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A NON-ARCHIMEDEAN ANALOGUE OF THE KOBAYASHI SEMI-DISTANCE AND ITS NON-DEGENERACY ON ABELIAN VARIETIES

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One way to state the Schwarz-Pick Lemma is to say that holomorphic maps from the unit disc to itself are distance decreasing in the hyperbolic (Poincaré) metric. The Kobayashi semi-distance on a complex analytic space is an intrinsically defined semi-distance with the property that holomorphic mappings are distance decreasing in the Kobayashi semi-distance and such that the Kobayashi semi-distance on the unit disc is just the hyperbolic distance coming from the Poincaré metric.

If d denotes the Kobayashi semi-distance on a complex analytic space X, it is possible that $d(x, y) = 0$ for two distinct points $x \neq y$ in X. For instance, if $X = \mathbf{C}$ is the complex plane, then $d(x, y) = 0$ for every x and y in $\mathbf{C}$. Therefore, if X is any analytic space and f is a non-constant holomorphic map from $\mathbf{C}$ into X, then $d(x, y) = 0$ for any two points x, y in the image of f by the distance decreasing property of holomorphic maps. Brody's Theorem, [Br], in its weakest formulation says that in the case that X is compact, this is the only way the Kobayashi semi-distance can degenerate. Namely,

THEOREM (BRODY). *Let X be a compact, complex analytic space. Then there exist two distinct points $x \neq y$ in X such that the Kobayashi semi-distance $d(x, y) = 0$ if and only if there exists a non-constant holomorphic map from $\mathbf{C}$ into X.*

In this paper, I define a non-Archimedean analogue of the Kobayashi semi-distance, and, using Berkovich's, [Ber], theory of non-Archimedean analytic spaces, I show that this semi-distance does not degenerate on Abelian varieties. In [Ch1] (or see [Ch2]), I showed that every non-Archimedean map from the affine line $\mathbf{A}^1$ into an Abelian variety must in fact be constant. Therefore, I view the main result of this paper as the first step in answering:

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QUESTION. *If X is a non-singular projective algebraic variety defined over an algebraically closed field complete with respect to a non-trivial, non-Archimedean valuation, is it true that there exists a non-constant analytic map from $\mathbf{A}^1$ into X if and only if the Kobayashi semi-distance on X is not an actual distance?*

Note that the proof of Brody's Theorem makes essential use of compactness, and there are counterexamples which show that some sort of compactness assumption is necessary; see Example 3 on page 79 of [L1]. Non-Archimedean algebraically closed fields are not locally compact, so the proof of Brody's Theorem does not immediately generalize to the non-Archimedean case.

In this paper, I will assume that the reader is acquainted with the main ideas in [Ber] or at the very least with the basic theory of rigid analysis as in [BGR]. For a more leisurely exposition of the results presented in this paper, I refer the reader to my thesis, [Ch 1].

1. Preliminaries

Throughout this paper, K will denote an algebraically closed field, complete with respect to a non-trivial, non-Archimedean absolute value $|\ |$.

The prototypical example of such a field is the p-adic field $\mathbf{C}_p$, which is the completion of the algebraic closure of $\mathbf{Q}_p$, which itself is the completion of $\mathbf{Q}$ with respect to the p-adic valuation $|\ |_p$ normalized so that $|p|_p = p^{-1}$. It is a theorem that $\mathbf{C}_p$ is algebraically closed, and the field $\mathbf{C}_p$ is the p-adic analogue of the complex numbers.

By *analytic space*, I will always mean analytic space in the sense of Berkovich as described in [Ber]. Berkovich's idea of analytic spaces has since been generalized in a couple of different directions. Berkovich himself gives a generalization in [Ber2]. For a different approach, a little more analogous to schemes, see [Sch]. Since I will mainly be concerned with algebraic varieties, Berkovich's original theory will be more than sufficient for my purposes. Recall that each point in an analytic space corresponds to a bounded multiplicative semi-norm on the space of functions on some affinoid neighborhood. I will use the notation $|\ |_x$ to denote the semi-norm corresponding to a point x in an analytic space X. Given an algebraic variety, one can provide it with an analytic structure in a completely straightforward manner; see [Ber] for details. I will denote both the algebraic variety and its analytification by the same symbol, and I doubt that this will lead to any confusion. Unless otherwise specified, all analytic spaces and maps will be defined over a fixed field K, algebraically closed and complete with respect to a non-trivial, non-Archimedean valuation. The symbol $\mathbf{A}^1$ denotes the affine line, and the symbol $\mathbf{A}^{1\times}$ denotes the affine line with the origin removed. The symbol $\mathbf{P}^n$ denotes projective n-space, and $\mathbf{P}^1$ therefore denotes the projective line. The symbol $\mathbf{B}^n$ will be used to denote the *closed unit n-ball*, which is the affinoid space associated to the Tate algebra $K < z_1, \ldots, z_n >$, which is the ring of formal power series in n-variables with coefficients in K whose absolute values tend to zero. The symbol $\mathbf{B}^1$ or simply $\mathbf{B}$ will denote the one dimensional closed unit ball. The *open unit n-ball* is then defined by

$$\mathring{\mathbf{B}}^n = \{x \in \mathbf{B}^n \colon |z_j|_x < 1 \text{ for } j = 1, \dots, n\}.$$

The one dimensional open unit ball is denoted by $\mathring{\mathbf{B}}^1$ or simply $\mathring{\mathbf{B}}$.

In this paper, I will always assume that any affinoid space or subdomain is strictly affinoid. I will also assume that all analytic spaces are reduced and separated.

Recall that to each point x in an analytic space X, one associates a field $\mathcal{K}(x)$, which is a complete, non-Archimedean extension of K, the field of definition for X. The field $\mathcal{K}(x)$ is obtained by first taking the ring of analytic functions on an affinoid neighborhood of x modulo the kernel of $|\ |_x$ and then completing the fraction field of this ring with respect to $|\ |_x$. For an analytic function f, the notation $f(x)$ denotes the image of f in $\mathcal{K}(x)$ under the canonical map. I denote by $X(K)$ the set of points $x \in X$ such that $\mathcal{K}(x) = K$. In the case that K is algebraically closed, this space corresponds to the set of points in the rigid analytic space associated to X. For example, $\mathbf{A}^1(K) = K$, and $\mathbf{A}^{1\times}(K) = K^\times$.

Throughout this paper, the use of the $\tilde{}$ character will be reserved to denote a residue class. For instance, $\widetilde{K}$ denotes the residue class field of K, the field defined by the elements in K of norm ≤ 1 modulo the elements in K of norm < 1. In the case that a is an element of K with norm ≤ 1, $\tilde{a}$ denotes the image of a in $\widetilde{K}$. We will also see that it is sometimes possible to associate to an analytic space X, a reduction $\widetilde{X}$, which is an algebraic variety defined over the residue class field $\widetilde{K}$. In this case, there will be a reduction map $\pi\colon X \to \widetilde{X}$, so given a point x in X, the notation $\tilde{x}$ is used to denote the image of x under the reduction map π. Affinoid spaces X have canonical reductions, and in the case that X is affinoid, the $\tilde{}$ notation will always refer to the canonical reduction.

Recall that by an *admissible affinoid cover* $\mathcal{U}$ of an analytic space X, one means a cover $\mathcal{U}$ consisting of affinoid subspaces U such that if V is any affinoid subspace of X, then $\mathcal{U}|_V = \{U \cap V\}$ is a *finite* covering of V. Let U be an affinoid space. Let V be an affinoid subdomain of U. If the induced morphism $\widetilde{V} \to \widetilde{U}$ is an open immersion, then V is called a *formal affinoid subdomain* of U. Now let X be an analytic space. An admissible, affinoid covering $\mathcal{U}$ of X is called *formal* if the intersection $U \cap V$ is a formal affinoid subdomain of U for every $U, V \in \mathcal{U}$. Given a formal covering $\mathcal{U}$ of X, one gets an algebraic variety $\widetilde{X_{\mathcal{U}}}$ over $\widetilde{K}$ and a reduction map

$$\pi_{\mathcal{U}}\colon X \to \widetilde{X_{\mathcal{U}}}.$$

If $\mathcal{U}$ and $\mathcal{V}$ are two formal affinoid coverings of X, then the reductions $\widetilde{X_{\mathcal{U}}}$ and $\widetilde{X_{\mathcal{V}}}$ are in general non-isomorphic. However, two formal coverings $\mathcal{U}$ and $\mathcal{V}$ are called equivalent if $U \cap V$ is a finite union of formal subdomains of both U and V for every $U \in \mathcal{U}$ and every $V \in \mathcal{V}$. Equivalent formal coverings give rise to isomorphic reductions.

By an *analytic group* G, one means a group object in the category of analytic spaces. This means that there exist three morphisms

(a) μ: $G \times G \to G$ (multiplication)
(b) i: $G \to G$ (inverse)
(c) e: $G \to G$ (identity)

satisfying the obvious relations. Note that G itself is not a group, but it follows easily that $G(K)$ is a group.

The two most important examples of analytic groups are the additive group $\mathbf{G}_a$, which as an analytic space is isomorphic to $\mathbf{A}^1$, and the multiplicative group $\mathbf{G}_m$, which as an analytic space is isomorphic to $\mathbf{A}^{1\times}$. Another important analytic group is the affinoid analytic group $\mathbf{G}_{m,1}$ associated to the Tate algebra $K < z, z^{-1} >$. The canonical reduction of $\mathbf{G}_{m,1}$ is the multiplicative group over the residue class field $\widetilde{K}$.

A *formal analytic space* X is an analytic space together with a fixed equivalence class of formal coverings. Such a space is denoted by $(X, \mathcal{U})$, where $\mathcal{U}$ is a formal covering of X representing its equivalence class. A morphism

$$\phi\colon\ (X, \mathcal{U}) \to (Y, \mathcal{V})$$

of formal analytic spaces is a morphism ϕ: $X \to Y$ of analytic spaces such that there exists a formal covering $\mathcal{U}'$ of X equivalent to $\mathcal{U}$ and a formal covering $\mathcal{V}'$ of Y equivalent to $\mathcal{V}$ such that for every $U \in \mathcal{U}'$, there exists a $V \in \mathcal{V}'$ such that $\phi(U) \subset V$. Recall that such a morphism induces a morphism

$$\tilde{\phi}\colon\ \widetilde{X_{\mathcal{U}}} \to \widetilde{Y_{\mathcal{V}}},$$

and this is the whole point of considering the category of formal analytic spaces.

One easily sees that products exist in the category of formal analytic spaces, so it makes sense to talk about *formal analytic groups*, which are group objects in the category of formal analytic spaces.

There is an obvious functor from the category of formal analytic spaces to the category of analytic spaces, and from the category of formal analytic groups to the category of analytic groups, namely the functor which simply forgets the formal cover. By a theorem of Bosch [Bos], this functor from the category of formal analytic groups to the category of analytic groups is fully faithful, so in particular *an analytic group can have at most one formal analytic group structure*. Of course it might be that some analytic groups cannot be made into formal analytic groups at all, and in fact, this is the case. Note that the problem is not in finding a formal covering for an analytic group G, but rather that not all analytic groups G have formal coverings such that the multiplication map from the product of G with itself will then be a morphism in the category of formal analytic spaces. This last property is equivalent to the existence of a formal covering $\mathcal{U}$ such that $\widetilde{G_{\mathcal{U}}}$ is a group variety.

The term *affine analytic torus*, or affine torus, refers to a product of multiplicative groups $\mathbf{G}_m$. (Actually, this should be called a split torus, but in this paper I will only be concerned with split tori.) The term *affinoid torus* will mean a product of the affinoid group $\mathbf{G}_{m,1}$. The *rank* of an affine or affinoid torus is by definition the number of copies of $\mathbf{G}_m$ or $\mathbf{G}_{m,1}$ in the product. Let T be an affine torus, and let Γ be a torsion free, discrete subgroup of $T(K)$. Then, Γ acts discretely and freely on T, so the quotient space $X_\Gamma = T/\Gamma$ is an analytic space. If the rank of Γ is equal to the rank of T, then X_Γ is called a *complete analytic torus*, or a complete torus.

2. A non-Archimedean analogue of the Kobayashi semi-distance

In this section I define a non-Archimedean analogue of the Kobayashi semi-distance. I also discuss its basic properties and give some basic examples and lemmas. Since the full set of points of a Berkovich analytic space X is not, in general, metrizable, I will define a semi-distance only on the set of K-points $X(K)$.

When one defines the Kobayashi semi-distance over the complex numbers, one starts with the Poincaré metric on the unit disc. The most important property of the Poincaré metric in this context is that holomorphic functions from the disc to the disc are distance decreasing, meaning that for such a map f, the distance from $f(z_1)$ to $f(z_2)$ is $\leq$ the distance from z_1 to z_2 for all pairs of points z_1, z_2 in the unit disc. To define a non-Archimedean analogue of the Kobayashi semi-distance, the first thing to notice is that analytic maps from the unit ball $\mathbf{B}$ to itself are distance decreasing in the standard non-Archimedean norm on $\mathbf{B}(K)$. We see this as follows.

PROPOSITION 2.1. *Let $f\colon \mathbf{B}^n \to \mathbf{B}^n$ be an analytic map, and let $\tilde{f}\colon \mathbf{A}^n_{\tilde{K}} \to \mathbf{A}^n_{\tilde{K}}$ denote the reduction of f, then f is an isomorphism if and only if $\tilde{f}$ is an isomorphism.*

Proof. See [BGR], 5.1.3/8. □

COROLLARY 2.2. *Let $f\colon \mathbf{B} \to \mathbf{B}$ be an analytic map given by*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Then, f is an isomorphism if and only if

$$|a_0| \leq 1, |a_1| = 1, \text{ and } |a_k| < 1, \text{ for all } k \geq 2.$$

Proof. The reduced map $\tilde{f}$, which is a polynomial in one variable, will be an isomorphism if and only if it has degree one. □

PROPOSITION 2.3. *If $f\colon \mathbf{B} \to \mathbf{B}$ is an analytic map, then for all $z, w \in \mathbf{B}(K)$,*

$$|f(z) - f(w)| \leq |z - w|.$$

Furthermore, if f is an analytic isomorphism, then

$$|f(z) - f(w)| = |z - w|$$

for all z, w in $\mathbf{B}(K)$.

Proof. Let $f(z)$ be given by $\sum a_k z^k$. Now, because f maps into **B**, one has $|a_k| \leq 1$ for all k. Therefore,

$$\begin{aligned} f(z) - f(w) &= \sum_{k=1}^{\infty} a_k(z^k - w^k) \\ &= \sum_{k=1}^{\infty} a_k(z - w)(z^{k-1} + wz^{k-2} + \cdots + w^{k-2}z + w^{k-1}) \\ &= (z - w)\sum_{k=1}^{\infty} a_k(z^{k-1} + wz^{k-2} + \cdots + w^{k-2}z + w^{k-1}). \end{aligned}$$

Since $|a_k| \leq 1$, $|z| \leq 1$, and $|w| \leq 1$, everything inside the last sum has norm ≤ 1. Hence,

$$|f(z) - f(w)| \leq |z - w|.$$

If, in addition, f is an isomorphism, then $|a_k| < 1$ for $k \geq 2$, and $|a_1| = 1$ by Corollary 2.2. This implies

$$|f(z) - f(w)| = |a_1||z - w| = |z - w|. \qquad \square$$

Remark. Lin Weng [We] has suggested an alternate definition for a hyperbolic distance on the "open" unit ball $\mathring{\mathbf{B}}(L)$ by defining

$$d_{\mathrm{hyp}}(a, b) = \log_q \frac{1 + |b - a|_p}{1 - |b - a|_p},$$

where L is a *finite* extension of $\mathbf{Q}_p$, q is the cardinality of the residue class field of L, and $\log_q$ means the logarithm with base q. At present, I prefer to stick to working with the "closed" unit ball and the standard norm.

Once the distance decreasing property of analytic maps is established, the non-Archimedean analogue of the complex situation can be developed by taking the unit ball together with its standard norm as the model. Let X be an analytic space, and let $x, y \in X(K)$. Suppose that there is a sequence of analytic maps

$$f_j\colon \mathbf{B} \to X, \qquad j = 1, \ldots, m,$$

and points $z_j, w_j \in \mathbf{B}(K)$ such that

$$f_1(z_1) = x, \quad f_m(w_m) = y, \quad \text{and} \quad f_j(w_j) = f_{j+1}(z_{j+1}), \quad j = 1, \ldots, m - 1.$$

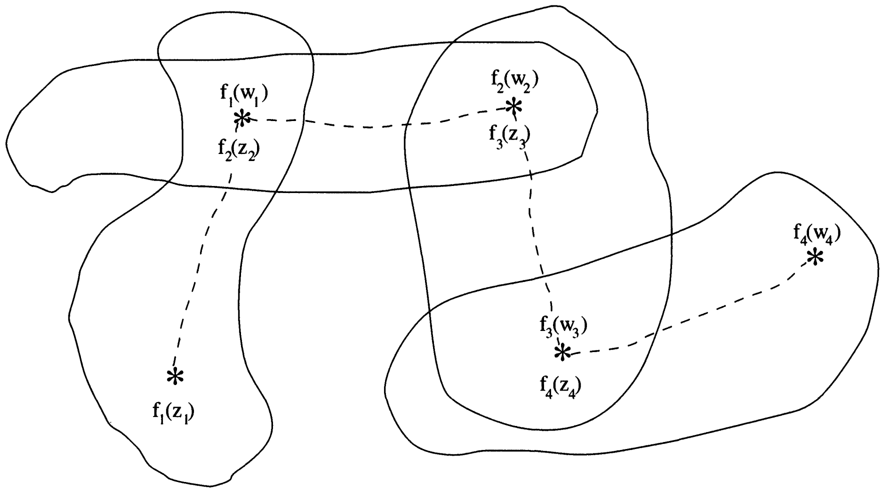

Figure 1

Such a sequence is called a *Kobayashi chain joining x and y* (Fig. 1).
The Kobayashi semi-distance is then defined by

$$d(x, y) = \inf \sum_{j=1}^{m} |z_j - w_j|,$$

where the infimum is taken over all Kobayashi chains (f_i, z_i, w_i) joining x and y. If there are no Kobayashi chains joining x and y, then define $d(x, y) = \infty$. The Kobayashi semi-distance is the only (semi-) distance used in this paper, so henceforth d will always refer to the Kobayashi semi-distance.

It is clear that d is symmetric in x and y, and that d satisfies the triangle inequality, though not necessarily the stronger non-Archimedean triangle inequality. It is, however, possible that $d(x, y) = 0$ but $x \neq y$, as we will see in the second example below.

Remark. In the case of connected complex analytic spaces, the Kobayashi semi-distance is always finite, but this is definitely not the case for non-Archimedean analytic spaces, as we shall see later.

Example 2.4. Let $X = \mathbf{B}$. Since $|f(z) - f(w)| \leq |z - w|$ for all $z, w \in \mathbf{B}(K)$ and all analytic maps f by Proposition 2.3, we see that $d(z, w) = |z - w|$ for all $z, w \in \mathbf{B}(K)$. Hence, the Kobayashi semi-distance on $\mathbf{B}(K)$ coincides with the distance induced from the standard norm on $\mathbf{B}(K)$.

Example 2.5. Let $X = \mathbf{A}^1$, and let $x, y \in \mathbf{A}^1(K) = K$. Let w be a non-zero element of $\mathbf{B}(K)$, and define

$$f\colon \mathbf{B} \to \mathbf{A}^1, \quad \text{by} \quad f(z) = x + \left(\frac{y-x}{w}\right) z.$$

Note that $f(0) = x$ and $f(w) = y$, so then $d(x, y) \leq |w|$. But, the choice of w was arbitrary, so w can be chosen with $|w|$ arbitrarily small, and hence

$$d(x, y) = 0 \text{ for every } x,\ y \in \mathbf{A}^1(K).$$

Notice that in this example we have used the fact that we can make arbitrarily large dilations of the disc when we map into $\mathbf{A}^1$.

PROPOSITION 2.6. *Let X and X' be analytic spaces, and let $f\colon X \to X'$ be an analytic map. Let d and d' denote the Kobayashi semi-distances on X and X' respectively. Then,*

$$d'(f(x), f(y)) \leq d(x, y)$$

for all $x, y \in X(K)$. In other words, analytic maps are distance decreasing in the Kobayashi semi-distance. If in addition, f is an analytic isomorphism, then f is an isometry for the Kobayashi semi-distance.

Proof. Composition with f makes any analytic map from $\mathbf{B}$ to X into a map from $\mathbf{B}$ to X'. Therefore, composition by f makes any Kobayashi chain connecting x and y in X into a Kobayashi chain of the same length connecting $f(x)$ and $f(y)$ in X'. The statement about isomorphisms follows by symmetry. □

COROLLARY 2.7. *Let X be an analytic space and $f\colon \mathbf{A}^1 \to X$ an analytic map. If $x, y \in X(K)$ are two points in the image of f, then $d(x, y) = 0$. Therefore, if X is an analytic space admitting a non-constant analytic map $f\colon \mathbf{A}^1 \to X$, then the Kobayashi semi-distance on X fails to be a distance.*

The next proposition relates the Kobayashi semi-distance on a product space to the Kobayashi semi-distance on each of the factor spaces.

PROPOSITION 2.8. *Let X and Y be analytic spaces, and let $X \times Y$ denote the product space. Let d_X, d_Y and $d_{X\times Y}$ denote the Kobayashi semi-distances on X, Y and $X \times Y$ respectively. For all $x, x' \in X(K)$ and all $y, y' \in Y(K)$, one has*

$$d_X(x, x') + d_Y(y, y') \geq d_{X\times Y}((x, y), (x', y')) \geq \max\{d_X(x, x'), d_Y(y, y')\}.$$

Proof. The second inequality follows from the fact that the projection maps

$$X \times Y \to X \quad \text{and} \quad X \times Y \to Y$$

are analytic, hence distance decreasing for the Kobayashi semi-distance by Proposition 2.6. The first inequality is trivial if either term on the left-hand side is infinite, so assume that both $d_X(x, x')$ and $d_Y(y, y')$ are finite. Map $X \to X \times Y$ by $\cdot \mapsto (\cdot, y)$ and $Y \to X \times Y$ by $\cdot \mapsto (x', \cdot)$, so that any Kobayashi chain connecting x to x' in X will connect (x, y) to (x', y) in $X \times Y$, and any Kobayashi chain connecting y to y' in Y will connect (x', y) and (x', y') in $X \times Y$. Then,

$$\begin{aligned} d_{X\times Y}((x,y),(x',y')) &\leq d_{X\times Y}((x,y),(x',y)) + d_{X\times Y}((x',y),(x',y')) \\ &\leq d_X(x,x') + d_Y(y,y'), \end{aligned}$$

by the triangle inequality. □

Example 2.9. If $X = \mathbf{B}^n = \mathbf{B} \times \cdots \times \mathbf{B}$, then

$$d((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \max_{1\leq j\leq n} |y_j - x_j|.$$

To see this, it suffices to assume that

$$x_1 = \cdots = x_n = 0 \quad \text{and} \quad |y_n| \geq |y_{n-1}| \geq \cdots \geq |y_2| \geq |y_1|$$

because isomorphisms preserve the Kobayashi semi-distance. Of course, we may also assume that $y_n \neq 0$. Now, let $f\colon \mathbf{B}^n \to \mathbf{B}^n$ be given by

$$z \mapsto \left(\frac{y_1}{y_n}z, \frac{y_2}{y_n}z, \ldots, \frac{y_{n-1}}{y_n}z, z\right).$$

Since f is distance decreasing,

$$d((0,\ldots,0),(y_1,\ldots,y_n)) = d(f(0), f(y_n)) \leq |y_n|.$$

But, the second inequality in Proposition 2.8 tells us that

$$|y_n| = \max_{1\leq j\leq n} |y_j| \leq d((0,\ldots,0),(y_1,\ldots,y_n)).$$

Therefore, $d((0,\ldots,0),(y_1,\ldots,y_n)) = |y_n|$ as desired.

PROPOSITION 2.10. *Let X be a non-singular analytic space. Then, the Kobayashi semi-distance d is a continuous function on $X(K) \times X(K)$.*

Proof. As usual, by the triangle inequality, it suffices to check that if $x_m \to x$ in $X(K)$, then $d(x_m, x) \to 0$. By the definition of non-singular, there is an analytic map $f\colon \mathbf{B}^n \to X$ such that $f(0) = x$ and such that f is an isomorphism onto its range. Hence, for m sufficiently large, let $z_m \in \mathbf{B}^n$ be the point such that $f(z_m) = x_m$. Then, the sequence z_m tends to zero, and hence $d(z_m, 0) \to 0$ because the Kobayashi distance on the n-ball is just the standard distance by Example 2.9. Because analytic maps are distance decreasing in the Kobayashi semi-distance,

$$d(x_m, x) = d(f(z_m), f(0)) \leq d(z_m, 0) \to 0,$$

so $d(x_m, x) \to 0$ as was to be shown. $\square$

LEMMA 2.11. *Let $\pi\colon \widehat{X} \to X$ be a finite analytic map which is a local isomorphism at every point of $\widehat{X}$. If the Kobayashi semi-distance on $\widehat{X}$ is an actual distance, then the Kobayashi semi-distance on X is also an actual distance.*

Proof. Let d and $\hat{d}$ denote the Kobayashi semi-distances on X and $\widehat{X}$ respectively. Let $x, y \in X(K)$ be two points such that $d(x, y) = 0$. Fix a lift $\hat{x}$ of x. Because $\mathbf{B}$ is contractible and π is a local isomorphism, any Kobayashi chain joining x to y in X will lift to a Kobayashi chain of the same length joining $\hat{x}$ to some lift $\hat{y}$ of y in $\widehat{X}$. Because $d(x, y) = 0$, there exists a sequence of Kobayashi chains joining x to y such that the length of the chains tends to zero. Because there are only finitely many points in $\widehat{X}$ lying above y, infinitely many of these chains must lift to Kobayashi chains joining $\hat{x}$ to a fixed lift $\hat{y}$ of y. Therefore, $\hat{d}(\hat{x}, \hat{y}) = 0$ for this particular lift $\hat{y}$. Because the Kobayashi semi-distance on $\widehat{X}$ is assumed to be a true distance, this implies $\hat{x} = \hat{y}$. Therefore, $x = y$, and the Kobayashi semi-distance on X must also be a true distance. $\square$

Remark. Later, when we get to Abelian varieties, we will have to consider infinite analytic covering maps. In that case, the above argument will not work, but because we have very specific knowledge about the structure of the spaces involved, we will be able to show what we want. It would be nice to answer the following question in general: *Is it true that the Kobayashi semi-distance on an analytic space X fails to be an actual distance if and only if the same thing is true for the universal covering space of X?*

Caution! I should remark at this point that an étale morphism over the complex numbers is also a topological covering map. This is definitely not true in the non-Archimedean case. For example, let A and A' be two Abelian varieties with Abelian reduction. Any non-trivial isogeny from A to A' will be an étale morphism in the sense of algebraic geometry. However, it will not be a covering map of the underlying Berkovich topological spaces. Indeed, if a' is the unique point in A' lying above the generic point of the reduction $\widetilde{A}'$, then the only point in the inverse image of a' is the unique point of A lying above the generic point of $\widetilde{A}$.

The following two lemmas show how the reduction of a formal analytic space can affect the Kobayashi semi-distance.

LEMMA 2.12. *Let X be a formal analytic space defined over an algebraically closed field complete with respect to a non-trivial, non-Archimedean valuation. Assume that $\widetilde{X}$, the reduction of X, does not contain any rational curves. Let Y be a* *connected analytic subspace of* $\mathbf{P}^1$, *and let* $f\colon \underset{\sim}{Y} \to X$ *be an analytic map. Then, the image of* f *lies above a single closed point of* $\widetilde{X}$.

Proof. First we show that the image of f lies entirely above the closed points in $\widetilde{X}$. Indeed, suppose there exists a point $y \in Y$ such that the point $x = f(y) \in X$ is such that $\tilde{x}$ is not a closed point of $\widetilde{X}$. Then, there would be a non-zero homomorphism from $\mathcal{K}(x)$ into $\mathcal{K}(y)$. This would imply

$$\mathcal{K}(\tilde{x}) \hookrightarrow \widetilde{\mathcal{K}(x)} \hookrightarrow \widetilde{\mathcal{K}(y)},$$

where the first inclusion follows directly from the definitions. However by Section 1.4.4 of [Ber], $\widetilde{\mathcal{K}(y)}$ is either $\widetilde{K}$ or the field of rational functions in one variable over $\widetilde{K}$, but since $\tilde{x}$ is assumed not to be closed, $\widetilde{\mathcal{K}(y)}$ must be the rational function field. This gives a non-constant map from $\mathbf{P}^1_{\widetilde{K}} \to \widetilde{X}$, contradicting the assumption that $\widetilde{X}$ contains no rational curves. Now since Y is connected, $f(Y)$ is also connected. The anti-continuity of reduction (Corollary 2.4.2 in [Ber]) then implies that the image of f cannot lie above more than one closed point in $\widetilde{X}$. □

LEMMA 2.13. *Let* X *be a formal analytic space with reduction* $\widetilde{X}$. *Assume that* $\widetilde{X}$ *is smooth and does not contain any rational curves. Then, the Kobayashi semi-distance on* X *is an actual distance.*

Proof. By Lemma 2.12, $f\colon \mathbf{B} \to X$ must lie entirely above a single closed point $\tilde{x} \in \widetilde{X}$ because $\widetilde{X}$ is assumed not to contain any rational curves. Therefore, any Kobayashi chain in X lies above a single closed point $\tilde{x} \in \widetilde{X}$. Since $\widetilde{X}$ is smooth, the inverse image of $\tilde{x}$ is isomorphic to $\mathring{\mathbf{B}}^n$ by Proposition 2.2 of [BL1], so the Kobayashi semi-distance on X is an actual distance by Example 2.9. □

Remark. If x and y are as in the lemma and $\tilde{x} \neq \tilde{y}$, then $d(x, y) = \infty$ even if X is connected because a Kobayashi chain cannot cross from the inverse image of one closed point to the inverse image of another closed point.

3. Non-Archimedean uniformization of Abelian varieties

Over the complex numbers, every Abelian variety can be realized as $\mathbf{C}^n$ modulo a lattice. Over non-Archimedean ground fields, this is not at all the case. In this section, I summarize the non-Archimedean uniformization theory of Abelian varieties. For the details, see the work of Bosch and Lütkebohmert [BL2] and Section 6.5 of [Ber].

A formal analytic group $(G, \mathcal{U})$ is said to have *Abelian reduction* if $\widetilde{G_{\mathcal{U}}}$ is an Abelian variety, and $(G, \mathcal{U})$ is said to have *semi-Abelian reduction* if $\widetilde{G_{\mathcal{U}}}$ is a semi-Abelian variety; a *semi-Abelian variety* is an extension of an Abelian variety by an affine torus. In view of Bosch's result, [Bos], mentioned earlier about the uniqueness of formal analytic group structures when they exist, an analytic group G will be said to have Abelian or semi-Abelian reduction if G can be given the structure of a formal analytic group with Abelian or semi-Abelian reduction.

THEOREM 3.1 (SEMI-ABELIAN REDUCTION). *Let A be an Abelian variety defined over an algebraically closed field K, complete with respect to a non-trivial, non-Archimedean valuation. Then, there exists a unique compact subgroup N of A such that N is an analytic domain in A and is a formal analytic group with semi-Abelian reduction. Furthermore, N contains a unique closed analytic subgroup T_1 in N such that T_1 is an affinoid torus fitting into an exact sequence*

$$1 \to T_1 \to N \to B \to 1,$$

where B is a formal analytic group with Abelian reduction, which is the analytification of an Abelian variety. Note that an affinoid torus is a formal analytic group, so the exact sequence above reduces to an exact sequence defining $\widetilde{N}$ as a semi-Abelian variety.

THEOREM 3.2 (Uniformization Theorem). *Let A be an Abelian variety defined over an algebraically closed field K, complete with respect to a non-trivial, non-Archimedean valuation. Let T_1, N and B be as in Theorem 3.1. Let T be an affine analytic torus with the same rank as T_1, and embed T_1 into T by*

$$T_1 \cong \mathbf{G}_{m,1} \times \cdots \times \mathbf{G}_{m,1} \hookrightarrow \mathbf{G}_m \times \cdots \times \mathbf{G}_m \cong T.$$

Then:

(a) $G = T \times N/\text{diagonal}$ *exists as an analytic quotient, and there is an exact sequence*

$$1 \to T \to G \to B \to 1,$$

so G is a semi-Abelian variety. Here "diagonal" refers to the image of T_1 along the diagonal in $T \times N$.

(b) *The immersion $N \hookrightarrow A$ extends uniquely to a surjective analytic group homomorphism*

$$\phi\colon G \to A,$$

which is also a topological covering map.

(c) $\Gamma = \ker\phi$ *is a discrete subgroup in $G(K)$, which is free and whose rank is equal to the rank of T_1.*

(d) *G is simply connected and $\pi_1(A) \cong \Gamma$.*

4. The Kobayashi semi-distance on Abelian varieties

The goal of this section is to show that the Kobayashi semi-distance on Abelian varieties is a genuine distance. The proof of this fact will essentially involve showing that on both Abelian varieties with Abelian reduction and on complete tori (the totally degenerate reduction case), the Kobayashi semi-distance is an actual distance. The next step is to apply the uniformization theorem for Abelian varieties to conclude the general case.

Example 4.1. The Kobayashi semi-distance on $\mathbf{A}^{1\times}$, the affine line minus a point, is a genuine distance. Furthermore, if $x, y \in \mathbf{A}^{1\times}(K)$ are such that $|x|_p \neq |y|_p$, then $d(x, y) = \infty$.

Proof. Let $x \in \mathbf{A}^{1\times}(K)$, and let $f\colon \mathbf{B} \to \mathbf{A}^{1\times}$ be an analytic map such that x is in the image of f. Write $f(z) = \sum_{k=0}^{\infty} c_k z^k$. Now since x is in the image of f, we have $\sup_k |c_k| \geq |x|$. On the other hand, since f does not have a zero, its Newton polygon cannot have any critical points, and therefore $|c_k| < |c_0|$ for all $k > 0$. Therefore, $|f(z)| = |c_0| = |x|$ for all $z \in \mathbf{B}(K)$. This implies that if $x, y \in \mathbf{A}^{1\times}(K)$ are such that $|x| \neq |y|$, then there cannot be a Kobayashi chain joining x to y, and hence $d(x, y) = \infty$. Furthermore, even if $|x| = |y| = r$, any Kobayashi chain joining x to y must be contained in $\mathbf{B}(r)$, the ball of radius r. Therefore, if we could find Kobayashi chains in $\mathbf{A}^{1\times}$ of arbitrarily small length joining x to y, then we could find Kobayashi chains in $\mathbf{B}(r)$ of arbitrarily small length joining x to y. But, by Example 2.4, the Kobayashi semi-distance on $\mathbf{B}(r) \cong \mathbf{B}$ is a genuine distance, so x must in fact equal y, and hence the Kobayashi semi-distance on $\mathbf{A}^{1\times}$ must also be a genuine distance. □

Applying Proposition 2.8 to the above, we get:

Example 4.2. If T is an affine analytic torus, then the Kobayashi semi-distance on T is a genuine distance.

We will also need this lemma about the Kobayashi semi-distance on affine tori.

LEMMA 4.3. *Let T be an affine analytic torus and Γ a discrete, torsion free subgroup of $T(K)$. Let $t \in T(K)$, and let γ be a non-trivial element in Γ. Then, $d(t, \gamma t) = \infty$.*

Proof. Write $T = \mathbf{G}_m \times \cdots \times \mathbf{G}_m$, and write $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $t = (t_1, \ldots, t_n)$. Because T is a product space, Proposition 2.8 says that

$$d(t, \gamma t) \geq \max_{1 \leq j \leq n} d(t_j, \gamma_j t_j),$$

where the Kobayashi semi-distance on the right is the Kobayashi semi-distance on $\mathbf{G}_m \cong \mathbf{A}^{1\times}$. Now since γ is not the identity, there is at least one j such that $|\gamma_j| \neq 1$. Indeed, if Γ had an element $\gamma = (\gamma_1, \ldots, \gamma_n)$ other than the identity such that $|\gamma_i| = 1$ for all i, then either γ would be a torsion element, or the subgroup generated by γ would have an accumulation point in the Berkovich analytic space T because it would lie in a compact subset of T. This would contradict the assumption that Γ is discrete and torsion free. Therefore, there is at least one j such that $|t_j|_p \neq |\gamma_j t_j|_p$. Then by Example 4.1, $d(t_j, \gamma_j t_j) = \infty$, and we are done. □

Example 4.4. If X is a complete analytic torus (not necessarily algebraic), then the Kobayashi semi-distance on X is a genuine distance.

Proof. By assumption, $X = T/\Gamma$, where T is an affine analytic torus and Γ is a discrete, torsion free subgroup of $T(K)$ with rank equal to the dimension of T. Furthermore, the natural map $T \to X$ is a topological covering map. Let d denote the Kobayashi semi-distance on X, and let $\hat{d}$ denote the Kobayashi semi-distance on T. Let x, y be two points in $X(K)$ such that $d(x, y) = 0$. Fix a lift $\hat{x}$ of x in T. Any Kobayashi chain joining x to y can be lifted to a Kobayashi chain of the same length joining $\hat{x}$ to some lift $\hat{y}$ of y because $T \to X$ is a covering map. *A priori*, the lifts of two different Kobayashi chains joining x to y may lift to chains joining $\hat{x}$ to two different lifts $\hat{y}$. However, Lemma 4.3 tells us that $\hat{d}(\hat{y}_1, \hat{y}_2) = \infty$ for two different lifts of y, so the triangle inequality implies that all lifts of Kobayashi chains joining x to y lift to chains joining $\hat{x}$ to the same $\hat{y}$. Therefore, $\hat{d}(\hat{x}, \hat{y}) = 0$, so by Example 4.2, $\hat{x} = \hat{y}$. Therefore, $x = y$, and we are done. □

Now we begin to study the Kobayashi semi-distance on arbitrary Abelian varieties.

LEMMA 4.5. *Let*

$$1 \longrightarrow T \longrightarrow G \xrightarrow{\ \phi\ } B \longrightarrow 1$$

be an algebraic extension of an Abelian variety B with Abelian reduction by an affine analytic torus T. Let $\pi\colon B \to \widetilde{B}$ be the reduction map for B. Let $\tilde{b}$ be a closed point in $\widetilde{B}$. Let $U = \pi^{-1}(\tilde{b}) \cong \mathring{\mathbf{B}}^n$ be the open set in B lying above $\tilde{b}$. Then, there is an analytic isomorphism

$$\mathring{\mathbf{B}}^n \times T \cong U \times T \cong \phi^{-1}(U).$$

Proof. Because all of the arrows in the above short exact sequence are algebraic morphisms and because T is a connected, solvable, algebraic group, we can apply Theorem 10 of [Ro] to conclude that there exists a non-empty Zariski open subset V of B and an algebraic morphism $\sigma\colon V \to G$ such that $\phi \circ \sigma = \mathrm{id}$.

Now, I claim that there exists a closed point $\tilde{b}_0$ in $\widetilde{B}$ such that $\pi^{-1}(\tilde{b}_0) \subset V$. To see this, let Z be the complement of V in B, and let W be a small affinoid neighborhood in B compatible with the reduction $\pi\colon B \to \widetilde{B}$ such that there exist analytic functions $f_1, \ldots, f_r$ on W so that

$$W \cap Z = \{w \in W\colon\ f_1(w) = \cdots = f_r(w) = 0\}.$$

Let $|\ |_{\sup}$ be the supremum semi-norm on W. Since V is not empty, we may assume that none of the f_i are identically zero. By multiplying each f_i by a non-zero constant, we may also assume that $|f_i|_{\sup} = 1$ for all i. Let

$$D(\tilde{f}_i) = \{\tilde{b} \in \widetilde{W} \subset \widetilde{B}\colon\ \tilde{f}_i(\tilde{b}) \neq 0\}.$$

Then, by Lemma 2.4.1 in [Ber] relating the topology of the reduction to the topology of the original space,

$$\pi^{-1}(D(\tilde{f}_i)) = \{b \in W\colon\ |f_i(b)|_{\sup} = 1\}.$$

Now if for every $\tilde{b}$, there is at least one b above $\tilde{b}$ such that $f(b) = 0$, then $\pi^{-1}(D(\tilde{f}_i)) = \emptyset$, and hence $D(\tilde{f}_i) = \emptyset$. Therefore, $\tilde{f}_i = 0$ for all i, and hence $f_i \equiv 0$ for all i, contradicting the assumption that V is not empty. Thus, there is indeed a closed point $\tilde{b}_0$ in $\widetilde{B}$ with $\pi^{-1}(\tilde{b}_0)$ contained in V.

Next, let $\tilde{b}$ be any closed point of $\widetilde{B}$. Because $\widetilde{B}$ is a group, translation by the appropriate group element gives us an automorphism $\tilde{\tau}$ such that $\tilde{\tau}(\tilde{b}) = \tilde{b}_0$. Now, $\tilde{\tau}$ comes from an automorphism τ given by a translation on B. Therefore, $\sigma \circ \tau|_{\pi^{-1}(\tilde{b})}$ gives the isomorphism

$$\phi^{-1}(\pi^{-1}(\tilde{b})) \cong \pi^{-1}(\tilde{b}) \times T.$$

The proof of the lemma is therefore completed by recalling that $\pi^{-1}(\tilde{b}) \cong \mathring{\mathbf{B}}^n$ by Proposition 2.2 of [BL1] since all closed points of $\widetilde{B}$ are smooth. $\square$

This brings us to the main result of this paper.

THEOREM 4.6. *The Kobayashi semi-distance on an Abelian variety A is a genuine distance.*

Proof. Let G be the universal cover of A, and let

$$1 \longrightarrow T \longrightarrow G \xrightarrow{\ \phi\ } B \longrightarrow 1$$

be the exact sequence from Theorem 3.2. Also, let Γ be the discrete group from Theorem 3.2. Recall that B has Abelian reduction, and let $\pi\colon B \to \widetilde{B}$ denote the reduction map.

First we will see that the Kobayashi semi-distance d on G is a distance. Let $x, y \in G(K)$ be two points such that $d(x, y) = 0$. If f is any analytic map from $\mathbf{B}$

into G, then by Lemma 2.12, the image of $\phi \circ f$ in B must lie entirely above a single closed point $\tilde{b}$ of $\widetilde{B}$ because $\widetilde{B}$ does not contain any rational curves. Let $\tilde{b}$ denote this closed point, and let $U = \pi^{-1}(\tilde{b})$. Therefore, any Kobayashi chain connecting x to y in G must be entirely contained in $\phi^{-1}(U)$. Hence, it suffices to verify that the Kobayashi semi-distance on $\phi^{-1}(U)$ is in fact a distance. By Lemma 4.5,

$$\phi^{-1}(U) \cong \mathring{\mathbf{B}}^n \times T.$$

Therefore, since the Kobayashi semi-distance on both $\mathring{\mathbf{B}}^n$ and T are distances, Proposition 2.8 implies that the Kobayashi semi-distance on $\phi^{-1}(U)$ is an actual distance, and by the above, we have therefore shown the same for G.

Now, let x be in $A(K)$, and let $\hat{x}$ and $\hat{x}'$ be two different lifts of x in G. We will show that $\hat{d}(\hat{x}, \hat{x}') = \infty$, where now the Kobayashi distance on G is denoted by $\hat{d}$, and d denotes the Kobayashi semi-distance on A. If $\hat{d}(\hat{x}, \hat{x}') \neq \infty$, then they can be joined by a Kobayashi chain, so must lie above a single closed point $\tilde{b}$ in $\widetilde{B}$ as above. By Lemma 4.5, we can then consider $\hat{x}$ and $\hat{x}'$ to be points in $\mathring{\mathbf{B}}^n \times T$. Furthermore, since we have assumed that $\hat{x}$ and $\hat{x}'$ are lifts of the same point in A, their images under the projection $\mathring{\mathbf{B}}^n \times T \to T$ differ by translation by an element of Γ. Proposition 2.8 then says that $\hat{d}(\hat{x}, \hat{x}')$ is greater than the Kobayashi distance between the images of $\hat{x}$ and $\hat{x}'$ in T, which is equal to infinity by Lemma 4.3.

Finally, let x and y be two points in $A(K)$ such that $d(x, y) = 0$. Fix a lift $\hat{x}$ of x in G. As in Example 4.4, any Kobayashi chain joining x to y in A will lift to a Kobayashi chain of the same length joining $\hat{x}$ to some lift $\hat{y}$ of y in G. From above, the Kobayashi distance between two different lifts in G of y is infinite, so by the triangle inequality, as in Example 4.4, every lift of a Kobayashi chain joining x to y lifts to a Kobayashi chain joining $\hat{x}$ to the same lift $\hat{y}$. Therefore, $\hat{d}(\hat{x}, \hat{y}) = 0$, and because the Kobayashi semi-distance on G is an actual distance, $\hat{x} = \hat{y}$. Therefore, $x = y$, and the proof of the theorem is complete. □

In conclusion, I point out what Theorem 4.6 tells us about the Kobayashi semi-distance on algebraic curves. If X is a smooth projective curve of positive genus, then X can be embedded in its Jacobian, so we get:

THEOREM 4.7. *Let X be an irreducible algebraic curve, and let $\widehat{X}$ be its normalization. Then, the Kobayashi semi-distance on X fails to be a distance if and only if $\widehat{X} \cong \mathbf{P}^1$ or $\widehat{X} \cong \mathbf{A}^1$ if and only if there exists a non-constant analytic map from $\mathbf{A}^1$ into X.*

Proof. If $\widehat{X} \cong \mathbf{P}^1$ or $\widehat{X} \cong \mathbf{A}^1$, then the Kobayashi semi-distance on X clearly fails to be a distance, and there are obviously non-constant analytic maps from $\mathbf{A}^1$ to X. Now assume that $\widehat{X} \not\cong \mathbf{P}^1$ and $\widehat{X} \not\cong \mathbf{A}^1$. Because $\widehat{X}$ is a normal curve, it is smooth. If $\widehat{X}$ has positive genus, then the Kobayashi semi-distance on $\widehat{X}$ is an actual distance because it is an analytic subspace of its projective completion, which is a subvariety of its Jacobian, so Theorem 4.6 implies that the Kobayashi semi-distance on $\widehat{X}$ is, in fact, a distance. Furthermore, Theorem 4.5.1 of [Ber] says that there are no non-constant analytic maps from $\mathbf{A}^1$ to $\widehat{X}$. If $\widehat{X}$ has genus 0, then $\widehat{X} \subseteq \mathbf{A}^{1\times}$ because of the assumption that $\widehat{X} \not\cong \mathbf{P}^1$ or $\mathbf{A}^1$. Therefore, the Kobayashi semi-distance on $\widehat{X}$ is a distance by Example 4.1, and any analytic map from $\mathbf{A}^1$ to $\widehat{X}$ must be constant. (This last fact is an elementary consequence of the theory of Newton polygons.) Finally, from the general properties of normalizations (see [G-R]), any non-constant analytic map from $\mathbf{B}$ or from $\mathbf{A}^1$ into X will lift to a non-constant analytic map to $\widehat{X}$. Therefore, the above implies that there can be no non-constant analytic maps from $\mathbf{A}^1$ into X. Similarly, since a Kobayashi chain in X will lift to a Kobayashi chain in $\widehat{X}$ of the same length and since the normalization map $\widehat{X} \to X$ is a finite morphism, we see, as in Lemma 2.11, that the Kobayashi semi-distance on X is, in fact, a distance, and we are done. □

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