + q_i$ is at most the number of 3-cells among $\{h_1, h_2, \ldots, h_i\}$ that k intersects,
(3) mutually disjoint polyhedral arc $\alpha_{i1}, \alpha_{i2}, \cdots,$ and α_{ip_i} on $\text{Bd}\, C_i$,
(4) mutually disjoint polyhedral arc $\beta_{i1}, \beta_{i1}, \cdots,$ and β_{iq_i} simultaneously straight in C_i,

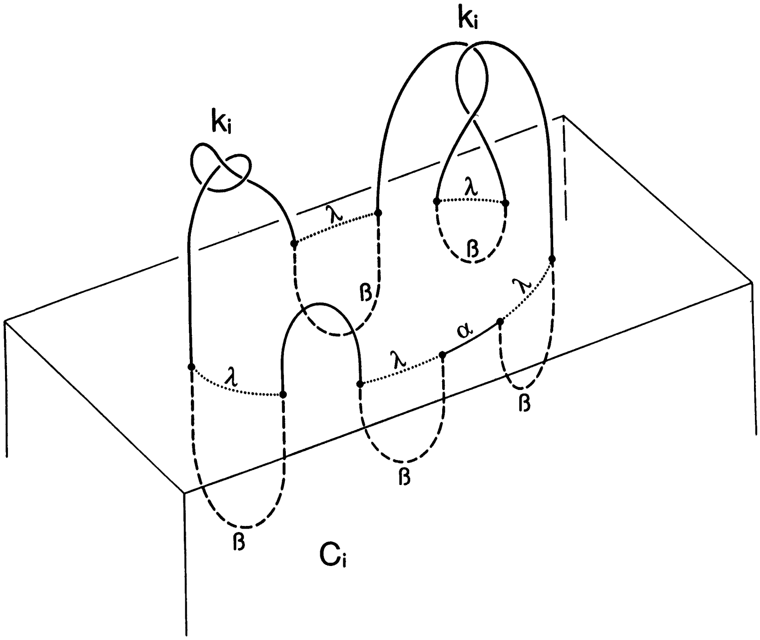

FIG. 1

(5) mutually disjoint polyhedral arcs $\lambda_{i1}, \lambda_{i2}, \cdots,$ and λ_{iq_i} on Bd C_i such that for $1 \leqq j \leqq q_i$, there is a polyhedral semispanning disc Δ_{ij} of C_i with Bd $\Delta_{ij} = \beta_{ij} \cup \lambda_{ij}$ and $\Delta_{i1}, \Delta_{i2}, \Delta_{iq_i}$ mutually disjoint, such that

(a) $k_i = (k - C_i) \cup (\bigcup_j \alpha_{ij}) \cup (\bigcup_j \beta_{ij})$,

(b) if x is an endpoint of some α_{ij}, then x is also an endpoint of some β_{iu}, and

(c) the α's and λ's are in general position relative to the boundaries of faces of H on Bd C_i.

If $p_i = 0$, there are no α's, and if $q_i = 0$, there are no β's. See Figure 1.

Now S_t is true. Let $k_t = k$. The arcs β_{t1} and λ_{t1}, and the disc Δ_{t1} were defined above. Let $p_t = 0$ and $q_t = 1$. Then $p_t + q_t = 1$, and note that k intersects at most one of the 3-cells $h_1, h_2, \cdots,$ and h_t.

Suppose now that $t < i < m - 1$ and S_{i-1} is true. We shall prove that S_i is true. Since S_{i-1} holds, there exist $k_{i-1}, p_{i-1}, q_{i-1}$, α's, β's, λ's, and Δ's as described in the statement of S_{i-1}.

We shall consider four cases . In each case, we may modify k_i and existing α's, β's, λ's, and Δ's. We may construct one additional α or one additional β, but not both.

Case 1. k and h_1 are disjoint.

In this case, no additional α's or β's are constructed. It follows in this case that k_{i-1} is disjoint from $h_i - D_i$.

There is a PL homeomorphism f_i: $D_i \to E_i$ such that $f_i | \text{Bd } D_i = \text{id}$. Then f_i: $D_i \to E_i$ extends to a PL homeomorphism $\hat{f}_i$ Bd: $C_{i-1} \to \text{Bd } C_i$ such that $\hat{f}_i | (\text{Bd } C_{i-1}) - D_i = \text{id}$. There is a PL homeomorphism F_i: $S^3 \to S^3$ such that

(1) $F_i(C_{i-1}) = C_i$,
(2) F_i extends $\hat{f}_i$, and
(3) except on a close neighborhood of h_i, $F_i = \text{id}$.

Let $p_i = p_{i-1}$ and $q_i = q_{i-1}$. If $1 \leqq j \leqq q_i$, let $\alpha_{ij} = F_i(\alpha_{i-1,j})$, $\lambda_{ij} = F_i(\lambda_{i-1,j})$, and $\Delta_{ij} = F_i(\Delta_{i-1,j})$. If $1 \leqq j \leqq q_i$, let $\beta_{ij} = F_i(\beta_{i-1,j})$. We may assume that the α's and λ's are in general position relative to the boundaries of faces of H on Bd C_i. Let $k_i = F_i(k_{i-1})$. We may assume, since $k \cap h_i = \phi$, that $F_i | k - k_i = \text{id}$.

Clearly k_{i-1} and k_i are of the same knot type in S^3. Since $p_i = p_{i-1}$, $q_i = q_{i-1}$, $k \cap h_i = \phi$, and $p_{i-1} + q_{i-1}$ is at most the number of cells among $h_1, h_2, \ldots,$ and h_{i-1} that k intersects, then $p_i + q_i$ is at most the number of cells among $h_1, h_2, \ldots,$ and h_i that k intersects. Thus S_i holds in this case.

Case 2. k intersects h_i but is disjoint from D_i.

In this case, one additional β is constructed. Let $\beta_{iq_i} = k \cap h_i$. Then β_{iq_i} is a polyhedral straight spanning arc of h_i with Bd β_{iq_i} in Int E_i. Let λ_{iq_i} be a polyhedral arc in Int E_i with Bd $\lambda_{iq_i} = \text{Bd } \beta_{iq_i}$. There is a polyhedral semi-spanning disc Δ_{iq_i} of h_i with Bd $\Delta_{iq_i} = \beta_{iq_i} \cup \lambda_{iq_i}$.

Let δ_i be a small polyhedral disc in Int D_i and disjoint from the α's and λ's. There is a piecewise linear homeomorphism F_i: $S^3 \to S^3$ such that

(1) $F_i(\delta_i) = D_i$,
(2) $F_i(C_{i-1}) = C_{i-1}$, and
(3) except on a close neighborhood of D_i, F_i is the identity and, in particular, $F_i | (\Delta_{iq_i} \cup k_i)$ is the identity.

Let $p_i = p_{i-1}$ and let $q_i = 1 + q_{i-1}$. If $1 \leqq j \leqq p_i$, let $\alpha_{ij} = F_i(\alpha_{i-1,j})$. If $1 \leqq j < q_i$, let $\beta_{ij} = F_i(\beta_{i-1,j})$, $\lambda_{ij} = F_i(\lambda_{i-1,j})$, and $\Delta_{ij} = F_i(\Delta_{i-1,j})$. We defined β_{iq_i}, λ_{iq_i}, and Δ_{iq_i} above. Let $k_i = k_{i-1}$. We use the homeomorphism F_i to adjust the α's, β's, and λ's that intersect D_i, but continue the argument with the original h's. We do not replace h_i by $F_i(h_i)$. Since $p_i + q_i = 1 + p_{i-1} + q_{i-1}$ and k intersects only one more 3-cell among $h_1, h_2, \cdots,$ and h_i than among h_1, h_2 and h_{i-1}, then condition (2) of S_i holds. It is easily seen that S_i holds in this case.

Case 3. k intersects D_i in exactly one point.

In this case, no additional α's or β's are constructed, but we extend an existing β.

Let x be the point common to k and D_i. It follows from condition 5(b) of S_{i-1} that x is an endpoint of a component of $k - \operatorname{Int} C_{i-1}$, and an endpoint of some β. There is an integer w such that $1 \leqq w \leqq q_{i-1}$ and x is an endpoint of $\beta_{i-1,w}$. Then x is also an endpoint of $\lambda_{i-1,w}$.

Let δ be a small polyhedral disc in $\operatorname{Int} D_i$ with x in $\operatorname{Int} \delta$, such that (1) $\delta \cap \lambda_{i-1,w}$ is an arc λ_0 with one endpoint z on $\operatorname{Bd} \delta$ and x as the other endpoint, and (2) δ intersects no α, no β other than $\beta_{i-1,w}$, and no λ other than $\lambda_{i-1,w}$.

There is a piecewise linear homeomorphism F_i: $S^3 \to S^3$ such that

(1) $F_i | \beta_{i-1,w} \cup (k \cap h_i)$ is the identity,
(2) $F_i(\delta) = D_i$,
(3) $F_i(C_i) = C_i$, and
(4) except on a close neighborhood of D_i, F_i is the identity.

Since $F_i(z)$ is on $\operatorname{Bd} D_i$, there is an arc λ' in E_i with endpoints $F_i(z)$ and the point common to E_i and k_i, and with $\operatorname{Int} \lambda'$ in $\operatorname{Int} E_i$. Since $k \cap h_i$ is straight in h_i, then $\lambda' \cup F_i(\lambda_0) \cup (k \cap h_i)$ bounds a polyhedral semispanning disc Δ' of h_i.

Since $\lambda_0 \subset \lambda_{i-1,w}$, it follows that $\Delta' \cup F_i(\Delta_{i-1,w})$ is a disc Δ_{iw}. Let

$$\lambda_{iw} = [F_i(\lambda_{i-1,w}) - F_i(\lambda_0)] \cup \lambda'.$$

Recall that $\beta_{i-1,w} = F_i(\beta_{i-1,w})$ and $k \cap h_i = k_i \cap h_i$. Now let $\beta_{iw} = \beta_{i-1,w} \cup (k \cap h_i)$. Then $\operatorname{Bd} \Delta_{iw} = \beta_{iw} \cup \lambda_{iw}$.

Let $p_i = p_{i-1}$ and $q_i = q_{i-1}$. If $1 \leqq j \leqq p_i$, let $\alpha_{ij} = F_i(\alpha_{i-1,j})$. If $1 \leqq j \leqq q_i$ and $j \neq w$, let $\beta_{ij} = F_i(\beta_{i-1,j})$, $\lambda_{ij} = F_i(\lambda_{i-1,j})$, and $\Delta_{ij} = F_i(\Delta_{i-1,j})$. Let

$$k_i = (k - \operatorname{Int} C_i) \cup \left(\bigcup_{j=1}^{p_i} \alpha_{ij} \right) \cup \left(\bigcup_{j=1}^{q_i} \beta_{ij} \right).$$

Note that $k_i = F_i(k_{i-1})$. Hence k_i and k_{i-1} are of the same knot type in S^3. As in Case 2, we use the homeomorphism F_i only to adjust the α's, β's, and λ's that intersect D_i.

It is easily verified that S_i holds in this case.

Case 4. k intersects D_i in two points.

In this case, we shall construct an additional α.

Let x and y be the points common to k and D_i. It is clear that $k \cap h_i = k_{i-1} \cap h_i$. By condition 5(b) of S_{i-1}, neither x nor y can be an endpoint of any α, and hence x and y are endpoints of β's. Since x and y lie in $\operatorname{Int} D_i$, there is a polygonal arc A from x to y and lying in $\operatorname{Int} D_i$.

We shall first adjust those α's that intersect A by pushing them off A, keeping C_{i-1} invariant. Suppose $1 \leqq j \leqq p_{i-1}$ and $\alpha_{i-1,j}$ intersects A. There is a piecewise linear homeomorphism g_j: $S^3 \to S^3$ such that (1) g_j fixes one endpoint of $\alpha_{i-1,j}$ and shortens $\alpha_{i-1,j}$ so that $g_j(\alpha_{i-1,j})$ is disjoint from A, (2) except on a close neighborhood of $\alpha_{i-1,j}$, g_j is the identity, and (3) $g_j(C_{i-1}) = C_{i-1}$. Let f_1: $S^3 \to S^3$ be the composite, in some order, of all such g_j's for the $\alpha_{i-1,j}$ that intersect A. Then for each arc α, $f_1(\alpha)$ is disjoint from A.

There is a piecewise linear homeomorphism f_2: $S^3 \to S^3$ such that (1) $f_2(D_i) = E_i$, (2) f_2 is the identity on $(\text{Bd}\, C_{i-1}) - (\text{Int}\, D_i)$, (3) $f_2(C_{i-1}) = C_i$, and (4) except on a close neighborhood of h_i, f_2 is the identity.

Let $F_i = f_2 \circ f_1$: $S^3 \to S^3$. Let $p_i = 1 + p_{i-1}$ and let $\alpha_{ip_i} = F_i(A)$. If $1 \leqq j \leqq p_{i-1}$, let $\alpha_{ij} = F_i(\alpha_{i-1,j})$. Let $q_i = q_{i-1}$. If $1 \leqq j \leqq q_i$, let $\beta_{ij} = F_i(\beta_{i-1,j})$, $\lambda_{ij} = F_i(\lambda_{i-1,j})$, and $\Delta_{ij} = F_i(\Delta_{i-1,j})$. Note that $p_i + q_i = 1 + p_{i-1} + q_{i-1}$. Let $k_i = F_i([k_{i-1} - (h_i \cap k)] \cup A)$.

Since $h_i \cap k$ is straight in h_i, then $A \cup (h_i \cap k)$ bounds a polyhedral semispanning disc of h_i. It follows easily that k_{i-1} and k_i are of the same knot type in S^3.

It is easily established that S_i holds in this case.

Thus if $t < i < m - 1$ and S_{i-1} is true, than S_i is true. Since S_t is true, it follows that S_{m-1} is true.

The situation involving h_m requires special treatment because of the possibility that $m = n$, in which case C_m is not defined.

Since m is the largest integer i such that k intersects h_i, it follows that k intersects D_m in two points x_m and y_m. Let A_m be a polygonal arc in Int D_m from x_m to y_m. By a procedure similar to that used in Case 4 above, we may use a piecewise linear homeomorphism $F_m = S^3 \to S^3$ to adjust the α's so that their images are disjoint from A_m, keeping the remainder of k_{m-1} pointwise fixed.

Since $k \cap h_m$ is straight in $h_m, (k \cap h_m) \cup A_m$ bounds a polyhedral semispanning disc B_m of h_m. Thicken B_m slightly relative to C_{m-1} to obtain a 3-cell B_m^* in h_m such that (1) B_m is a spanning disc of B_m^*, (2) $B_m^* \cap D_m$ is a disc having A_m as a spanning arc, (3) $k \cap h_m$ is a spanning arc of $(\text{Bd}\, B_m^*) - \text{Int}(B_m^* \cap D_m)$, and (4) B_m^* is a close (closed) neighborhood of B_m.

Let $p_m = 1 + p_{m-1}$ and let $q_m = q_{m-1}$. If $1 \leqq j < p_m$, let $\alpha_{mj} = F_m(\alpha_{m-1,j})$, and let $\alpha_{mp_m} = k \cap h_m$. If $1 \leqq j \leqq q_m$, let $\beta_{mj} = F_m(\beta_{m-1,j})$, $\lambda_{mj} = F_m(\lambda_{m-1,j})$, and $\Delta_{mj} = F_m(\Delta_{m-1,j})$. Let $l = F_m(k_{m-1})$. Clearly l and k_{m-1} have the same knot type in S^3.

Let $C = C_{m-1} \cup B_m^*$. Then C is a polyhedral 3-cell in S^3 and $l \subset C$.

Since $p_{i-1} + q_{i-1}$ is at most the number of 3-cells among $h_1, h_2, \ldots,$ and h_{m-1} that intersect k, and $l \subset h_1 \cup h_2 \cup \cdots \cup h_m$, then clearly $p_m + q_m \leqq |\pi(k, H)|$.

Now l and k are of the same knot type in S^3, since $k = k_0, k_1, k_2, \cdots, k_{m-1}$, and l all have the same knot type.

Now for each integer j with $1 \leqq j \leqq p_m$, let $\alpha_j = \alpha_{mj}$, and if $1 \leqq j \leqq q_m$, let $\beta_j = \beta_{mj}$. It is clear that $p_m = q_m$, and let $r = p_m = q_m$. Since for $1 \leqq j \leqq r$, β_j lies on the boundary of the polyhedral semispanning disc Δ_{mj}

of C, and $\Delta_{m1}, \Delta_{m2}, \cdots$, and Δ_{mr} are disjoint, then the β's are simultaneously straight in C. Further, each of $\alpha_1, \alpha_2, \cdots$, and α_r lies on Bd C and

$$l = \left(\bigcup_{j=1}^{r} \alpha_j \right) \cup \left(\bigcup_{j=1}^{r} \beta_j \right).$$

It follows that l is in r-bridge position on C. Further, since $p_m + q_m \leqq |\pi(k, H)|$, then $2r \leqq |\pi(k, H)|$.

Thus the knot l has the properties that (1) k and l are of the same knot type in S^3 and (2) for some positive integer r such that $2r \leqq |\pi(k, H)|$ and some polyhedral 3-cell C in S^3, l is in r-bridge position on C. It was pointed out above that this leads to a contradiction. Hence H is nonshellable.

We shall conclude this section by showing that the result of Theorem 1 is, in a sense, sharp. See also [7].

THEOREM 2. *Suppose that k is a nontrivial knot in S^3. Then there exists a shellable polyhedral cell partitioning H of S^3 such that k is compatible with H and $|\pi(k, H)| = 2\,\mathrm{br}\,k$.*

Proof. Let $r = \mathrm{br}\,k$. Then there exists a polyhedral 3-cell C in S^3 such that k is in r-bridge position on C. Hence there exist mutually disjoint polyhedral arcs α_1, α_2 and α_r on Bd C and mutually disjoint polyhedral arcs $\beta_1, \beta_2, \cdots$, and β_r simultaneously straight in C such that $k = (\bigcup_{i=1}^{r} \alpha_i) \cup (\bigcup_{i=1}^{r} \beta_i)$. Since $\beta_1, \beta_2, \cdots$, and β_r are simultaneously straight in C, there exist mutually disjoint polyhedral semispanning discs $D_1, D_2, \cdots$, and D_r of C such that if $1 \leqq i \leqq r$, then $\beta_i \subset \mathrm{Bd}\, D_i$.

Let $B = S^3 - \mathrm{Int}\, C$; B is a polyhedral 3-cell in S^3. If $1 \leqq i \leqq r$, adjust α_i by pushing Int α_i slightly into Int B. We may do this so that the adjusted $\alpha_1, \alpha_2, \cdots$, and α_r are polyhedral and simultaneously straight in B. We may assume that this adjustment is made by a piecewise linear homeomorphism f: $S^3 \to S^3$ that is the identity on each of $\beta_1, \beta_2, \cdots$, and β_r. Since $f(\alpha_1), f(\alpha_2)$ and $f(\alpha_r)$ are simultaneously straight in B, there are mutually disjoint polyhedral semispanning discs $E_1, E_2, \ldots$, and E_r of B such that if $1 \leqq i \leqq r, f(\alpha_i) \subset \mathrm{Bd}\, E_i$. We may assume that if $1 \leqq i \leqq r$ and $1 \leqq j \leqq r$, then $D_i \cap \mathrm{Bd}\, B$ and $E_j \cap \mathrm{Bd}\, B$ are in relative general position on Bd B.

Thicken $f(\alpha_i), f(\alpha_2), \ldots$, and $f(\alpha_r)$ slightly relative to B to obtain mutually disjoint polyhedral 3-cells $F_1^*, F_2^*, \cdots$, and F_r^* such that if $1 \leqq j \leqq r$, then (1) $F_j^* \cap \mathrm{Bd}\, B$ is the union of two disjoint discs, $F_j^* \cap E_j$ is a disc, and $f(\alpha_j)$ is a straight spanning arc of F_j^*, and (2) if $1 \leqq i \leqq r$, $D_i \cap F_j^*$ is empty or an arc.

Suppose $1 \leqq j \leqq r$. Let $E_j' = \mathrm{Cl}(E_j - F_j^*)$. Cut E_j' into narrow strips $E_{j1}, E_{j2}, \cdots$, and E_{jn_j}, cutting in a direction normal to Bd B, so that if $1 \leqq k \leqq n_j$, E_{jk} intersects at most one of the D's, and then in an interior point of $E_{jk} \cap \mathrm{Bd}\, B$. We may assume that if $1 \leqq i \leqq r$, then $E_{jk} \cap D_i$ is empty or a point. We assume $E_{j1}, E_{j2}, \cdots$, and E_{jn_j} counted in order so that any two consecutive ones intersect in an arc.

If $1 \leqq j \leqq r$, thicken $E_{j1}, E_{j2}, \cdots$, and E_{jn_j} very slightly to obtain polyhedral 3-cells $E^*_{j1}, E^*_{j2}, \cdots$, and $E^*_{jn_j}$ such that (1) if $1 \leqq k \leqq n_j$, $E^*_{jk} \cap \operatorname{Bd} B$ is a disc, $E^*_{jk} \cap F^*_j$ is a disc, E^*_{jk} intersects any neighboring E^* in a disc, and if $1 \leqq i \leqq r$, then $E^*_{jk} \cap D_i$ is empty or an arc. In addition, if $k < n_j$, then $E^*_{jk} \cap E^*_{j,k+1} \cap D_i = \phi$.

Thicken $D_1, D_2, \cdots$, and D_r very, very slightly relative to C to obtain mutually disjoint polyhedral 3-cells $D^*_1, D^*_2, \cdots$, and D^*_r such that if $1 \leqq i \leqq r$, then (1) $D^*_i \cap \operatorname{Bd} C$ is a disc and β_i is a straight spanning arc of D^*_i, (2) if $1 \leqq j \leqq r$, $D^*_i \cap F^*_j$ is empty or a disc, (3) if $1 \leqq j \leqq r$ and $1 \leqq k \leqq n_j$, then (a) $E^*_{jk} \cap D^*_i$ is empty or a disc and (b) is $k < n_j$, then $E^*_{jk} \cap E^*_{j,k+1} \cap D^*_i = \phi$.

Let

$$C_0 = \operatorname{Cl}\left(C - \bigcup_{i=1}^{r} D^*_i\right) \text{ and } B_0 = \operatorname{Cl}\left(B - \bigcup_{j=1}^{r}\left(F^*_j \cup \left(\bigcup_{k=1}^{n_j} E^*_{jk}\right)\right)\right).$$

Then C_0 and B_0 are polyhedral 3-cells. There is a polyhedral cell partitioning $\{B_1, B_2, \cdots, B_q\}$ of B_0 such that (1) if $1 \leqq i \leqq q$ and X is either C_0 or one of the D^*, the E^*, or the F^*, then $B_q \cap X$ is empty or a disc, and (2) if $i < q$, $[(\operatorname{Bd} B_0) \cup B_1 \cup \cdots \cup B_i] \cap B_{i+1}$ is a disc. Let T consist of C_0, the D^*, the E^*, the $F^*, B_1, B_2, \cdots$, and B_q. We may construct the B's so that T is a polyhedral cell partitioning of S^3.

Let $\langle C_0, D^*_1, D^*_2, \cdots, D^*_r, E^*_{11}, E^*_{12}, \cdots, E^*_{1n_1}, F^*_1, E^*_{21}, E^*_{22}, \cdots, E^*_{2n_2}, F^*_2, \cdots, E^*_{r1}, \cdots, E^*_{rn_r}, F^*_r, B_1, B_2, \cdots, B_q\rangle$ be an ordering of K. It is easily seen that this is a shelling of T. Let this shelling be denoted by $\langle t_1, t_2, \cdots, t_m\rangle$.

If $1 \leqq i \leqq m$, let $h_i = f^{-1}(t_i)$. Let $H = \langle h_1, h_2, \cdots, h_m\rangle$. Then H is a polyhedral cell partitioning of S^3, H is shellable, and by construction, k is compatible with H. Clearly $|\pi(k, H)| = 2r = 2 \operatorname{br} k$. □

3. Weak compatibility

In this section, we shall establish a variant of the main result, Theorem 1 above, in which we weaken the conditions regarding how the knot is placed relative to the 2-skeleton of the partitioning.

Suppose H is a polyhedral cell partitioning of S^3 and k is a knot in S^3. Then k is *weakly compatible with* H if and only if (1) k and 2-skel H are in relative general position in S^3, and (2) if $h \in H$ and k intersects h, then the arcs which form the components of $h \cap k$ are simultaneously straight in h.

Suppose H and k are as above. Then the *partitioning of* k *induced by* H is $\pi(k, H) = \{\alpha$: *for some cell* h *of* H, α *is a component of* $h \cap k\}$; $|\pi(k, H)|$ denotes the number of arcs in the partitioning of k induced by H.

THEOREM 3. *Suppose H is a polyhedral cell partitioning of S^3, k is a knot in S^3, and k is weakly compatible with H. If $|\pi(k, H)| < \operatorname{br} k$, then H is not shellable.*

Proof. Suppose H is shellable. By Lemma 4 below, there is a shellable cell partitioning F of S^3 such that (1) k is compatible with F and (2) $|\pi(k, F)| = 2|\pi(k, H)|$. Since by hypothesis, $|\pi(k, h)| < \operatorname{br} k$, then $|\pi(k, F)| < 2 \operatorname{br} k$. This contradicts Theorem 1. Thus H is not shellable. □

LEMMA 4. *Suppose H is a shellable polyhedral cell partitioning of S^3 and k is a knot in S^3 weakly compatible with H. Then there is a shellable polyhedral cell partitioning F of S^3 such that k is compatible with F and $|\pi(k, F)| = 2|\pi(k, H)|$.*

Before proving Lemma 4, we shall give some preliminaries. A *partitioning* of a 2-sphere or disc X^2 is a finite covering $\mathscr{P}$ of X^2 by discs-with-holes such that (1) if 2, 3, or more sets of $\mathscr{P}$ intersect, their common part is an arc, point, or empty, respectively, and (2) if $D \in \mathscr{P}$ and $D \cap \operatorname{Bd} X^2 \neq \phi$, then $D \cap \operatorname{Bd} X^2$ is an arc. If each element of $\mathscr{P}$ is a disc, then $\mathscr{P}$ is a *disc partitioning* of X^2. A *shelling* of a *disc partitioning* $\mathscr{D}$ of X^2 is a counting $\langle D_1, D_2, \cdots, D_n\rangle$ of $\mathscr{D}$ such that if $1 \leqq i < n$, then $D_1 \cup D_2 \cup \cdots \cup D_i$ is a disc. A disc partitioning $\mathscr{D}$ of X^2 is *shellable* if and only if it has a shelling.

It follows from theorems of plane topology that every disc partitioning of either a 2-sphere or a disc is shellable. However, we sometimes need a special kind of shelling of a disc. A *ring partitioning* of a disc D^2 is a disc partitioning $\mathscr{D}$ of D^2 obtaining by dividing D^2 into concentric annuli, one of which contains $\operatorname{Bd} D^2$, and a central disc, and then dividing the annuli into discs by using crossing arcs in the annuli. A *ring shelling* of $\mathscr{D}$ is a counting of $\mathscr{D}$ which first counts the discs of the outer ring in order, then those of the next inward ring in order, $\cdots$, and finally the central disc. Note that for such a counting $\langle D_1, D_2, \cdots, D_n\rangle$, if $1 < i < n$, then $(D_1 \cup D_2 \cup \cdots \cup D_{i-1}) \cap D_i$ is an arc.

A *cell partitioning* of a 3-cell C^3 is a finite covering K of C^3 by 3-cells such that (1) if 2, 3, 4, or more sets of K intersect, their common part is a disc, an arc, a point, or empty, respectively, and (2) $\{k \cap \operatorname{Bd} C^3\colon k \in K \textit{ and } k \cap \operatorname{Bd} C^3 \neq \phi\}$ is a disc partitioning $\mathscr{D}$ of $\operatorname{Bd} C^3$. A cell partitioning K of C^3 is *shellable* if and only if it has a counting $\langle k_1, k_2, \cdots, k_m\rangle$ such that if $1 \leqq i \leqq m$, then $k_1 \cup k_2 \cup \cdots \cup k_i$ is a 3-cell. Such a counting $\langle k_1 \cup k_2, \cdots, k_m\rangle$ is a *shelling* of K.

Proof of Lemma 4. Since H is shellable, there is a shelling $\langle h_1, h_2, \cdots, h_n\rangle$ of H. By Lemma 5, if $1 < i < n$, then $(h_1 \cup h_2 \cup, \cdots, h_{i-1}) \cap h_i$ is a disc.

In the constructions of this proof, we shall assume that polyhedral sets are in relative general position, in a sense appropriate to the context.

Let N be a close tubular neighborhood of (2-skel H) canonically constructed. If v is any vertex of H, $N(v)$ is a small polyhedral ball about v. Let

$$N^0 = \cup\{N(v): v \text{ is a vertex of } H\}.$$

If e is any edge of H, $N(e)$ is a thin polyhedral 3-cell obtained by thickening $e - \operatorname{Int} N^0$ relative to N^0. Let

$$N^1 = N^0 \cup (\cup\{N(e): e \text{ is an edge of } H\}.$$

If f is any face of H, $N(f)$ is a polyhedral 3-cell obtained by thickening $f - \operatorname{Int} N^1$ slightly, relative to N^1. Then

$$N = N^1 \cup (\cup\{N(f): f \text{ is a face of } H\}.$$

We may assume that $N^1 \cap k = \phi$, and for each face f of H, the number of components of $k \cap N(f)$ equals the number of points of $k \cap f$. Let $\mathscr{N}$ be the family of all the sets $N(v)$, $N(e)$, and $N(f)$ constructed above.

Suppose Σ is any set which is a union of faces of H. Let $N(\Sigma)$ be the union of the sets $N(x)$ where x is a vertex, an edge, or a face of H lying in Σ. Note that $N(\Sigma)$ is a tubular neighborhood of Σ.

Suppose that $1 < i < n$. Let

$$N_i^* = N(\operatorname{Bd} h_1) \cup N(\operatorname{Bd} h_2) \cup \cdots \cup N(\operatorname{Bd} h_i).$$

Let

$$D_i = (h_1 \cup h_2 \cup \cdots \cup h_{i-1}) \cap h_i.$$

Note that $N(\operatorname{Bd} h_i) \cap N_{i-1}^* = N(D_i)$. Let N_i be the union of all the sets of $\mathscr{N}$ lying in $N(\operatorname{Bd} h_i)$ but not contained in $N(D_i)$. Note that N_i is a 3-cell. Now $N_i \cap N_{i-1}^* = N_i \cap N(D_i)$, and this set is an annulus A_i. If we define $N_1 = N(\operatorname{Bd} h_i)$, and for each j, $1 < j < n$, define N_j as above, then $N_{i-1}^* = N_1 \cup N_2 \cdots \cup N_{i-1}$.

In our construction of the partitioning F, we shall partition $N_1, N_2, \cdots$, and N_{n-1} so that these partitionings fit together in specified ways. We accordingly pay special attention to the annuli $A_2, A_3, \cdots$, and A_{n-1}. If $2 \leqq i < n$, let λ_i be the boundary curve of A_i lying in h_i, and let μ_i be the other.

If $1 \leqq i \leqq n$, let $h_i' = h_i - \operatorname{Int} N$. We may construct N so that if $1 \leqq i \leqq n$, the components of $k \cap h_i'$ are simultaneously straight in h_i'. If $1 \leqq i \leqq n$, let $\alpha_{i1}, \alpha_{i2}, \cdots$, and α_{im_i} be the components of $k \cap h_i'$. Let $\Delta_{i1}, \Delta_{i2}, \cdots$, and Δ_{im_i} be mutually disjoint polyhedral semispanning discs in h_i' such that if $1 \leqq j \leqq m_i$, $\alpha_{ij} \subset \operatorname{Bd} \Delta_{ij}$. Let $\Delta_{i1}^*, \Delta_{i2}^*, \cdots$, and $\Delta_{im_i}^*$ be mutually disjoint polyhedral 3-cells in h_i' such that if $1 \leqq j \leqq m_i$, Δ_{ij}^* is obtained by a slight thickening of Δ_{ij} relative to $\operatorname{Bd} h_i'$, α_{ij} is a straight spanning arc of Δ_{ij}^*, and

$\Delta^*_{ij} \cap \text{Bd } h'_i$ is a disc on $\text{Bd } \Delta^*_{ij}$. Let $h^*_i = \text{Cl}(h'_i - \bigcup_{j=1}^{m_i} \Delta^*_{ij})$. Note that if $1 \leqq j \leqq m_i$, then $h^*_i \cap \Delta^*_{ij}$ is a disc.

Note that $\{(\text{Bd } h'_i) \cap N(x)$: x is a vertex, edge, or face of H lying in $\text{Bd } h_i\}$ is a disc partitioning $\mathscr{P}_i$ of $\text{Bd } h'_i$. We may assume that if $1 \leqq j \leqq m_i$, Δ^*_{ij} is disjoint from each $N(v)$ where v is a vertex of H on $\text{Bd } h_i$, and that both $\text{Bd } \Delta_{ij}$ and $((\text{Bd } \Delta^*_{ij}) \cap \text{Bd } h')$ are in general position on $\text{Bd } H^*_i$ relative to the boundary of each disc δ of $\mathscr{P}_i$. We may also assume that for each such disc δ, and each j, $1 \leqq j \leqq m_i$, each component of $(\text{Bd } \delta) \cap \Delta^*_{ij}$ contains exactly one point of $(\text{Bd } \delta) \cap \Delta_{ij}$. If $l > i$, note that $A_l \cap \text{Bd } N_i$ is ϕ, a disc, or is all of A_l. Note that if $A_l \subset \text{Bd } N_i$, then $A_l \cap A_i = \phi$.

Suppose $2 \leqq k < n$. We shall now construct a partitioning of A_k into discs. Let Y_k be the set of all points p of $\text{Bd } A_k$ such that either (1) for some $l \neq k$, p lies on $\text{Bd } A_l$, or (2) for some pair s and t with $1 \leqq s \leqq n$ and $1 \leqq t \leqq m_s$, p lies on $\text{Bd}(\Delta^*_{st} \cap \text{Bd } h'_s)$. There exist mutually disjoint polyhedral crossing arcs $\beta_{k1}, \beta_{k2}, \cdots$, and β_{kr_k} of A_k such that if $B_{k1}, B_{k2}, \cdots$, and B_{kr_k} are the discs obtained by partitioning A_k using the β's, then (1) no endpoint of any β lies in Y_k, (2) if $1 \leqq l \leqq r_k$, then neither $B_{kl} \cap \lambda_k$ nor $B_{kl} \cap \mu_k$ contains two distinct points of Y_k, and (3) if $s < k$, $A_k \cap \text{Bd } A_s$ is disjoint from each of the β's.

Let

$$M_1 = h_1 \cup N_1, M_2 = M_1 \cup h_2 \cup N_2, \cdots, M_i = M_{i-1} \cup h_i \cup N_i, \cdots,$$
$$\text{and } M_{n-1} = M_{n-2} \cup h_{n-1} \cup N_{n-1}.$$

Thus if $1 \leqq i < n$, then $M_i = \bigcup_{j=1}^{i}(h_j \cup N_j)$.

We are now prepared to construct the partitioning F. We shall construct a cell partitioning F_1 of M_1, extend this to a cell partitioning F_2 of $M_2, \cdots$, and finally a partitioning F_{n-1} of M_{n-1}. We then construct F.

To construct F_1, our primary concern is to partition N_1. Let $\Sigma_1 = \text{Bd } h'_1$ and let Σ'_1 be the boundary component of N_1 distinct from Σ_1. Σ_1 and Σ'_1 are both 2-spheres, and $\text{Bd } N_1 = \Sigma_1 \cup \Sigma'_1$. Now N_1 is homeomorphic to $\Sigma_1 \times [0, 1]$, and we may construct a polyhedral product structure, denoted by $\Sigma_1 \times [0, 1]$, identifying Σ_1 and Σ'_1 with $\Sigma_1 \times \{0\}$ and $\Sigma_1 \times \{1\}$, respectively. We may assume that each component of $k \cap N_1$ is a product fiber.

The sets $\Delta^*_{1j} \cap \Sigma_1$, $1 \leqq j \leqq m_1$, together with $\Sigma_1 \cap (\text{Bd } h^*_1)$, form a polyhedral partitioning $\mathscr{E}_1$ of Σ_1. Let $\mathscr{E}'_1$ be the collection consisting of, for each $s > 1$, (a) each nonempty set $\Delta^*_{st} \cap \Sigma'_1$, $1 \leqq t \leqq m_s$, (b) each nonempty set $(\text{Bd } h^*_s) \cap \Sigma'_1$ and (c) each nonempty set $B_{sq} \cap \Sigma'_1$, $1 \leqq q \leqq r_s$. *$\mathscr{E}'_1$ is a polyhedral partitioning of Σ'_1.*

Let π_1 be projection onto Σ_1 in the product structure described above for N_1. We may assume that (1-skel $\mathscr{E}_1$) and π(1-skel $\mathscr{E}'_1$) are in relative general position on Σ_1. Then there exists a shellable disc partitioning $\mathscr{D}_1 = \langle D_{11}, D_{12}, \cdots, D_{1\nu_1}\rangle$ of Σ_1 such that if $D \in \mathscr{D}_1$, then (1) if E is either a set of $\mathscr{E}_1$, or for some set E' of $\mathscr{E}'_1$, $E = \pi(E')$, then $D \cap E$ is empty or a disc,

(2) Bd D and $(\cup\{\text{Bd } E\colon E \in \mathscr{E}_1\}) \cup (\cup\{\text{Bd } \pi(E)\colon E \in \mathscr{E}_1'\})$ are in relative general position on Σ_1, and (3) Bd D and k are disjoint. If $1 \leqq l \leqq \nu_1$, let $X_{1l} = D_{1l} \times [0, 1]$; we may assume that X_{1l} is polyhedral.

Let F_1 be the set consisting of h_1^*, the Δ_{1j}^* for $1 \leqq j \leqq m_1$, and the X_{1l} for $1 \leqq l \leqq \nu_1$. We shall show that F_1 is a cell partitioning of M_1. For any j, $1 \leqq j \leqq m_1, h_1^* \cap \Delta_{1j}^*$ is a disc, and if $1 \leqq l \leqq \nu_1$, then since $D_{1l} \cap \text{Bd } h_1^*$ is empty or a disc, $X_{1l} \cap h_1^*$ is empty or a disc. Any two distinct Δ_{1j}^* are disjoint. If $1 \leqq j \leqq m_1$ and $1 \leqq l \leqq \nu_1$, then $D_{1l} \cap \Delta_{1j}^*$ is empty or a disc, and thus $X_{1l} \cap \Delta_{1j}^*$ is empty or a disc. Finally, if $1 \leqq l < u \leqq \nu_1$, and $D_{1l} \cap D_{1u} \neq \phi$, then it is an arc, and hence $X_{1l} \cap X_{1u}$ is a disc. To show that the common part of three intersecting sets of F_1 is an arc, we use the facts that $\mathscr{D}_1$ is a disc partitioning of Σ_1, and the Δ_{ij}^* are products in a collar for Σ_1. A similar argument holds for the case of four elements of F_1.

Next we shall show that F_1 is shellable. Let

$$\langle h_1^*, \Delta_{11}^*, \Delta_{12}^*, \cdots, \Delta_{1m_1}^*, X_{11}, X_{12}, \cdots, X_{1\nu_1}\rangle$$

be a counting of F_1. We shall show that this is a shelling of F_1. Clearly it suffices to show that if we take a set in the counting after the first, and intersect that set with the union of those that precede it, we get a disc. Since the Δ_{1j}^* are mutually disjoint, this holds for h_1^* and all of the Δ_{1j}^*. Now $h_1^* \cup \Delta_{11}^* \cup \cdots \cup \Delta_{1m_1}^*$ is a 3-cell, h_1', and $\Sigma_1 = \text{Bd } h_1'$. Since $X_{11} \subset \Sigma_1$, then $X_{11} \cap h_1'$ is a disc. Suppose $1 \leqq l < \nu_1$, and $h_1' \cup X_{11} \cup \cdots \cup X_{1l}$ is a 3-cell Z_{1l}. Since $X_{1,l+1} \cap \Sigma_1$ is the disc $D_{1,l+1}$, and $D_{1,l+1}$ intersects $\cup_{t=1}^{l} D_{1t}$ in an arc α, then $Z_{1l} \cap X_{1,l+1}$ is the union of the two discs $D_{1,l+1}$ and $(\alpha \times [0, 1])$, along the arc α. Hence $Z_{1l} \cap X_{1,l+1}$ is a disc. Now $D_{1\nu_1} \cap (\cup_{t=1}^{\nu_1 - 1} D_{1t}) = \text{Bd } D_{1\nu_1}$, and $Z_{1,\nu_1 - 1} \cap X_{1\nu_1}$ is the union of the disc $D_{1\nu_1}$ and the annulus $(\text{Bd } D_{1\nu_1}) \times [0, 1]$, along the boundary of $D_{1\nu_1}$. Thus, $Z_{1,\nu_1 - 1} \cap X_{1\nu_1}$ is a disc. Hence the indicated counting is a shelling of F_1.

We shall now extend F_1 to a shellable cell partitioning F_2 of M_2. Our primary concern is with partitioning N_2. Recall that N_2 is a 3-cell. Let $\Sigma_2 = (\text{Bd } N_2) \cap (\text{Bd } h_2')$. Recall that $A_2 = (\text{Bd } N_2) \cap (\text{Bd } N_1)$. Let $\Sigma_2' = (\text{Bd } N_2) - \text{Int}(A_2 \cup \Sigma_2)$. Σ_2 and Σ_2' are discs, and $\text{Bd } N_2 = \Sigma_2 \cup A_2 \cup \Sigma_2'$. Now N_2 is homeomorphic to $\Sigma_2 \times [0, 1]$, and we may construct a polyhedral product structure, denoted by $\Sigma_2 \times [0, 1]$, identifying Σ_2, Σ_2', and A_2 with $\Sigma_2 \times \{0\}$, $\Sigma_2 \times \{1\}$, and $(\text{Bd } \Sigma_2) \times [0, 1]$, respectively. We may assume that each of the crossing arcs $\beta_{21}, \beta_{22}, \cdots$, and β_{2r_2} of A_2 are fibers in the product structure, and so is each component of $k \cap N_2$.

The sets $\Delta_{2j}^* \cap \Sigma_2$, $1 \leqq j \leqq m_2$, together with $\Sigma_2 \cap (\text{Bd } h_2^*)$, form a polyhedral partitioning $\mathscr{E}_2$ of Σ_2. Let $\mathscr{E}_2'$ be the collection consisting of, for each $s > 2$, (a) each nonempty set $\Delta_{st}^* \cap \Sigma_2'$, $1 \leqq t \leqq m_s$, (b) each nonempty set $(\text{Bd } h_s^*) \cap \Sigma_2'$, and (c) each nonempty set $B_{sq} \cap \Sigma_2'$, $1 \leqq q \leqq r_s$. $\mathscr{E}_2'$ is a polyhedral partitioning of Σ_2'.

Let π_2 be projection onto Σ_2 in the product structure described above for N_2. We may assume that (1-skel $\mathscr{E}_2$) and π(1-skel $\mathscr{E}_2'$) are in relative general position of Σ_2. There exists a ring partitioning

$$\mathscr{D}_2 = \{D_{21}, D_{22}, \cdots, D_{2r_2}, \cdots, D_{2\nu_2}\}$$

of Σ_2 such that (1) if $D \in \mathscr{D}_2$, then (a) if E is either a set of $\mathscr{E}_2$ or for some set E' of $\mathscr{E}'_2$, $E = \pi(E')$, then $D \cap E$ is empty or a disc, and (b) (Bd D) and k are disjoint, (2) the central disc $D_{2\nu_2}$ of $\mathscr{D}_2$ is disjoint from every Δ^*_{st}, $2 < s \leqq n$, and $1 \leqq t \leqq m_s$, and from every A_s, $2 \leqq s < n$, and (3) $\langle D_{21}, D_{22}, \cdots, D_{2r_2}, \cdots, D_{2\nu_2}\rangle$ is a ring shelling of $\mathscr{D}_2$. We may also assume that the outer ring Ω_2 of $\mathscr{D}_2$ is narrow, and the discs of Ω_2 are $D_{21}, D_{22}, \cdots$, and D_{2r_2} where for $1 \leqq w \leqq r_2$,

$$D_{2w} \cap (\text{Bd}\,\Sigma_2) = B_{2w} \cap (\text{Bd}\,\Sigma_2).$$

If $1 \leqq l \leqq \nu_2$, let $X_{2l} = D_{2l} \times [0, 1]$; we may assume X_{2l} is polyhedral.

Let F_2 be the set consisting of the sets of F_1 together with h^*_2, the Δ^*_{2j} for $1 \leqq j \leqq m_2$, and the X_{2l} for $1 \leqq l \leqq \nu_2$. We shall show that F_2 is a cell partitioning of M_2. A part of this proof may be gotten by modifying the argument above that F_1 is a cell partitioning of M_1. We need only consider how sets of F_2 in M_1 intersect those in $h'_2 \cup N_2$. Any set in $h^*_2, \Delta^*_{21}, \ldots,$ and $\Delta^*_{2m_2}$ is disjoint from each set of $h^*_1, \Delta^*_{11}, \ldots,$ and $\Delta^*_{1m_1}$. If F is either h^*_2 or for some j, $1 \leqq j \leqq m_2$, is Δ^*_{2j}, then by construction of $\mathscr{D}_1$, if $D \in \mathscr{D}_1$, $D \cap F$ is empty or a disc. Hence if $X \in F_1$, then $F \cap X$ is ϕ or a disc.

Now suppose $F \in F_2$, $F \subset N_2$, and F intersects M_1. Then F intersects A_2, and hence for some disc D_{2q} of the outer ring of $\mathscr{D}_2$, $F = D_{2q} \times [0, 1]$. Further, by the construction of the product structure $\Sigma_2 \times [0, 1]$, $F \cap A_2 = B_{2q}$. By construction of $\mathscr{D}_1$, if a set D of $\mathscr{D}_1$ intersects B_{2q}, their common part is a disc. It follows that if $X \in F_1$ and X intersects F, then $X \cap F$ is a disc. It now follows that F_2 is a cell partitioning of M_2.

Next we shall show that F_2 is shellable. Let $\langle h^*_1, \Delta^*_{11}, \cdots, \Delta^*_{1m_1}, X_{11}, \cdots, X_{1\nu_1}, X_{21}, X_{22}, \cdots, X_{2r_2}, \cdots, X_{2,\nu_1-1}, \Delta^*_{21}, \cdots, \Delta^*_{2m_2}, h^*_2, X_{2\nu_2}\rangle$ be a counting of F_2. Note that we count the "plug" $X_{2\nu_2}$ last. We shall show that this counting is a shelling of F_2. We only need to consider those sets of F_2 not in F_1. Recall that $\bigcup\{F\colon F \in F_1\} = M_1$.

Since $X_{21} \cap A_2$ is the disc B_{21}, it follows that $X_{21} \cap M_1 = B_{21}$. If $1 < l < r_2$, then $X_{2l} \cap A_2$ is the disc B_{2l}, and since $B_{2l} \cap B_{2,l-1}$ is an arc, then $X_{2l} \cap X_{2,l-1}$ is a disc. These discs intersect in one of the β's, and hence X_{2l} intersects the union of those before it in the counting in a disc. X_{2r_2} intersects M_1 in a disc B_{2r_2}, intersects X_{2,r_2-1} in a disc, intersects X_{21} in a disc, and the union of these three is a disc. A similar argument holds for each other ring in the partitioning $\mathscr{D}_2$. Thus $h_1 \cup N_1 \cup (\bigcup_{l=1}^{\nu_2-1} X_{2l})$ is a 3-cell Z_{2,ν_2-1}. Clearly for each j, $1 \leqq j \leqq m_2$, $Z_{2,\nu_2-1} \cap \Delta^*_{2j}$ is a disc. $[(Z_{2,\nu_2-1}) \cup (\bigcup_{j=1}^{m_2}\Delta^*_{2j})] \cap h^*_2$ is the disc (Bd h^*_2) $-$ Int $D_{2\nu_2}$. It follows that the union of all the cells of F_2 except $X_{2\nu_2}$ is a 3-cell W_{2,ν_2-1}. $X_{2\nu_2} \cap W_{2,\nu_2-1}$ is the union of the disc $D_{2\nu_2}$ and the annulus (Bd $D_{2\nu_2}$) $\times [0, 1]$, and is a disc. Hence F_2 is shellable.

We continue this process, constructing a shellable cell partitioning F_3 of M_3 that extends F_2, a shellable cell partitioning F_4 of M_4 that extends F_3, and so on. Suppose $2 < i < n$, F_{i-1} has been constructed, and is a shellable cell partitioning of M_{i-1}. We construct F_i by first partitioning N_i, using a ring partitioning in a manner analogous to that of the construction of F_2. As part of this, we construct a "plug" for N_i. Then we define F_i to consist of the cells of F_{i-1}, the cells partitioning N_i, the Δ^*_{ij}, and h^*_i.

To prove that F_i is a cell partitioning of M_i, we may modify the arguments given for F_1 and F_2. The main additional point to be considered involves cells F of F_i lying in N_i and intersecting Bd A_s for some $s < i$. Then F intersects A_i, and for some j, $1 \leqq j \leqq r_i$, $F \cap A_i = B_{ij}$. Suppose F intersects a cell F' of F_s such that F' lies in N_s and intersects A_s. Then A_i intersects Bd A_s in a single spanning arc of A_i. For some k, $1 \leqq k \leqq r_s$, $F' \cap A_s = B_{sk}$, and F' is a slight thickening, relative to N_s, of B_{sk}. It follows that $F \cap F'$ is a disc. In showing that no four sets of F_i have any arc in common, we may use the fact that $\mathscr{N} \cup \{h'_i\colon 1 \leqq i \leqq n\}$ is a cell partitioning of S^3.

We order F_i by first counting the cells of F_{i-1}, following the given shelling, then the cells of F_i in N_i, using a ring shelling of the ring partitioning involved, except that we do not count the "plug." We then count the Δ^*_{ij}, then h^*_i, and finally the "plug." It is easily seen that this is a shelling of F_i.

Suppose that we have defined F_{n-1}. Note that F_{n-1} covers $S^3 -$ Int h'_n. We now define F to be the set consisting of the cells of F_{n-1}, the Δ^*_{nj} for $1 \leqq j \leqq m_n$, and h^*_n. Clearly F is a cell partitioning of S^3. To obtain a shelling for F, we first count F_{n-1} as above, then $\Delta^*_{n1}, \Delta^*_{n2}, \cdots$, and Δ_{nm_n}, and finally h^*_n.

Finally, k is compatible with F and $|\pi(k, F)| = 2|\pi(k, H)|$. To see this, note that (1) each Δ^*_{ij} contains exactly one spanning arc lying on k, and this arc is straight in Δ^*_{ij}, and (2) if $p \in k \cap$ (2-skel H), there is some X_{st} such that $k \cap X_{st}$ is a single spanning arc of X_{st} containing p, and this arc is straight in X_{st}. □

LEMMA 5. *If* $\langle h_1, h_2, \cdots, h_n \rangle$ *is a shelling of a polyhedral cell partitioning of* S^3, *and* $1 < i < n$, *then* $(h_1 \cup h_2 \cup \cdots \cup h_{i-1}) \cap h_i$ *is a disc.*

Proof. Suppose $1 < i < n$, and let $H_{i-1} = h_1 \cup h_2 \cup \cdots \cup h_{i-1}$, $H_i = H_{i-1} \cup h_i$, and $D_i = H_{i-1} \cap h_i$. Then both H_{i-1} and H_i are 3-cells, and each component of D_i is a punctured disc.

Suppose D_i is not connected, and let A and B be distinct components of D_i. There is a polyhedral simple closed curve J on Bd H_{i-1} disjoint from D_i and separating A from B on Bd H_{i-1}. Let Δ be the disc on Bd H_{i-1} bounded by J and containing A. Let α be a polyhedral spanning arc of H_{i-1} from a point x of Int A to a point y of Int B. Let β be a polyhedral spanning arc of h_i from x to y. Then $\alpha \cup \beta$ and J are disjoint polyhedral simple closed curves in S^3, and, by considering Δ, can be seen to be linked in

S^3. Since $\alpha \cup \beta \subset \text{Int } H_i$ and $J \cap \text{Int } H_i = \phi$, this is a contradiction. Hence D_i is a punctured disc.

Suppose Bd D_i is not connected. Suppose K and L are distinct boundary curves of D_i, and let U and V be the components of (Bd H_{i-1}) $- D_i$ bounded by K and L, respectively. Then $\overline{U}$ is a disc, and U and V lie on Bd H_i. Let α' be a polyhedral spanning arc of H_{i-1} from a point x' of U to a point y' of V, and let β' be a polyhedral spanning arc of $S^3 - \text{Int } H_i$ from x' to y'. Then $\alpha' \cup \beta'$ and K are disjoint polyhedral simple closed curves in S^3, and, by considering $\overline{U}$, can be seen to be linked. Since $K \subset h_i$ and $\alpha' \cup \beta') \cap h_i = \phi$, this is a contradiction. Hence D_i is a disc. □

4. Nonshellable cell partitionings of S^3

In this section, we shall describe two examples whose constructions use the ideas of the preceding part of this paper. The first is a nonshellable cell partitioning of S^3. The second is a nest of cell partitionings of S^3, each partitioning of the nest being nonshellable.

To construct the first example, let k be a trefoil knot in S^3. It is known that k has bridge number 2 [8]. Let T be a polyhedral tubular neighborhood of k; T is a solid torus. Divide T into three polyhedral chambers T_1, T_2, and T_3 by using polyhedral meridional disc of T, each intersecting k in exactly one point. See Figure 2. It is easy to construct a polyhedral cell partitioning H of S^3 which includes T_1, T_2, and T_3 among the 3-cells of H. Clearly, k is

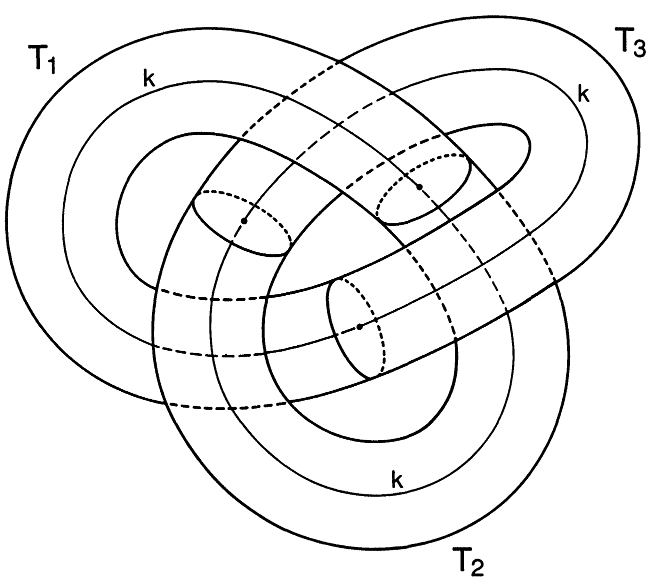

FIG. 2

compatible with H, and $\pi(k, h) = 3$. By Theorem 1, H is nonshellable. For additional examples of nonshellable cell partitionings of S^3, see [2].

For the second example, we shall need some definitions. If A and B are two coverings of a set X, then *B refines A* if and only if each set of B lies in some set of A. A *nest* of polyhedral cell partitionings of S^3 is a sequence $\{H_1, H_2, H_3, \cdots\}$ of polyhedral cell partitionings of S^3 such that (1) for each positive integer n, H_{n+1} refines H_n, (2) if m and n are positive integers and $m > n$, then for each cell h of H_n, the set of all 3-cells of H_m lying in h is a cell partitioning of h, (3) as $n \to \infty$, (mesh H_n) $\to 0$, and (4) certain natural general position conditions are satisfied (see [1]). The conditions of (4) can be obtained by the standard type of small adjustment, so we shall not consider them here.

The second example is a polyhedral nest $\{H_1, H_2, H_3, \cdots\}$ of cell partitionings of S^3 such that for each positive integer n, H_n is nonshellable.

Let H_1 be the cell partitioning of the example above. Let k_1 denote the trefoil knot used in that construction. The knot k_1 lies in a tubular neighborhood T of k_1, and T is cut into three 3-cells T_1, T_2, and T_3. Let $W_1 = T$. Note that $\operatorname{br} k_1 = 2$ and $|\pi(k_1, H_1)| = 3$. As noted above, H_1 is nonshellable. We may make the construction of H_1 so that if L is the arc length of the knot k_1, then (mesh H_1) $< \frac{1}{2}L$.

If $i = 1, 2$, or 3, replace $k_1 \cap T_i$ by a polygonal spanning arc knotted in a trefoil, with the same endpoints as $k_1 \cap T_i$. This yields a knot k_2 in $\operatorname{Int} W_1$. See Figure 3.

Now k_2 is the composite of the trefoil k_1 with three other trefoil knots, one in each of T_1, T_2, and T_3. With the aid of the following lemma from [8] we may show that $\operatorname{br} k_2 = 2 + 3$.

LEMMA 5. *If s and t are two knots in S^3 and $s\#t$ is a composite of s and t, then* $\operatorname{br}(s\#t) = (\operatorname{br} s) + (\operatorname{br} t) - 1$.

Let W_2 be a very close tubular neighborhood of k_2. For each $i, i = 1, 2$, or 3, divide $W_2 \cap T_i$ into three 3-cells by meridional discs in W_2. If $i = 1, 2$, or 3, let the resulting 3-cells of W_2 in T_i be denoted by T_{i1}, T_{i2}, and T_{i3}.

We may make the construction of k_2 and W_2 such that if $i = 1, 2$, or 3, then (diam T_{ij}) $< \frac{1}{4}L$. It is easy to construct a polyhedral cell partitioning H_2 of S^3 such that (1) H_2 refines H_1 (2) if $h \in H_1$, the cells of H_2 in h form a partitioning of h, and (3) (mesh H_2) $< \frac{1}{4}L$.

Continue this process. Suppose that n is a positive integer and H_n has been constructed. Then there exist a knot k_n and a tubular neighborhood W_n of k_n. W_n is divided into 3^n 3-cells, each of which belongs to H_n and each of which has diameter less than $L/2^n$. If T is one of these 3-cells, then $k_n \cap T$ is a straight spanning arc of T. Further,

$$\operatorname{br} k_n = 2 + 3 + 3^2 + \cdots + 3^{n-1}.$$

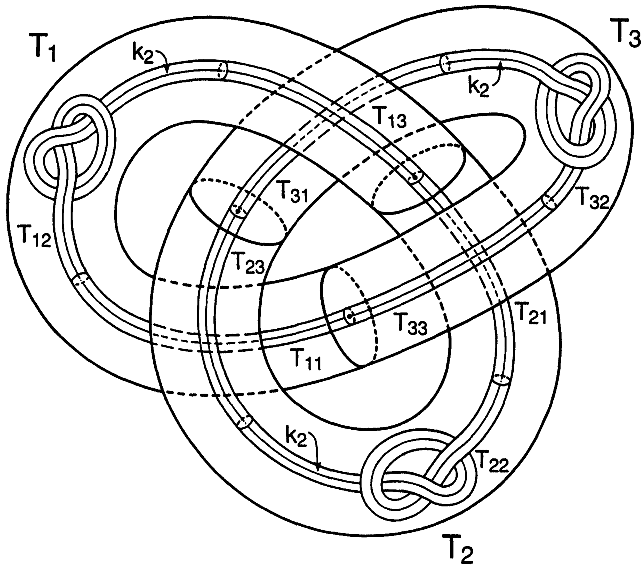

FIG. 3

Since $2\,\mathrm{br}\,k_n = 3^n + 1$ and $|\pi(k_n, H_n)| = 3^n$, then, by Theorem 1, H_n is nonshellable.

For each such 3-cell T, replace $k_n \cap T$ by a polygonal spanning arc knotted in a trefoil, with the same endpoints as $k_n \cap T$, and such that it can be cut into three subarcs, each of diameter less than $L/2^{n+1}$. This yields a knot k_{n+1} in Int W_n. Then k_n is the composite of k_n and 3^n trefoil knots. By Lemma 5,

$$\mathrm{br}\,k_{n+1} = \frac{3^n + 1}{2} + 3^n = \frac{3^{n+1} + 1}{2}.$$

Let W_{n+1} be a close tubular neighborhood of k_{n+1}. Cut W_{n+1} into 3^{n+1} 3-cells by using discs on the 2-skeleton of H_n and additional meridional discs of W_{n+1}, so that each 3-cell T as above contains exactly three of the 3-cells from W_{n+1}. We may make this construction so that each of the resulting 3-cells from W_{n+1} has diameter less than $L/2^{n+1}$.

There is a polyhedral cell partitioning H_{n+1} of S^3 that includes the 3-cells from W_{n+1} constructed above and has mesh less than $L/2^{n+1}$. Since $2\,\mathrm{br}\,k_{n+1} = 3^{n+1} + 1$ and $|\pi(k_{n+1}, H_{n+1})| = 3^{n+1}$, it follows by Theorem 1 that H_{n+1} is nonshellable.

Thus by induction, there exist polyhedral cell partitionings $H_1, H_2, H_3, \cdots$ of S^3 as described above. It is easily verified that $\{H_1, H_2, H_3, \cdots\}$ is a nest of polyhedral cell partitionings of S^3. By construction, for each positive integer n, H_n is nonshellable.

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The Pennsylvania State University
University Park, Pennsylvania