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# TANGENTIAL HARMONIC APPROXIMATION ON RELATIVELY CLOSED SETS

STEPHEN J. GARDINER

## 1. Introduction

Let $\Omega$ be an open set in Euclidean space $\mathbb{R}^n$ ($n \geq 2$) and $E$ be a relatively closed subset of $\Omega$. A subset $A$ of $\Omega$ will be called *$\Omega$-bounded* if its closure $\bar{A}$ is a compact subset of $\Omega$. We use $\hat{E}$ to denote the union of $E$ with all $\Omega$-bounded (connected) components of $\Omega \setminus E$. As usual, $A^{\circ}$ and $\partial A$ will denote respectively the interior and boundary of a set $A$. Also, $C(A)$ will denote the collection of all real-valued continuous functions on $A$, and $\mathscr{H}(A)$ (resp. $\mathscr{S}^{+}(A)$) will denote the collection of functions which are harmonic (resp. positive and superharmonic) on some open set containing $A$. We will say that the pair $(\Omega, E)$ *satisfies the $(K, L)$-condition* if, for each compact subset $K$ of $\Omega$, there is a compact subset $L$ of $\Omega$ which contains every $\Omega$-bounded component of $\Omega \setminus (E \cup K)$ whose closure intersects $K$. The Alexandroff compactification of $\Omega$ will be denoted by $\Omega^*$. We note that $\Omega^* \setminus E$ is connected if and only if $\hat{E} = E$ and that, if this is the case, then $\Omega^* \setminus E$ is locally connected if and only if $(\Omega, E)$ satisfies the $(K, L)$-condition. The following result was recently established by Armitage and Goldstein [3, Theorem 1.1].

THEOREM A. *Let $\Omega$ be a connected open set in $\mathbb{R}^n$ which possesses a Green function $G(\cdot, \cdot)$, let $E$ be a relatively closed subset of $\Omega$ and let $P \in \Omega$. If $\Omega^* \setminus E$ is connected and locally connected, then for each $h$ in $\mathscr{H}(E)$ and for each positive number $\varepsilon$ there exists $H$ in $\mathscr{H}(\Omega)$ such that*

$$|H(X) - h(X)| < \varepsilon \min\{1, G(P, X)\} \quad (X \in E).$$

Using Theorem A and material from [9] we will prove the following. The reader is referred to Helms [12] or Doob [7] for an account of thin sets and the fine topology.

THEOREM 1. *Let $\Omega$ be a connected open set in $\mathbb{R}^n$ and $E$ a relatively closed proper subset of $\Omega$. The following are equivalent.*

(a) *For each $h$ in $\mathscr{H}(E)$ and each positive number $\varepsilon$, there exists $H$ in $\mathscr{H}(\Omega)$ such that $|H - h| < \varepsilon$ on $E$.*

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(b) *For each h in* $\mathscr{H}(E)$ *and each s in* $\mathscr{S}^+(\hat{E})$, *there exists H in* $\mathscr{H}(\Omega)$ *such that* $0 < H - h < s$ *on E.*

(c) $\Omega \setminus \hat{E}$ *and* $\Omega \setminus E$ *are thin at the same points of E, and* $(\Omega, E)$ *satisfies the* $(K, L)$*-condition.*

The equivalence of (a) and (c) above is due to the author [9, Theorem 4]; condition (b) is new. Clearly Theorem 1 includes Theorem A, and it permits faster approximation if we impose restrictions on the set $E$. A simple example is given below to illustrate this comment and also the sharpness of the speed of approximation in (b).

*Example* 1. Let $\Omega = \mathbb{R}^n$, let $a_m > 0$ $(m = 1, \dots, n-1)$ and define

$$\omega = (-a_1, a_1) \times \cdots \times (-a_{n-1}, a_{n-1}) \times \mathbb{R} \quad \text{and} \quad \alpha = (\pi/2)\left(\sum_{m=1}^{n-1} a_m^{-2}\right)^{1/2}$$

Then:

(i) Given any pair $(\mathbb{R}^n, E)$ such that condition (c) of Theorem 1 holds and $E \subset \omega$, any $h$ in $\mathscr{H}(E)$ and any positive number $\varepsilon$, there exists $H$ in $\mathscr{H}(\mathbb{R}^n)$ such that

$$0 \le H(X) - h(X) < \varepsilon \exp(-\alpha |x_n|) \quad (X = (x_1, \dots, x_n) \in E);$$

(ii) The above statement becomes false if $\alpha$ is replaced by any larger number.

The exponential decay of the error in (i) above compares favourably with Theorem A, where the maximum error is $\varepsilon \log(1/(1 + |X|))$ when $n = 2$, or $\varepsilon(1 + |X|)^{2-n}$ when $n \ge 3$, and also with Theorem 1.1 of [2] where the maximum error is of the form $\varepsilon(1 + |X|)^{-a}$ for any predetermined choice of the number $a$.

Further applications of Theorem 1 are given in Theorems 2, 3 and 5 below.

THEOREM 2. *Let* $\Omega$ *be a connected open set in* $\mathbb{R}^n$ *and E a relatively closed proper subset of* $\Omega$. *The following are equivalent.*

(a) *For each h in* $C(E) \cap \mathscr{H}(E^\circ)$ *and each positive number* $\varepsilon$, *there exists H in* $\mathscr{H}(\Omega)$ *such that* $|H - h| < \varepsilon$ *on E.*

(b) *For each h in* $C(E) \cap \mathscr{H}(E^\circ)$ *and each s in* $\mathscr{S}^+(\hat{E})$, *there exists H in* $\mathscr{H}(\Omega)$ *such that* $0 < H - h < s$ *on E.*

(c) $\Omega \setminus \hat{E}$ *and* $\Omega \setminus E^\circ$ *are thin at the same points of E, and* $(\Omega, E)$ *satisfies the* $(K, L)$*-condition.*

Again the equivalence of (a) and (c) above is known (see [9, Theorem 5]), the new feature being condition (b). Theorem 2 improves Theorem 1.2 in [3] in the same way that Theorem 1 improves Theorem A. We note that

Shaginyan [15] has announced a result related to Theorem 2, but no proof has yet appeared.

In the next three results we investigate which pairs $(\Omega, E)$ permit approximation with an error function which decays arbitrarily quickly.

THEOREM 3. *Let $\Omega$ be a connected open set in $\mathbb{R}^n$ and $E$ a relatively closed proper subset of $\Omega$. The following are equivalent.*

(a) *For each $h$ in $\mathscr{H}(E)$ and each continuous function $\varepsilon: E \to (0,1]$, there exists $H$ in $\mathscr{H}(\Omega)$ such that $0 < H - h < \varepsilon$ on $E$.*

(b) *The pair $(\Omega, E)$ satisfies:*

(i) *$\Omega \setminus \hat{E}$ and $\Omega \setminus E$ are thin at the same points of $E$, and*

(ii) *for each compact subset $K$ of $\Omega$ there is a compact subset $L$ of $\Omega$ which contains every $\Omega$-bounded component of $\Omega \setminus (E \cup K)$ whose closure intersects $K$ and also every fine component of the fine interior of $E$ that intersects $K$.*

The following result solves a problem posed by Boivin and Gauthier [5].

THEOREM 4. *Let $\Omega$ be a connected open set in $\mathbb{R}^n$ and $E$ a relatively closed proper subset of $\Omega$. The following are equivalent.*

(a) *For each $h$ in $C(E) \cap \mathscr{H}(E^\circ)$ and each continuous function $\varepsilon: E \to (0,1]$, there exists $H$ in $\mathscr{H}(E)$ such that $|H - h| < \varepsilon$ on $E$.*

(b) *For each $h$ in $C(E) \cap \mathscr{H}(E^\circ)$ and each continuous function $\varepsilon: E \to (0,1]$, there exists $H$ in $\mathscr{H}(E)$ such that $0 < H - h < \varepsilon$ on $E$.*

(c) *The pair $(\Omega, E)$ satisfies:*

(i) *$\Omega \setminus E$ and $\Omega \setminus E^\circ$ are thin at the same points of $E$, and*

(ii) *for each compact subset $K$ of $\Omega$ there is a compact subset $L$ of $\Omega$ which contains every component of $E^\circ$ that intersects $K$.*

Finally, the above results can be combined to obtain the following slight refinement (the new feature is the introduction of condition (b)) of a recent result of Goldstein and the author [10]. The proof given in that paper is more direct, and independent of [3].

THEOREM 5. *Let $\Omega$ be a connected open set in $\mathbb{R}^n$ and $E$ a relatively closed proper subset of $\Omega$. The following are equivalent.*

(a) *For each $h$ in $C(E) \cap \mathscr{H}(E^\circ)$ and each continuous function $\varepsilon: E \to (0,1]$, there exists $H$ in $\mathscr{H}(\Omega)$ such that $|H - h| < \varepsilon$ on $E$.*

(b) *For each $h$ in $C(E) \cap \mathscr{H}(E^\circ)$ and each continuous function $\varepsilon: E \to (0,1]$, there exists $H$ in $\mathscr{H}(\Omega)$ such that $0 < H - h < \varepsilon$ on $E$.*

(c) *The pair $(\Omega, E)$ satisfies:*

(i) *$\Omega \setminus \hat{E}$ and $\Omega \setminus E^\circ$ are thin at the same points of $E$, and*

(ii) *for each compact subset $K$ of $\Omega$ there is a compact subset $L$ of $\Omega$ which contains every $\Omega$-bounded component of $\Omega \setminus (E \cup K)$ whose closure intersects $K$ and also every component of $E^\circ$ that intersects $K$.*

The proofs of Theorem 1–5 are given in §2–6, and the details of Example 1 can be found in §7. In §8, we discuss the possibility of adding a third equivalent condition to Theorem 3 corresponding to condition (a) of Theorem 5.

## 2. Proof of Theorem 1

**2.1.** We will require the following lemmas.

LEMMA 1. *Suppose that condition* (c) *of Theorem* 1 *holds. Then, for each* $h$ *in* $\mathscr{H}(E)$ *and for each* $s$ *in* $\mathscr{S}^+(\hat{E})$ *there exists* $H$ *in* $\mathscr{H}(\hat{E})$ *such that* $|H - h| < s$.

LEMMA 2. *Let* $\Omega$ *be an open set in* $\mathbb{R}^n$, *let* $E$ *be a relatively closed subset of* $\Omega$, *and suppose that* $\Omega^* \setminus E$ *is connected and locally connected. Then there is a sequence* $(K_m)$ *of compact subsets of* $\Omega$ *such that* $\bigcup_m K_m = \Omega$ *and such that, for each* $m$, *we have* $K_m \subset K_{m+1}^\circ$ *and the set* $\Omega^* \setminus (E \cup K_m)$ *is connected.*

LEMMA 3. *Let* $\Omega$ *be an open set in* $\mathbb{R}^n$, *let* $E$ *be a relatively closed subset of* $\Omega$, *and suppose that* $\Omega^* \setminus E$ *is connected and locally connected. Then, for each* $h$ *in* $\mathscr{H}(E)$ *and each* $s$ *in* $\mathscr{S}^+(\Omega)$, *there exists* $H$ *in* $\mathscr{H}(\Omega)$ *such that* $|H - h| < s$ *on* $E$.

To prove Lemma 1 we follow the argument of [9, §§7.1, 7.2], replacing each occurrence of $\varepsilon$ by $s(X)$.

Lemma 2 is elementary and so its proof is left to the reader.

In proving Lemma 3 we may, be considering each component of $\Omega$ separately, assume that $\Omega$ is connected. If $\Omega$ does not have a Green function, then $n = 2$ and $s$ is constant by Myrberg's theorem [12, Theorem 8.33]. The conclusion of the lemma then follows from a result of Gauthier, Goldstein and Ow [11, Theorem 3] (which is a special case of the known equivalence of (a) and (c) in Theorem 1). If $\Omega$ possesses a Green function $G(\cdot, \cdot)$, then let $B$ be a closed ball in $\Omega$ with centre $P$ and choose a number $a$ in the interval $[1, \infty)$ such that

$$a > \sup\{G(P, X) \colon X \in \Omega \setminus B\}.$$

Given $s$ in $\mathscr{S}^+(\Omega)$ we define

$$b = \inf\{s(X)/\min\{a, G(P, X)\} \colon X \in B\}.$$

It follows that

$$s(X) \geq b \min\{a, G(P,X)\} \geq b \min\{1, G(P,X)\} \quad (X \in \Omega).$$

Hence Lemma 3 follows in this case from Theorem A.

**2.2.** Theorem 1 will now be proved. Clearly (b) implies (a), and we know from [9, Theorem 4] that (a) implies (c), so it remains to prove that (c) implies (b).

Suppose that (c) holds, let $h$ be a harmonic function on an open set $\omega_1$ which contains $E$ and let $s$ be a positive superharmonic function on an open set $\omega_2$ which contains $\hat{E}$. If $\hat{E} = \Omega$, then $E = \Omega$ by the thinness assumption in (c), which yields a contradiction. So $\hat{E} \neq \Omega$. Replacing $s$ by its reduced function (réduite) relative to a closed ball contained in $W \setminus \hat{E}$, for each component $W$ of $\omega_2$, we can assume $s$ to be harmonic on $\omega_2$. Also, we can assume that $\inf_E s = 0$, for otherwise the desired inequality in (b) is a consequence of the known equivalence of (a) and (c). It will be enough to show that there exists $H$ in $\mathscr{H}(\Omega)$ such that $|H - h| < s$ on $E$. For, if this can be done, then (since $h + s/2$ is in $\mathscr{H}(E)$) we can find $H$ in $\mathscr{H}(\Omega)$ such that

$$|H - (h + s/2)| < s/2 \quad \text{on } E,$$

and deduce that $0 < H - h < s$ on $E$. In view of Lemma 1 we can assume that $\hat{E} = E$. Thus $\Omega^* \setminus E$ is connected and (by (c)) locally connected.

Let $\omega$ be an open set such that

$$E \subset \omega \subset \overline{\omega} \subset \omega_1 \cap \omega_2,$$

let $(K_m)$ be a sequence of compact subsets of $\Omega$ as in Lemma 2, and let

$$a = \inf\{s(X): X \in \overline{\omega} \cap K_1\}.$$

For each $k$ in $\mathbb{N}$ we define the sets

$$D_k = \{X \in \omega_2: s(X) \leq 2^{-k}a\} \quad \text{and} \quad \Omega_k = \Omega \setminus (\partial\omega \cap D_k),$$

and the integer

$$m(k) = \sup\{m \in \mathbb{N}: K_m \cap \overline{\omega} \cap D_k = \emptyset\}.$$

We observe that, for a fixed choice of $m$, the set $K_m \cap \overline{\omega} \cap D_k$ is void for all sufficiently large values of $k$. Hence $m(k) \to \infty$ as $k \to \infty$.

Since $\Omega^* \setminus (E \cup K_{m(k)})$ is connected (see Lemma 2), so too is the set $\Omega^*_{k+1} \setminus (E \cup K_{m(k)})$. The latter set is also locally connected. For, if this were not the case, then there would be a compact subset $L$ of $\Omega_{k+1}$ (and hence of $\Omega$) for which the $\Omega_{k+1}$-bounded components $\{V_j\}$ of $\Omega_{k+1} \setminus (E \cup K_{m(k)} \cup L)$ do not have $\Omega_{k+1}$-bounded union. Thus there is a sequence of points $(X_l)$ such that each $X_l$ belongs to some $V_j$, and such that $(X_l)$ converges to the Alexandroff point for $\Omega_{k+1}$. However, in view of the fact that $\Omega^* \setminus E$ is locally connected, we know that $\bigcup_j V_j$ is $\Omega$-bounded, and so there is a subsequence of $(X_l)$ which converges to some point of $E$. Since $E \cap \partial\Omega_{k+1} \subseteq E \cap \partial\omega = \emptyset$, we obtain a contradiction. Thus $\Omega^*_{k+1} \setminus (E \cup K_{m(k)})$ is locally connected, as claimed.

Next we define a function on $\Omega_k$ by

$$s_k(X) = \begin{cases} s(X) & \text{if } X \in \omega \cap D_k, \\ 2^{-k}a & \text{elsewhere in } \Omega_k. \end{cases}$$

Clearly $s_k$ is positive and superharmonic on $\Omega_k$, and satisfies $s_k = \min\{s, 2^{-k}a\}$ on $\omega$. Further, $s_{k+1} \le s_k$ on $\Omega_k$.

We now define a sequence $(h_k)$ of harmonic functions inductively as follows. By Lemma 3 there exists $h_1$ in $\mathscr{H}(\Omega_1)$ such that $|h_1 - h| < 2^{-1}s_1$ on $E$. Given $h_k$ in $\mathscr{H}(\Omega_k)$, we use Lemma 3 to obtain $h_{k+1}$ in $\mathscr{H}(\Omega_{k+1})$ such that

$$|h_{k+1} - h_k| < 2^{-k-1}s_{k+1} \quad \text{on } E \cup K_{m(k)}.$$

On $K_{m(k)}$ we have

$$|h_l - h_k| \le \sum_{j=k}^{l-1} |h_{j+1} - h_j| < \sum_{j=k}^{l-1} 2^{-j-1}s_{j+1} < 2^{-k}s_{k+1} \le 2^{-2k-1}a$$

when $l > k$. Hence the sequence $(h_k)$ converges locally uniformly on $\Omega$ to a harmonic function $H$. On $E$ we have

$$|h_k - h| \le \sum_{j=2}^{k} |h_j - h_{j-1}| + |h_1 - h| < \sum_{j=1}^{k} 2^{-j}s_j < s,$$

and hence $|H - h| \le s$. This completes the proof of Theorem 1.

## 3. Proof of Theorem 2

The following lemma may be deduced from [3, Theorem 1.2] in the same way that Lemma 3 was deduced from Theorem A.

LEMMA 4. *Let $\omega$ be an open set in $\mathbb{R}^n$ and let $E$ be a relatively closed subset of $\omega$ such that $\omega \setminus E$ and $\omega \setminus E^\circ$ are thin at the same points of $E$. Then, for each $h$ in $C(E) \cap \mathscr{H}(E^\circ)$ and each $s$ in $\mathscr{S}^+(\omega)$, there exists $H$ in $\mathscr{H}(E)$ such that $|H - h| < s$ on $E$.*

Theorem 2 will now be deduced from Theorem 1 and Lemma 4. Clearly (b) implies (a), and we know from [9, Theorem 5] that (a) implies (c). Suppose now that (c) holds, let $h$ be in $C(E) \cap \mathscr{H}(E^\circ)$ and let $s$ be a positive superharmonic function on some open set $\omega$ which contains $\hat{E}$. We note that $E$ is relatively closed in $\omega$. Also, since $E^\circ \subseteq E \subseteq \hat{E}$, we know that $\Omega \setminus \hat{E}$, $\Omega \setminus E$ and $\Omega \setminus E^\circ$ (and, of course, $\omega \setminus E$ and $\omega \setminus E^\circ$) are all thin at the same points of $E$. Thus we can apply Lemma 4 to obtain $h_1$ in $\mathscr{H}(E)$ such that $|h_1 - h| < s/2$ on $E$. Next we apply Theorem 1 to obtain $h_2$ in $\mathscr{H}(\Omega)$ such that $|h_2 - h_1| < s/2$ on $E$, and hence $|h_2 - h| < s$ on $E$. In view of what was said in the second paragraph of §2.2, this is enough to show that there exists $H$ in $\mathscr{H}(\Omega)$ such that $0 < H - h < s$. Thus (b) is established.

## 4. Proof of Theorem 3

**4.1.** We will assume throughout Sections 4–6 that $\Omega$ is a connected open set in $\mathbb{R}^n$ and that $E$ is a relatively closed proper subset of $\Omega$.

LEMMA 5. *If the pair $(\Omega, E)$ satisfies condition* (b) *of Theorem* 3, *then so does the pair $(\Omega, \hat{E})$.*

To prove Lemma 5, suppose that $(\Omega, E)$ satisfies condition (b) of Theorem 3 and let $F = \hat{E}$. Then $\hat{F} = F$ and so $(\Omega, F)$ trivially satisfies condition (b)(i). We now define a function $f$ on the fine interior of $F$ by assigning it the value 0 at points which also belong to $E$, and the value $k$ on the $k$th $\Omega$-bounded component of $\Omega \setminus E$. To see that $f$ is finely continuous, we need only check that $f^{-1}(\{0\})$ is finely open. In fact, if $Y \in f^{-1}(\{0\})$, then $Y \in E$. Since $\Omega \setminus F$ is thin at $Y$, so also is $\Omega \setminus E$ by hypothesis, and hence $f^{-1}(\{0\})$ is a fine neighbourhood of $Y$, as required. It follows that each fine component of the fine interior of $\hat{E}$ is either a fine component of the fine interior of $E$ or an $\Omega$-bounded component of $\Omega \setminus E$. Since $(\Omega, E)$ satisfies condition (b)(ii) it is now clear that $(\Omega, F)$ satisfies this condition also.

**4.2.** One implication in Theorem 3 will be deduced from Theorem 1 using the following lemma.

LEMMA 6. *Suppose that the pair $(\Omega, E)$ has the following property.*

$(*)$ *For each compact subset $K$ of $\Omega$ there is a compact subset $L$ of $\Omega$ which contains every fine component of the fine interior of $E$ that intersects $K$.*

*Then, for each continuous function* $\varepsilon\colon E \to (0, 1]$, *there exists* $s$ *in* $\mathscr{S}^+(E)$ *such that* $s < \varepsilon$ *on* $E$.

In proving Lemma 6 we can assume, by deleting a closed ball contained in $\Omega \setminus E$, if necessary, that $\Omega$ possesses a Green function. Using condition $(*)$ we can construct a sequence $(K_m)$ of compact subsets of $\Omega$ such that $K_1^\circ \neq \varnothing$ and $\cup_m K_m = \Omega$, and such that, for each $m$,

(I) $K_m \subset K_{m+1}^\circ$, and

(II) every fine component $V$ of the fine interior of $E$ which satisfies $\overline{V} \cap K_m \neq \varnothing$ also satisfies $\overline{V} \subset K_{m+1}^\circ$.

Next we define $A(l, m) = K_m \setminus K_l^\circ$ whenever $l < m$, and also

$$F(m; k) = \{X \in A(m, m+3)\colon \operatorname{dist}(X, E) \geq 1/k\} \quad (m, k \in \mathbb{N})$$

and

$$\delta_m = \inf\{\varepsilon(X)\colon X \in E \cap A(m, m+1)\} \quad (m \in \mathbb{N}).$$

Now let $P \in K_1^\circ$, let $g$ denote the Green function for $\Omega$ with pole at $P$, and let $R_g^C$ denote the reduced function of $g$ relative to a set $C$ in $\Omega$. We observe (see [7, 1.VI.3(e)]) that

$$R_g^{F(m;k)}(X) \uparrow R_g^{A(m,m+3)\setminus E}(X) \quad (k \to \infty;\ X \in \Omega).$$

If $u$ is a positive superharmonic function on $\Omega$ which satisfies $u(X) \geq g(X)$ when $X \in A(m, m+3) \setminus E$, then the same inequality holds for points $X$ of the set

$$\{X \in E\colon A(m, m+3) \setminus E \text{ is not thin at } X\}. \tag{1}$$

In view of (II) above and the fine minimum principle (see [7, 1.XI.19] or [8, Chapter III]), it follows that $u \geq g$ on $A(m+1, m+2)$ and hence

$$R_g^{F(m;k)}(X) \uparrow g(X) \quad (k \to \infty;\ X \in E \cap A(m+1, m+2)).$$

Dini's theorem implies that this convergence is uniform, so there exists $k_m$ in $\mathbb{N}$ such that

$$g(X) \geq R_g^{F(m;k_m)}(X) > g(X) - \delta_{m+1} \quad (X \in E \cap A(m+1, m+2)). \tag{2}$$

We now define the set

$$F_1 = \bigcup_{m=1}^{\infty} F(m; k_m).$$

Clearly $F_1$ is a relatively closed subset of $\Omega$ such that $K_1^\circ \cap F_1 = \varnothing$ and $E \subset \Omega \setminus F_1$. We also define the positive number

$$\delta_0 = \inf\{\varepsilon(X): X \in E \cap K_2\}$$

and the function

$$s_1(X) = 2^{-1} \min\{g(X) - R_g^{F_1}(X), 2^{-1}\delta_0\} \quad (X \in \Omega \setminus F_1). \tag{3}$$

We observe that $s_1$ is non-negative and superharmonic on $\Omega \setminus F_1$, and positive on the component of $K_1^\circ$ which contains $P$. Also, since

$$R_g^{F_1}(X) \geq R_g^{F(m;k_m)}(X) \quad (X \in \Omega; m \in \mathbb{N}),$$

it follows from (2) and (3) that $s_1(X) < 2^{-1}\varepsilon(X)$ on $E$.

The above argument can be repeated, with $(K_m)_{m \geq l}$ in place of $(K_m)_{m \geq 1}$, to obtain a relatively closed subset $F_l$ of $\Omega$ satisfying $K_l^\circ \cap F_l = \varnothing$ and $E \subset \Omega \setminus F_l$, and also a non-negative superharmonic function $s_l$ on $\Omega \setminus F_l$ which is positive on the component of $K_l^\circ$ which contains $P$ and which satisfies $s_l < 2^{-l}\varepsilon$ on $E$. If we define $F = \cup_l F_l$, and $s = \Sigma_l s_l$ on $\Omega \setminus F$, then $F$ is a relatively closed subset of $\Omega$ satisfying $E \subset \Omega \setminus F$, and $s$ is a positive superharmonic function on $\Omega \setminus F$ satisfying $s < \varepsilon$ on $E$ as required.

**4.3.** To prove Theorem 3 we recall from Lemma 5 that, if the pair $(\Omega, E)$ satisfies condition (b), then so does the pair $(\Omega, \hat{E})$. Further, any continuous function $\varepsilon$: $E \to (0, 1]$ has a continuous extension to $\hat{E}$ which also takes values in $(0, 1]$. Thus Lemma 6, applied to the pair $(\Omega, \hat{E})$, shows that there exists $s$ in $\mathscr{S}^+(\hat{E})$ such that $s < \varepsilon$ on $E$. It now follows from Theorem 1 that (a) holds. Conversely, if (a) holds, then Theorem 1 shows that $(\Omega, E)$ satisfies (b)(i) and the $(K, L)$-condition. It remains to show that condition $(*)$ of Lemma 6 also holds.

We establish this by contradiction. Suppose that condition $(*)$ fails to hold. Then there is a compact subset $K$ of $\Omega$, a sequence $(V_k)$ of fine components (not necessarily distinct) of the fine interior of $E$ which satisfy $V_k \cap K \neq \varnothing$, and a sequence $(X_k)$ of points such that $X_k \in V_k$ for each $k$ and such that $(X_k)$ converges to the Alexandroff point for $\Omega$. Now let $U$ be an $\Omega$-bounded connected open set which contains $K$, and define $\omega = U \cup (\cup_k V_k)$. Then $\omega$ is a finely connected finely open set. Let $u$ be the fine regularized reduced function of the constant function 1 relative to $U$ in the finely open set $\omega$ (see [8, §11]), and let $\delta_k = 2^{-k}u(X_k)$. Since $\omega$ is finely connected, we know (see [8, Theorem 12.6]) that $\delta_k > 0$ for each $k$, and so we can choose a continuous function $\varepsilon: E \to (0, 1]$ such that $\varepsilon(X_k) = \delta_k$ for each $k$.

If we define $h \equiv 0$, then by hypothesis there exists $H$ in $\mathscr{H}(\Omega)$ such that $0 < H < \varepsilon$ on $E$. We define the positive number

$$a = \inf\{H(X): X \in \overline{U} \cap E\}$$

and the function

$$v(X) = \begin{cases} \min\{H(X), a/2\} & (X \in \Omega \setminus U) \\ a/2 & (X \in U) \end{cases}$$

so that $v$ is positive and superharmonic on an open set containing $\omega$. Hence $v \geq (a/2)u$ on $\omega$, and so

$$2^{-k}u(X_k) = \delta_k = \varepsilon(X_k) > v(X_k) \geq (a/2)u(X_k) \quad (k \in \mathbb{N}),$$

a contradiction. Thus condition $(*)$ must hold, and the proof of Theorem 3 is complete.

## 5. Proof of Theorem 4

**5.1.** We begin by establishing the following analogue of Lemma 6.

LEMMA 7. *Suppose the pair* $(\Omega, E)$ *satisfies condition* (c) *of Theorem* 4 *and* $\varepsilon: E \to (0, 1]$ *is continuous. Then there exists* $s$ *in* $\mathscr{S}^+(E)$ *such that* $s < \varepsilon$ *on* $E$.

The proof of Lemma 7 is similar in pattern to that of Lemma 6, so we will refer to §4.2 for some of the argument. As before, we can assume that $\Omega$ possesses a Green function. Using condition (c) of Theorem 4 we can construct a sequence $(K_m)$ of compact subsets of $\Omega$ such that $K_1^\circ \neq \varnothing$ and $\bigcup_m K_m = \Omega$, and such that, for each $m$,

(I) $K_m \subset K_{m+1}^\circ$, and
(II) every component $V$ of $E^\circ$ which satisfies $\overline{V} \cap K_m \neq \varnothing$ also satisfies $\overline{V} \subset K_{m+1}^\circ$.

Further, let $A(l, m)$, $F(m; k)$, $\delta_m$, $P$ and $g$ be as defined in §4.2, and let $\hat{R}_g^C$

denote the regularized reduced function (balayage) of $g$ relative to a set $C$ in $\Omega$. We observe that

$$\hat{R}_g^{F(m;k)}(X) \uparrow \hat{R}_g^{A(m,m+3)\setminus E}(X) \quad (k \to \infty;\ X \in \Omega).$$

If $u$ is a positive superharmonic function on $\Omega$ which satisfies $u \geq g$ on $A(m, m+3) \setminus E$, then the same inequality holds on the set described in (1). Since condition (c) of Theorem 4 holds, we know that $\Omega \setminus E$ and $\Omega \setminus E^\circ$ are thin at the same points of $E$. Further, the set of points of $\partial E$ where $\Omega \setminus E^\circ$ is thin is a polar set. It follows (see [7, 1.VI.3(c)]) that

$$\hat{R}_g^{F(m;k)}(X) \uparrow \hat{R}_g^{S}(X) \quad (k \to \infty;\ X \in \Omega),$$

where

$$S = [A(m, m+3) \setminus E] \cup [(A(m, m+3))^\circ \setminus E^\circ].$$

Condition (II) above, the minimum principle and Dini's theorem allow us to conclude that there exists $k_m$ such that

$$g(X) \geq \hat{R}_g^{F(m;k_m)}(X) > g(X) - \delta_{m+1} \quad (X \in E \cap A(m+1, m+2)).$$

The remainder of the proof of Lemma 7 now proceeds exactly as the part of the proof of Lemma 6 which follows (2).

**5.2.** To prove Theorem 4, we observe from Lemmas 4 and 7 that (c) implies (b). Clearly (b) implies (a). If (a) holds, then (c)(i) follows from work of Keldyš [13], Deny [6] and Labrèche [14] (or see [4, §8]) on local uniform harmonic approximation. It remains to establish (c)(ii), and we will do this by refining an argument in [10, §4]. We will require the following result of Armitage, Bagby and Gauthier [1].

THEOREM B. *Let $\omega$ be an unbounded connected open set in $\mathbb{R}^n$. Then there exists a continuous function $\varepsilon_\omega$: $[0, \infty) \to (0, 1]$ with the following property: if $h \in \mathscr{H}(\omega)$ and $|h(X)| \leq \varepsilon_\omega(|X|)$ on $\omega$, then $h \equiv 0$.*

Suppose now that condition (c)(ii) fails to hold. Then there is a sequence $(V_k)$ of components (not necessarily distinct) of $E^\circ$ and two sequences $(X_k), (Y_k)$ of points such that $X_k$, $Y_k \in V_k$ for each $k$, such that $(X_k)$ converges either to a point $P$ in $\partial\Omega$ or to the Alexandroff point $*$ for $\mathbb{R}^n$, and such that $(Y_k)$ converges to a point $Q$ in $\partial E \cap \Omega$. By using a Kelvin transformation centered at $P$, if necessary, we can suppose that $X_k \to *$. (The transformed pair $(\Omega', E')$ would still satisfy (a) but not (c).) We can also assume that $Q$ is the origin $O$, and that $|Y_k| < k^{-1}$ for each $k$. As in the proof of Lemma 6 we can assume, without loss of generality, that $\Omega$ possesses a Green function. Next, let $(B_k)$ be a sequence of pairwise disjoint closed balls in $\Omega \setminus E$ with centers $Z_k$ such that $Z_k \to O$, let $v_k$ be the capacitary potential on $\Omega$ valued 1 on $B_k$, and define

$$h(X) = \sum_{k=1}^{\infty} 2^{-k} v_k(X) \quad (X \in \Omega).$$

It is easy to see that $h \in C(E) \cap \mathscr{H}(E^\circ)$. For each $k$, let $\omega_k$ be the unbounded connected open set defined by

$$\omega_k = \left( \bigcup_{m=k}^{\infty} V_m \right) \cup \left\{ X: 0 < |X| < k^{-1} \text{ and } X \notin \bigcup_m B_m \right\}$$

and let $\varepsilon_{\omega_k}$ be as in Theorem B. We define $\delta$: $[0, \infty) \to (0, 1]$ by

$$\delta(t) = \min\{\varepsilon_{\omega_1}(t), \ldots, \varepsilon_{\omega_k}(t)\} \quad (t \in [k-1, k); k \in \mathbb{N}),$$

and let $\varepsilon$: $[0, \infty) \to (0, 1]$ be a continuous function satisfying $\varepsilon \leq \delta$. From our hypothesis that (a) holds, we know that there is a harmonic function $H$ on an open set $W$ which contains $E$ such that $|H - h| < \varepsilon$ on $E$. There exists $k'$ such that $W$ contains the closed ball $B$ of centre $O$ and radius $1/k'$. If we define

$$a = \sup\{|H(X) - h(X)|/\varepsilon_{\omega_{k'}}(|X|): X \in \overline{\omega}_{k'} \text{ and } |X| \leq k'\},$$

it follows that

$$|(H(X) - h(X))/(a+1)| < \varepsilon_{\omega_{k'}}(|X|) \quad (X \in \omega_{k'}).$$

Hence, by Theorem B, $H \equiv h$ on $\omega_{k'}$. This contradicts the mean value property of $H$ on a neighbourhood of any $B_k$ contained in the set $\{X: 0 < |X| < 1/k'\}$. Hence condition (c)(ii) must hold, as required.

## 6. Proof of Theorem 5

**6.1.** First we give the following analogue of Lemma 5.

LEMMA 8. *If the pair* $(\Omega, E)$ *satisfies condition* (c) *of Theorem* 5, *then so does the pair* $(\Omega, \hat{E})$.

To prove Lemma 8, suppose that $(\Omega, E)$ satisfies condition (c) of Theorem 5 and let $F = \hat{E}$. Since $E^\circ \subset F^\circ \subset F$, it is certainly true that $\Omega \setminus \hat{F}$ (which equals $\Omega \setminus F$) and $\Omega \setminus F^\circ$ are thin at the same points of $E$, and thus clearly also at the same points of $F$. Hence $(\Omega, F)$ satisfies condition (c)(i) of Theorem 5. Secondly, if $V$ is an $\Omega$-bounded component of $\Omega \setminus E$, then condition (c)(i) implies that $\partial V \subseteq \partial \hat{E}$ (see [9, §7.1]), and so $V$ is also a component of $F^\circ$. It follows that $(\Omega, F)$ satisfies condition (c)(ii) of Theorem 5, as required.

**6.2.** To prove Theorem 5 we suppose that (c) holds and use Lemmas 7 and 8 (extending $\varepsilon$ continuously to $\hat{E}$ as in §4.3) to observe that there exists $s$ in $\mathscr{S}^+(\hat{E})$ such that $s < \varepsilon$ on $E$. It now follows from Theorem 2 that (b) holds. Clearly (b) implies (a). If (a) holds, then it follows from Theorem 2 that $(\Omega, E)$ satisfies (c)(i) and the $(K, L)$-condition. This, together with Theorem 4, shows that (c)(ii) also holds. The proof of Theorem 5 is now complete.

## 7. Details of Example 1

Let $a_m > 0$ $(m = 1, \ldots, n-1)$, and let $\alpha$ be as in Example 1. For each $\delta$ in $(0, 1]$ we define the set

$$\omega_\delta = (-\delta a_1, \delta a_1) \times \cdots \times (-\delta a_{n-1}, \delta a_{n-1}) \times \mathbb{R}$$

and the functions

$$u_\delta(X) = \cos(\pi x_1/(2\delta a_1)) \ldots \cos(\pi x_{n-1}/(2\delta a_{n-1}))\exp(\alpha x_n/\delta),$$
$$s_\delta(X) = \varepsilon \min\{u_\delta(x_1, \ldots, x_n), u_\delta(x_1, \ldots, x_{n-1}, -x_n)\}.$$

Then $u_\delta$ is positive and harmonic on $\omega_\delta$, and so $s_1 \in \mathscr{S}^+(\omega_1)$. (In fact, $s_\delta$ is a potential on $\omega_\delta$.) Assertion (i) of Example 1 now follows immediately from Theorem 1.

To prove (ii), let $\beta > \alpha$ and choose $\delta$ in $(0, 1)$ close enough to 1 so that $\alpha/\delta < \beta$. Let $P$ be the point $(a_1, 0, \ldots, 0)$ in $\mathbb{R}^n$ and let

$$h(X) = \log(1/|X - P|) \ (n = 2), \quad h(X) = |X - P|^{2-n} \ (n \geq 3),$$

so that certainly $h \in \mathscr{H}(\overline{\omega_{(1+\delta)/2}})$. Now suppose that $H$ is a harmonic function on $\mathbb{R}^n$ such that $H - h \geq 0$ on $\overline{\omega_{(1+\delta)/2}}$. We must have $H - h > 0$ on $\omega_{(1+\delta)/2}$, for otherwise $H \equiv h$ on $\omega_{(1+\delta)/2}$ and hence on $\mathbb{R}^n \setminus \{P\}$, which contradicts the fact that $H \in \mathscr{H}(\mathbb{R}^n)$. If we define

$$c = \inf\{(H(X) - h(X))/s_\delta(X) \colon X = (x_1, \ldots, x_{n-1}, 0) \in \omega_\delta\},$$

then $H - h \geq cs_\delta$ on $\omega_\delta$ by the Maria-Frostman domination principle [12, Theorem 8.43], and so

$$H(0,\dots,0,x_n) - h(0,\dots,0,x_n) \geq c\exp(-\alpha|x_n|/\delta) \quad (x_n \in \mathbb{R}).$$

This establishes (ii).

## 8. An open problem

In view of Theorem 5 it seems plausible that the assertion below is equivalent to conditions (a) and (b) of Theorem 3.

*For each h in* $\mathscr{H}(E)$ *and each continuous function* $\varepsilon\colon E \to (0,1]$, *there exists* $H$ *in* $\mathscr{H}(\Omega)$ *such that* $|H - h| < \varepsilon$ *on* $E$.

It would be possible to prove this by imitating the reasoning of [10, §4] if the following generalization of Theorem B is valid.

*For each unbounded finely connected finely open subset* $\omega$ *of* $\mathbb{R}^n$ *there is a continuous function* $\varepsilon_\omega\colon [0,\infty) \to (0,1]$ *so that if* $h \in \mathscr{H}(\overline{\omega})$ *and* $|h(X)| \leq \varepsilon_\omega(|X|)$ *on* $\omega$, *then* $h \equiv 0$ *on* $\omega$.

We do not know whether this latter assertion is true.

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McGill University
Montreal, Quebec, Canada
University College
Dublin, Ireland