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A HECKE CORRESPONDENCE THEOREM FOR MODULAR INTEGRALS WITH RATIONAL PERIOD FUNCTIONS

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1. Introduction

In the 1930's Erich Hecke used the Mellin transform and its inverse to demonstrate a systematic relationship between automorphic forms and Dirichlet series [5], [6]. In particular, entire modular forms on the full modular group $\Gamma(1) = SL(2, \mathbf{Z})$ correspond to Dirichlet series which satisfy a functional equation.

In [2], Eichler introduced generalized abelian integrals which he obtained by integrating modular forms of positive weight. An Eichler integral satisfies a modular relation with a polynomial period function. In [8] and [9], Marvin Knopp generalized Eichler integrals and developed the theory of modular integrals with rational period functions.

In [9], Knopp shows that an entire modular integral with a rational period function corresponds to a Dirichlet series which satisfies Hecke's functional equation, provided the rational period function has poles only at 0 or ∞. Knopp also proves a converse theorem, from which it follows that if the rational period function has any other poles the corresponding Dirichlet series does not satisfy the same functional equation.

In [4], Hawkins and Knopp prove a Hecke correspondence theorem in which a modular integral with an arbitrary rational period function corresponds to a Dirichlet series which satisfies a more general functional equation. In this case the functional equation for the Dirichlet series contains an additional remainder term which arises from the poles of the rational period function which are not at 0 or ∞. Hawkins and Knopp formulate their results for modular integrals on the theta group, Γ_θ, a subgroup of index 3 in $\Gamma(1)$. The theta group has a single group relation and any rational period function on Γ_θ must satisfy a corresponding relation. This relation in turn imposes a relation on the remainder term in the functional equation for the corresponding Dirichlet series.

In this paper we present a Hecke correspondence theorem for modular integrals of weight $2k \in 2\mathbf{Z}^+$ with rational period functions on the *full* modular group $\Gamma(1)$. The modular group has a second group relation which imposes more structure (than Γ_θ) on any modular integral, forcing its rational period function to satisfy a second relation. This in turn imposes more structure on the remainder term in the functional equation for the corresponding Dirichlet series. We will modify the characterization of rational

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period functions on $\Gamma(1)$ given by Choie and Zagier [1] in order to emphasize the second relation. We will show that a remainder term associated with $\Gamma(1)$ must satisfy a second relation which arises from the second relation for the rational period function and we will write the remainder term and its second relation explicitly.

2. Modular integrals

We will consider $\Gamma(1) = SL(2, \mathbf{Z})$ to be a group of linear fractional transformations acting on $\mathcal{H}$, the upper half plane, by putting $Mz = \frac{az+b}{cz+d}$ for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ and $z \in \mathcal{H}$. With this interpretation we identify an element M with its negative $-M$. $\Gamma(1)$ is generated by $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and satisfies the group relations

$$T^2 = (ST)^3 = I.$$

Suppose F is a function holomorphic in $\mathcal{H}$ and has the Fourier expansion

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}, \quad z \in \mathcal{H}. \tag{1}$$

Let $2k \in 2\mathbf{Z}$. If for every $z \in \mathcal{H}$, F satisfies the modular relation

$$z^{-2k} F\left(\frac{-1}{z}\right) = F(z) + q(z), \tag{2}$$

where $q(z)$ is a rational function, we say that F is an *entire modular integral of weight* $2k$ *on* $\Gamma(1)$ *with rational period function* q. If $q \equiv 0$, then F is an entire modular *form* of weight $2k$ on $\Gamma(1)$. Using the *slash operator* $F \mid_{2k} M = F \mid M$ defined by

$$(F \mid M)(z) = (cz + d)^{-2k} F(Mz),$$

we may rewrite (2) as

$$F \mid T = F + q. \tag{3}$$

The group relation $T^2 = I$ implies that a rational period function q satisfies the relation

$$q \mid T + q = 0, \tag{4}$$

and the relation $(ST)^3 = I$ implies that q satisfies the second relation

$$q \mid (ST)^2 + q \mid ST + q = 0. \tag{5}$$

Knopp [7, Section II] showed that (4) and (5) characterize the set of rational period functions for a given weight.

3. Rational period functions

This section summarizes the results we need concerning rational period functions on $\Gamma(1)$. It also describes modifications to Choie and Zagier's characterization [1] in order to emphasize the second relation (5).

In [9], Marvin Knopp proves that the poles of a rational period function on $\Gamma(1)$ occur only at 0, ∞, or at real quadratic irrationalities. He also shows that when the weight $2k$ is positive and the period function has only rational poles, it is of the form

$$q(z) = \begin{cases} c_1(1 - z^{-2}) + c_2 z^{-1}, & k = 1 \\ c(1 - z^{-2k}), & k > 1, \end{cases} \tag{6}$$

where c, c_1, and c_2 are complex numbers. The function $c(1 - z^{-2k})$ (for *any* $k \in \mathbf{R}$) is the period function for the trivial modular integral $F(z) \equiv -c$. The function $c_2 z^{-1}$ is a multiple of the period function for $E_2(z)$, the Eisenstein series of weight 2 on $\Gamma(1)$.

In [3], Hawkins describes the pole set of a rational period function and shows that it is the disjoint union of irreducible systems of poles. If $q(z)$ has a pole at a fixed quadratic irrational number α, an *irreducible system of poles*, $P(\alpha)$, is the minimal set of quadratic irrational numbers which must be poles of $q(z)$ because of (4) and (5). Hawkins also observes a connection between irreducible pole sets and indefinite binary quadratic forms.

We will use the following definitions and properties of quadratic forms which can be found in [13]. Let A, B and C be relatively prime integers such that $D = B^2 - 4AC$ is positive and not a square. Then $Q(x, y) = Ax^2 + Bxy + Cy^2$ is called a *primitive indefinite binary quadratic form* of *discriminant* D. We also denote $Q(x, y)$ by $Q = [A, B, C]$.

Given a quadratic form $Q(x, y)$ of discriminant D and a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, define

$$\hat{Q}(x, y) = (Q \circ M)(x, y) = Q(ax + by, cx + dy),$$

which is another quadratic form of the same discriminant D. Suppose that Q and $\hat{Q}$ are two binary quadratic forms of discriminant D. We will say that Q and $\hat{Q}$ are *equivalent in the narrow sense*, and write $Q \sim \hat{Q}$, if there is an element M of $\Gamma(1)$ such that $\hat{Q} = Q \circ M$. The relation $Q \sim \hat{Q}$ is an equivalence relation (narrow equivalence) on the set of quadratic forms of a given discriminant. Let $\mathcal{A}$ denote a narrow equivalence class of binary quadratic forms and define another (not necessarily distinct) equivalence class of forms,

$$\theta\mathcal{A}^{-1} = \{[-A, -B, -C]\colon [A, B, C] \in \mathcal{A}\}.$$

The binary quadratic form $Q = [A, B, C]$ is associated with the real quadratic irrational number $\alpha = \alpha_Q = \frac{B+\sqrt{D}}{2A}$, one of the roots of $Q(z, -1) = Az^2 - Bz + C$.

We will write $Q \leftrightarrow \alpha$ to denote this correspondence. The other root $\alpha' = \frac{B-\sqrt{D}}{2A}$ is associated with the negative form $-Q = [-A, -B, -C]$.

A quadratic form is said to be *simple* if $A > 0 > C$. Quadratic irrational numbers associated with simple forms are also said to be *simple*. A quadratic irrationality α is simple if and only if $\alpha > 0 > \alpha'$, where α' is the algebraic conjugate of α. A quadratic form is said to be *reduced* if $A > 0$, $C > 0$, and $B > A + C$. *Reduced* quadratic irrational numbers are those associated with reduced forms. A quadratic irrationality α is reduced if and only if $\alpha > 1 > \alpha' > 0$. Each narrow equivalence class $\mathcal{A}$ contains a finite, positive number of reduced forms [13].

Hawkins proves in [3] that an irreducible pole set $P(\alpha)$ corresponds to the set of reduced quadratic forms in a narrow equivalence class. Since any class $\mathcal{A}$ of forms corresponds to a unique irreducible pole set we will also denote the pole set by $P(\mathcal{A})$.

Choie and Zagier [1] establish a connection between the simple quadratic forms in an equivalence class $\mathcal{A}$ and the pole set $P(\mathcal{A})$. Let $\mathcal{Z}_{\mathcal{A}}$ denote the set of simple quadratic irrational numbers which are associated with $\mathcal{A}$.

LEMMA 1 (CHOIE AND ZAGIER). $P(\mathcal{A}) = \mathcal{Z}_{\mathcal{A}} \cup T\mathcal{Z}_{\mathcal{A}}$.

It is worth noting that $\mathcal{Z}_{\mathcal{A}}$ is the set of positive poles in $P(\mathcal{A})$ and that $T\mathcal{Z}_{\mathcal{A}} = \{-1/\alpha \mid \alpha \in \mathcal{Z}_{\mathcal{A}}\}$ is the set of negative poles in $P(\mathcal{A})$.

Choie and Zagier show that any rational period function q of weight $2k$ has the form

$$q(z) = \sum_{\mathcal{A}} C_{\mathcal{A}} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}}} (q_\alpha(z) - q_{T\alpha}(z)) + q_0'(z). \tag{7}$$

The outer sum is on the (finite number of) classes of binary quadratic forms which correspond to the irreducible pole sets of q. Each $C_{\mathcal{A}}$ is a complex number which depends only on the class $\mathcal{A}$. The functions q_α and $q_{T\alpha}$ are the principal parts of q at α and $T\alpha$, respectively, normalized so the coefficients of $(z-\alpha)^{-k}$ in q_α and $(z - T\alpha)^{-k}$ in $q_{T\alpha}$ are both one. The function q_0' is a rational function with a pole only at zero of order at most $2k$.

A complete description of the rational period function $q(z)$ requires explicit expressions for the functions q_α and $q_{T\alpha}$. Let $PP_\alpha[f]$ denote the principle part of $f(z)$ at $z = \alpha$. Choie and Zagier prove the following lemma.

LEMMA 2 (CHOIE AND ZAGIER). *Let α be a quadratic irrationality, α' its conjugate. Then*

$$q_\alpha(z) = PP_\alpha\left[\frac{(\alpha-\alpha')^k}{(z-\alpha)^k(z-\alpha')^k}\right] = PP_\alpha\left[\frac{D^{k/2}}{(az^2-bz+c)^k}\right], \tag{8}$$

where $[a, b, c]$ is the binary quadratic form associated to α and D is the discriminant of $[a, b, c]$.

The proof of Lemma 3 also shows that under the same assumptions,

$$q_{\alpha'}(z) = PP_{\alpha'}\left[\frac{(\alpha'-\alpha)^k}{(z-\alpha)^k(z-\alpha')^k}\right] = PP_{\alpha'}\left[\frac{(-1)^k D^{k/2}}{(az^2-bz+c)^k}\right], \tag{9}$$

an expression which we will use later. We may write q_α and $q_{\alpha'}$ in a more explicit way, using the partial fraction decomposition

$$\begin{aligned}\frac{1}{(az^2-bz+c)^k} &= \frac{1}{a^k}\sum_{l=1}^{k}\binom{2k-1-l}{k-l}\frac{(\alpha'-\alpha)^{l-2k}(-1)^k}{(z-\alpha)^l}\\ &\quad+\frac{1}{a^k}\sum_{l=1}^{k}\binom{2k-1-l}{k-l}\frac{(\alpha-\alpha')^{l-2k}(-1)^k}{(z-\alpha')^l}.\end{aligned}$$

We have

$$q_\alpha(z) = \sum_{l=1}^{k}\binom{2k-1-l}{k-l}\frac{(\alpha-\alpha')^{l-k}(-1)^{l-k}}{(z-\alpha)^l}, \tag{10}$$

and

$$q_{\alpha'}(z) = \sum_{l=1}^{k}\binom{2k-1-l}{k-l}\frac{(\alpha-\alpha')^{l-k}}{(z-\alpha')^l}. \tag{11}$$

We will modify the characterization of rational period functions given by Choie and Zagier in order to emphasize the second relation. We begin with an alternative way to express an irreducible pole set $P(\mathcal{A})$. Let $\mathcal{Z}'_{\mathcal{A}}$ denote $\{\alpha' \colon \alpha \in \mathcal{Z}_{\mathcal{A}}\}$.

LEMMA 3. $P(\mathcal{A}) = \mathcal{Z}_{\mathcal{A}} \cup \mathcal{Z}'_{\theta\mathcal{A}^{-1}}$.

Proof. A routine argument of containment in both directions shows that $T\mathcal{Z}_{\mathcal{A}} = \mathcal{Z}'_{\theta\mathcal{A}^{-1}}$. This, along with Lemma 3, completes the proof. □

We will rewrite the part of a rational period function which corresponds to the poles in $\mathcal{Z}_{\mathcal{A}}$ which are between zero and one. The following lemma will allow us to distinguish these poles when using the associated quadratic forms.

LEMMA 4. *Suppose that α is a simple quadratic irrational number associated with the quadratic form $[a, b, c]$. Then*

(*i*) $\alpha > 1$ *if and only if* $b > a + c$, *and*
(*ii*) $0 < \alpha < 1$ *if and only if* $b < a + c$.

The proof of Lemma 3 is a routine exercise in using inequalities.

The next lemma will allow us to rewrite those poles in $P(\mathcal{A})$ which are images under $(ST)^2$ of other poles in $P(\mathcal{A})$.

LEMMA 5. *For every equivalence class $\mathcal{A}$ of quadratic forms we have*

(*i*) $\{\beta \mid \beta \in \mathcal{Z}_{\mathcal{A}}, 0 < \beta < 1\} = \left\{(ST)^2\alpha' \mid \alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}}, \alpha > 1\right\}$, *and*
(*ii*) $\left\{\beta' \mid \beta \in \mathcal{Z}_{\theta\mathcal{A}^{-1}}, 0 < \beta < 1\right\} = \left\{(ST)^2\alpha \mid \alpha \in \mathcal{Z}_{\mathcal{A}}, \alpha > 1\right\}$.

Proof. Statement (i) is equivalent to statement (ii). An argument which involves containment in both directions and some tedious manipulations shows that statement (i) is true. □

Choie and Zagier observe, in a somewhat different form [1, page 95], that for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$,

$$q_{M\alpha} = q_\alpha \mid M^{-1} - PP_{a/c}\left[q_\alpha \mid M^{-1}\right], \tag{12}$$

where $PP_{a/c}\left[q_\alpha \mid M^{-1}\right](z)$ has a pole at $z = a/c$ of order at most $2k - 1$. If $M = (ST)^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, then $M^{-1} = ST = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $a/c = 0$. Thus we have

$$q_{(ST)^2\alpha} = q_\alpha \mid ST - PP_0\left[q_\alpha | ST\right], \tag{13}$$

where $PP_0\left[q_\alpha|ST\right]$ has a pole at $z = 0$ of order at most $2k - 1$.

We may now rewrite (7) in the desired form. By the proof of Lemma 3,

$$\begin{aligned} q &= \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}}} q_\alpha - \sum_{\alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}}} q_{\alpha'} \right\} + q_0' \\ &= \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} q_\alpha + \sum_{\substack{\beta \in \mathcal{Z}_{\mathcal{A}} \\ 0 < \beta < 1}} q_\beta - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha > 1}} q_{\alpha'} - \sum_{\substack{\beta \in \mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ 0 < \beta < 1}} q_{\beta'} \right\} + q_0'. \end{aligned}$$

Lemma 5 and (13) imply that

$$\begin{aligned} \sum_{\substack{\beta \in \mathcal{Z}_{\mathcal{A}} \\ 0 < \beta < 1}} q_\beta &= \sum_{\substack{\alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha > 1}} q_{(ST)^2\alpha'} \\ &= \sum_{\substack{\alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha > 1}} \left(q_{\alpha'} \mid ST - PP_0\left[q_{\alpha'}|ST\right]\right), \end{aligned}$$

and

$$\sum_{\substack{\beta\in\mathcal{Z}_{\theta\mathcal{A}^{-1}}\\0<\beta<1}} q_{\beta'} = \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\\alpha>1}} q_{(ST)^2\alpha}$$

$$= \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\\alpha>1}} (q_\alpha \mid ST - PP_0\,[q_\alpha|ST]) .$$

This gives us

$$q' = \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\\alpha>1}} q_\alpha \mid (I-ST) - \sum_{\substack{\alpha\in\mathcal{Z}_{\theta\mathcal{A}^{-1}}\\\alpha>1}} q_{\alpha'} \mid (I-ST) \right\} + q_0, \tag{14}$$

where

$$q_0 = q_0' - \sum_{\substack{\alpha\in\mathcal{Z}_{\theta\mathcal{A}^{-1}}\\\alpha>1}} PP_0\,[q_{\alpha'}|ST] + \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\\alpha>1}} PP_0\,[q_\alpha|ST]$$

is a rational function with a pole only at zero of order at most $2k$.

The expression (14) highlights the second relation (5) because each of the terms $q_\alpha \mid (I-ST)$ or $q_{\alpha'} \mid (I-ST)$ by itself satisfies the second relation. Since q must satisfy the second relation, q_0 must satisfy it as well.

We will write $q(z)$ in a more explicit form. Let

$$q_{\alpha,l}(z) = \frac{1}{(z-\alpha)^l}, \qquad q_{\alpha',l}(z) = \frac{1}{(z-\alpha')^l}. \tag{15}$$

Put $\beta' = (ST)^2\alpha$ and $\beta = (ST)^2\alpha'$. Then

$$(q_{\alpha,l} \mid ST)(z) = \frac{(\beta')^l}{z^{2k-l}(z-\beta')^l}, \qquad (q_{\alpha',l} \mid ST)(z) = \frac{\beta^l}{z^{2k-l}(z-\beta)^l}. \tag{16}$$

With this notation, (10) and (11) are

$$q_\alpha(z) = \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k}(-1)^{l-k} q_{\alpha,l}, \tag{17}$$

and

$$q_{\alpha'}(z) = \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k} q_{\alpha',l}. \tag{18}$$

Substituting (17) and (18) into (14) we have

$$q = \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} q_{\alpha,l} \mid (I - ST) \right.$$

$$\left. - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} q_{\alpha',l} \mid (I - ST) \right\} + q_0, \tag{19}$$

where q_0 may be written as

$$q_0(z) = \sum_{m=0}^{2k} \frac{b_m}{z^m} \tag{20}$$

with the b_m, $m = 0, 1, \ldots, 2k$ complex constants. Finally, if we use (15), (16), and (20) in (19), we may write the rational period function as

$$q(z) = \sum_{\mathcal{A}} C_{\mathcal{A}}$$

$$\times \left\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} \left(\frac{1}{(z-\alpha)^l} - \frac{(\beta')^l}{z^{2k-l}(z-\beta')^l} \right) \right.$$

$$\left. - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} \left(\frac{1}{(z-\alpha')^l} - \frac{\beta^l}{z^{2k-l}(z-\beta)^l} \right) \right\}$$

$$+ \sum_{m=0}^{2k} \frac{b_m}{z^m}. \tag{21}$$

4. The direct Hecke theorem

In this section we prove that a modular integral of positive, even weight $2k$ on $\Gamma(1)$ leads to a Dirichlet series with a functional equation. We derive an explicit form for the remainder term in the functional equation, which is based on (21).

Let $F(z)$ be an entire modular integral on $\Gamma(1)$ of weight $2k \in 2\mathbf{Z}^+$ with rational period function $q(z)$. We may assume without loss of generality that $F(z)$ is a *cusp* modular integral, *i.e.*, that $a_0 = 0$ in the Fourier expansion (1).

Write $z = x + iy$ with $x, y \in \mathbf{R}$. It can be shown [7, 622-623] that F satisfies

$$|F(z)| \leq K\left(|z|^{\alpha} + y^{-\beta}\right), \quad z \in \mathcal{H} \tag{22}$$

for some positive real numbers K, α and β. It follows that the coefficients a_n in the Fourier expansion (1) for F satisfy

$$a_n = \mathcal{O}(n^{\beta}), \quad n \to +\infty. \tag{23}$$

This, with $a_0 = 0$ in (1), implies that

$$F(iy) = \mathcal{O}(e^{-2\pi y}), \quad y \to +\infty. \tag{24}$$

Because of (22) and (24) we may consider the *Mellin transform of* F,

$$\Phi(s) = \int_0^{\infty} F(iy) y^s \frac{dy}{y}, \tag{25}$$

a function of $s = \sigma + it$. For $\sigma > \beta + 1$, we can integrate term by term to get

$$\Phi(s) = (2\pi)^{-s}\Gamma(s)\phi(s), \tag{26}$$

where

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{27}$$

is the *Dirichlet series associated with* F. The bound on the growth of the coefficients (23) implies that sum in (27) converges absolutely and uniformly on compact subsets of the right half plane $\sigma > \beta + 1$, so that $\phi(s)$ is analytic there.

Using the modular relation (3) we have

$$\begin{aligned}\int_0^1 F(iy) y^s \frac{dy}{y} &= \int_1^{\infty} F\left(\frac{-1}{iy}\right) y^{-s} \frac{dy}{y} \\ &= i^{2k} \int_1^{\infty} F(iy) y^{2k-s} \frac{dy}{y} + i^{2k} \int_1^{\infty} q(iy) y^{2k-s} \frac{dy}{y}.\end{aligned}$$

Thus

$$\Phi(s) = D(s) + E(s),$$

where

$$D(s) = \int_1^{\infty} F(iy)[y^s + i^{2k} y^{2k-s}] \frac{dy}{y} \tag{28}$$

and

$$E(s) = i^{2k} \int_1^{\infty} q(iy) y^{2k-s} \frac{dy}{y}. \tag{29}$$

It is not hard to see that $D(s)$ is entire and satisfies the functional equation

$$D(2k-s) - i^{2k} D(s) = 0. \tag{30}$$

From (21) we know that $q(z) = \mathcal{O}(1)$ as $|z| \to \infty$. Thus the integral defining $E(s)$ in (29) converges in the right half plane $\sigma > 2k$. Hawkins and Knopp prove in [4] that $E(s)$, and hence $\Phi(s)$, has a meromorphic continuation to the s-plane with, at worst, simple poles at integer points $m \le 2k$. They also show that $\Phi(s)$ is bounded in every lacunary vertical strip of the form

$$S(\sigma_1, \sigma_2; t_0)\colon\ \sigma_1 \le \sigma \le \sigma_2,\ \ |t| \ge t_0 > 0, \tag{31}$$

where σ_1, σ_2, and t_0 are real numbers.

Since $\Phi(s)$ has a meromorphic continuation to the whole s-plane we may write the functional equation which is suggested by (30),

$$\Phi(2k-s) - i^{2k}\Phi(s) = R(s), \tag{32}$$

where $R(s)$ is a meromorphic function which we will call the *remainder term*. Then by (30) we have

$$R(s) = E(2k-s) - i^{2k} E(s), \tag{33}$$

from which it is clear that $R(s)$ depends only on the rational period function q and not on the modular integral F. The expression (33) (or (32)) implies that $R(s)$ satisfies the (first) relation

$$R(2k-s) + i^{2k} R(s) = 0, \tag{34}$$

which was first observed by Hawkins and Knopp [4].

We will find an explicit expression for $R(s)$ using (33) and (21). This will give meaning to the functional equation (32) and it will enable us to prove a converse theorem. Put $E_a(s) = E(2k-s)$ and $E_b(s) = -i^{2k}E(s)$, so that

$$R(s) = E_a(s) + E_b(s). \tag{35}$$

By (29) we have

$$E_a(s) = i^{2k} \int_1^\infty q(iy) y^s \frac{dy}{y}, \tag{36}$$

and

$$E_b(s) = -\int_1^\infty q(iy) y^{2k-s} \frac{dy}{y}. \tag{37}$$

If we use the first relation (4) to replace $q(iy)$ in (37) we have

$$\begin{aligned} E_b(s) &= i^{2k} \int_1^\infty q\left(\frac{-1}{iy}\right) y^{-s} \frac{dy}{y} \\ &= i^{2k} \int_0^1 q(iy) y^s \frac{dy}{y}. \end{aligned} \tag{38}$$

A simple calculation shows that the parts of $E_a(s)$ and $E_b(s)$ which arise from q_0 cancel each other, so that q_0 contributes nothing to the remainder term. As a result, we may write

$$R(s) = \hat{E}_a(s) + \hat{E}_b(s), \tag{39}$$

where

$$\hat{E}_a(s) = i^{2k} \int_1^\infty \{q(iy) - q_0(iy)\} y^s \frac{dy}{y},$$

and

$$\hat{E}_b(s) = i^{2k} \int_0^1 \{q(iy) - q_0(iy)\} y^s \frac{dy}{y}.$$

We will use (21) to write $q - q_0$ as the sum of two functions which we can consider separately. Let

$$\begin{aligned} q^{(1)}(z) = \sum_{\mathcal{A}} C_{\mathcal{A}} \Bigg\{ & \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} \left(\frac{1}{(z-\alpha)^l} \right) \\ & - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta \mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} \left(\frac{1}{(z-\alpha')^l} \right) \Bigg\}, \end{aligned} \tag{40}$$

and

$$\begin{aligned} q^{(2)}(z) = \sum_{\mathcal{A}} C_{\mathcal{A}} \Bigg\{ & \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} \left(\frac{-(\beta')^l}{z^{2k-l}(z-\beta')^l} \right) \\ & - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta \mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} \left(\frac{-\beta^l}{z^{2k-l}(z-\beta)^l} \right) \Bigg\}, \end{aligned} \tag{41}$$

where the outer sum in each expression is on the equivalence classes which correspond to the irreducible pole sets of $q(z)$. Then by (21) we have

$$q - q_0 = q^{(1)} + q^{(2)},$$

so that

$$\begin{aligned} \hat{E}_a(s) &= i^{2k} \int_1^\infty q^{(1)}(iy) y^s \frac{dy}{y} + i^{2k} \int_1^\infty q^{(2)}(iy) y^s \frac{dy}{y} \\ &= \hat{E}_a^{(1)}(s) + \hat{E}_a^{(2)}(s), \end{aligned}$$

and

$$\begin{aligned}\hat{E}_b(s) &= i^{2k}\int_0^1 q^{(1)}(iy)y^s\frac{dy}{y} + i^{2k}\int_0^1 q^{(2)}(iy)y^s\frac{dy}{y}\\ &= \hat{E}_b^{(1)}(s) + \hat{E}_b^{(2)}(s).\end{aligned}$$

This, with (39), gives us

$$R(s) = \hat{E}_a^{(1)}(s) + \hat{E}_a^{(2)}(s) + \hat{E}_b^{(1)}(s) + \hat{E}_b^{(2)}(s). \tag{42}$$

The integral for $\hat{E}_a^{(1)}(s)$ converges for $\sigma < 1$ and the integral for $\hat{E}_b^{(1)}(s)$ converges for $\sigma > 0$. Thus, for $0 < \sigma < 1$, we have

$$R^{(1)}(s) = \hat{E}_a^{(1)}(s) + \hat{E}_b^{(1)}(s) \tag{43}$$

$$= i^{2k}\int_0^\infty q^{(1)}(iy)y^s\frac{dy}{y}. \tag{44}$$

In a similar way, for $2k-1 < \sigma < 2k$, we have

$$R^{(2)}(s) = \hat{E}_a^{(2)}(s) + \hat{E}_b^{(2)}(s) \tag{45}$$

$$= i^{2k}\int_0^\infty q^{(2)}(iy)y^s\frac{dy}{y}. \tag{46}$$

As a result the determination of $R(s)$ reduces to the evaluation of the integrals

$$\begin{aligned}
R_{\alpha,l}^{(1)}(s) &= i^{2k}n\int_0^\infty q_{\alpha,l}(iy)y^s\frac{dy}{y} = i^{2k}\int_0^\infty \frac{y^s}{(iy-\alpha)^l}\frac{dy}{y},\\
R_{\alpha,l}^{(2)}(s) &= i^{2k}\int_0^\infty (q_{\alpha,l}|ST)(iy)y^s\frac{dy}{y} = i^{2k}(\beta')^l\int_0^\infty \frac{y^s}{(iy)^{2k-l}(iy-\beta')^l}\frac{dy}{y},\\
R_{\alpha',l}^{(1)}(s) &= i^{2k}\int_0^\infty q_{\alpha',l}(iy)y^s\frac{dy}{y} = i^{2k}\int_0^\infty \frac{y^s}{(iy-\alpha')^l}\frac{dy}{y},\\
R_{\alpha',l}^{(2)}(s) &= i^{2k}\int_0^\infty (q_{\alpha',l}|ST)(iy)y^s\frac{dy}{y} = i^{2k}\beta^l\int_0^\infty \frac{y^s}{(iy)^{2k-l}(iy-\beta)^l}\frac{dy}{y},
\end{aligned} \tag{47}$$

with $\beta = (ST)^2\alpha'$, $\beta' = (ST)^2\alpha$, and $1 \le l \le k$.

The evaluation of these integrals involves exponential functions of the form $z^a = e^{a\log z}$, where $\log z = \log|z| + i\arg z$ for $z \in \mathbf{C}$. We will take the principal branch for each logarithm, using the convention that $-\pi \le \arg z < \pi$.

In order to evaluate the integrals in (47) we use the representation for the beta function [10, page 13]

$$B(a,b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}dt,$$

valid for Re $a > 0$ and Re $b > 0$. Let δ be a nonzero real number and change variables by putting $y = i\delta t$. If we use a contour integral to move the path of integration to the positive real axis, we have

$$B(a,b) = i^b \delta^b \int_0^\infty \frac{y^a}{(y+i\delta)^{a+b}} \frac{dy}{y}.$$

We replace b with $b-a$ and rearrange to get

$$\int_0^\infty \frac{y^a}{(iy-\delta)^b} \frac{dy}{y} = i^{a-2b}\delta^{a-b} B(a, b-a), \tag{48}$$

for $0 < \text{Re } a < \text{Re } b$, $\delta \in \mathbf{R}$, $\delta \neq 0$.

Using (48) to evaluate the integrals in (47), we have

$$\begin{aligned} R^{(1)}_{\alpha,l}(s) &= i^{s+2k-2l}\alpha^{s-l} B(s, l-s), \\ R^{(2)}_{\alpha,l}(s) &= i^{s-2k}(\beta')^{s-2k+l} B(s-2k+l, 2k-s), \\ R^{(1)}_{\alpha',l}(s) &= i^{s+2k-2l}(\alpha')^{s-l} B(s, l-s), \\ R^{(2)}_{\alpha',l}(s) &= i^{s-2k}\beta^{s-2k+l} B(s-2k+l, 2k-s). \end{aligned} \tag{49}$$

Now we may use (43) and (45) in (42) to write

$$R(s) = R^{(1)}(s) + R^{(2)}(s). \tag{50}$$

The expressions (44) and (40), along with the notation of (47), imply that

$$R^{(1)} = \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} R^{(1)}_{\alpha,l} \right.$$
$$\left. - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta \mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} R^{(1)}_{\alpha',l} \right\}.$$

Similarly, (46) and (41) imply that

$$R^{(2)} = \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} \left(-R^{(2)}_{\alpha,l}\right) \right.$$
$$\left. - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta \mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} \left(-R^{(2)}_{\alpha',l}\right) \right\}.$$

Using these expressions in (50) we have

$$R = \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}} \\ \alpha>1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k}(-1)^{l-k}(R^{(1)}_{\alpha,l} - R^{(2)}_{\alpha,l}) - \sum_{\substack{\alpha\in\mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha>1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k}(R^{(1)}_{\alpha',l} - R^{(2)}_{\alpha',l}) \right\}. \tag{51}$$

Then, by (49), we have the expression

$$R(s) = \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}} \\ \alpha>1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k}(-1)^{l-k} \times (i^{s+2k-2l}\alpha^{s-l}B(s,l-s) - i^{s-2k}(\beta')^{s-2k+l}B(s-2k+l,2k-s)) - \sum_{\substack{\alpha\in\mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha>1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k} \times (i^{s+2k-2l}(\alpha')^{s-l}B(s,l-s) - i^{s-2k}\beta^{s-2k+l}B(s-2k+l,2k-s)) \right\}. \tag{52}$$

We can replace β and β', using $\beta = (ST)^2\alpha' = \frac{-1}{\alpha'-1}$ and $\beta' = (ST)^2\alpha = \frac{-1}{\alpha-1}$. After simplifying, the result is

$$R(s) = \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}} \\ \alpha>1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k} \times \left(i^{s}\alpha^{s-l}B(s,l-s) - i^{-s}(\alpha-1)^{2k-s-l}B(s-2k+l,2k-s)\right) - \sum_{\substack{\alpha\in\mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha>1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k}(-1)^{l-k} \times \left(i^{s}(\alpha')^{s-l}B(s,l-s) - i^{-s}(\alpha'-1)^{2k-s-l}B(s-2k+l,2k-s)\right) \right\}. \tag{53}$$

We have proved the following theorem.

THEOREM 6. *Suppose that $F(z)$ is an entire modular integral of weight $2k \in 2\mathbf{Z}^+$ on $\Gamma(1)$, with rational period function $q(z)$ given by* (21); *suppose that F has the Fourier expansion* (1) *with zero constant term, so that* (24) *holds. Let $\Phi(s)$ be defined by* (25) *for $\sigma > \beta$.*

Then for $\sigma > \beta + 1$, $\Phi(s)$ is also given by (26) *and* (27), *and*
(*a*) *$\Phi(s)$ has a meromorphic continuation to the whole s-plane with, at worst, simple poles at integer points $m \leq 2k$. $\Phi(s)$ is represented for $\sigma > \beta$ by*

$$\Phi(s) = D(s) + E(s).$$

$D(s)$ is given by (28) *and is entire, and $E(s)$ is given by* (29) *and has a meromorphic continuation to the whole s-plane. Furthermore,*
(*b*) *$\Phi(s)$ is bounded in every lacunary vertical strip of the form* (31), *and*
(*c*) *$\Phi(s)$ satisfies the functional equation* (32) *where $R(s)$ is given by* (53).

5. The second relation

We have already observed that the remainder term $R(s)$ satisfies one relation, (34). In this section we will describe a second relation which $R(s)$ must satisfy. This second relation follows from the fact that the corresponding rational period function $q(z)$ satisfies (5).

Suppose that α is one of the poles of $q(z)$ which is denoted by α or α' in (21). (In order to simplify the notation we will suppress the prime if α is negative.) Let $\beta = (ST)^2\alpha$ and $\gamma = (ST)^2\beta$, so that $\alpha = (ST)^2\gamma$. Then β represents one of the poles of $q(z)$ which is denoted by β or β' in (21). Fix l, $1 \leq l \leq k$ and put

$$\begin{aligned} q_1(z) &= \frac{1}{(z-\alpha)^l}, \\ q_2(z) &= (q_1|ST)(z) = \frac{\beta^l}{z^{2k-l}(z-\beta)^l}, \\ q_3(z) &= (q_1|(ST)^2)(z) = \frac{(\gamma-1)^l}{(z-1)^{2k-l}(z-\gamma)^l}. \end{aligned} \tag{54}$$

Then by (21) $q - q_0$ is a linear combination of terms of the form

$$\hat{q} = q_1 - q_2,$$

and $|ST$ maps $q_1 \to q_2 \to q_3 \to q_1$. Each term $\hat{q}$ satisfies the second relation (5), since

$$\hat{q}|ST = q_2 - q_3$$

and

$$\hat{q}|(ST)^2 = q_3 - q_1.$$

In order to write the terms of $R(s)$ which correspond to q_1, q_2 and q_3 we put

$$R_j(s) = i^{2k} \int_0^\infty q_j(iy) y^s \frac{dy}{y} \tag{55}$$

for $j = 1, 2, 3$. Then $R(s)$ is a linear combination of terms of the form

$$\hat{R} = R_1 - R_2.$$

We know that R_1 and R_2 are meromorphic in the entire s-plane, since they have been written explicitly in (49). Since the integral defining $R_3(s)$ converges in the strip $0 < \sigma < 2k$, we have $R_1(s)$, $R_2(s)$ and $R_3(s)$ all defined at least in the strip $0 < \sigma < 2k$.

Let a, b and ϕ be real numbers, let r, m and l be nonnegative integers, and put

$$R_{r,m,l}(s; a, b, \phi) = i^{2k} \int_0^\infty \frac{y^s}{(iy-a)^r (iy-b)^m (iy-\phi)^l} \frac{dy}{y}. \tag{56}$$

The parameters a, b and ϕ will represent poles of terms of the rational period function, and r, m and l will denote the respective orders of the poles. The region of convergence for the integral in (56) depends on the values of a, b, ϕ, r, m, and l. With this notation we have

$$\begin{aligned} R_1(s) &= R_{0,0,l}(s; 0, 1, \alpha), \\ R_2(s) &= \beta^l R_{2k-l,0,l}(s; 0, 1, \beta), \\ R_3(s) &= (\gamma - 1)^l R_{0,2k-l,l}(s; 0, 1, \gamma), \end{aligned} \tag{57}$$

with $\beta = (ST)^2\alpha$, $\gamma = (ST)^2\beta$ and $\alpha = (ST)^2\gamma$.

Define the mapping ρ by

$$\rho(R_{r,m,l}(s; 0, 1, \phi)) = i^{2k} R_{r,m,l}(2k - s; -1, 0, \phi - 1). \tag{58}$$

It is clear that ρ is linear, i.e., that

$$\rho(a_1 R_1 + a_2 R_2) = a_1 \rho(R_1) + a_2 \rho(R_2)$$

for any constants a_1 and a_2 and functions R_1 and R_2 of the form (56). The mapping ρ is the image of the mapping $|ST$ on the modular integral side of the correspondence.

The next lemma will allow us to rewrite the right hand side of (58).

LEMMA 7.

$$R_{r,m,l}(2k - s; -1, 0, \phi - 1) = i^{-2k-2m}((ST)^2\phi)^l R_{2k-r-m-l,r,l}(s; 0, 1, (ST)^2\phi)$$

The proof uses the integral definition (56), a change of variables, and some simple manipulations.

Using Lemma 7 we may write the mapping ρ in an alternative way as

$$\rho(R_{r,m,l}(s;0,1,\phi)) = i^{-2m}((ST)^2\phi)^l R_{2k-r-m-l,r,l}(s;0,1,(ST)^2\phi). \tag{59}$$

We can now use ρ to state a relation which $\hat{R}$ satisfies and which reflects the fact that $\hat{q}$ satisfies (5).

THEOREM 8. *Let R_1, R_2 and R_3 be given by* (55) *and* (54). *Suppose that $\hat{R} = R_1 - R_2$, and suppose that ρ is the mapping defined by* (58). *Then $\hat{R}$ satisfies the relation*

$$\hat{R} + \rho(\hat{R}) + \rho^2(\hat{R}) = 0.$$

Proof. We first show that ρ: $R_1 \to R_2 \to R_3 \to R_1$. Using (59) we have

$$\begin{aligned}\rho(R_1(s)) &= \rho(R_{0,0,l}(s;0,1,\alpha))\\ &= \beta^l R_{2k-l,0,l}(s;0,1,\beta)\\ &= R_2(s).\end{aligned}$$

Also,

$$\begin{aligned}\rho(R_2(s)) &= \beta^l \rho(R_{2k-l,0,l}(s;0,1,\beta))\\ &= \beta^l\gamma^l R_{0,2k-l,l}(s;0,1,\gamma)\\ &= (\gamma-1)^l R_{0,2k-l,l}(s;0,1,\gamma)\\ &= R_3(s),\end{aligned}$$

since $\beta = ST\gamma = \frac{\gamma-1}{\gamma}$. Finally,

$$\begin{aligned}\rho(R_3(s)) &= (\gamma-1)^l \rho(R_{0,2k-l,l}(s;0,1,\gamma))\\ &= (\gamma-1)^l\alpha^l i^{2l} R_{0,0,l}(s;0,1,\alpha)\\ &= R_{0,0,l}(s;0,1,\alpha)\\ &= R_1(s),\end{aligned}$$

since $\gamma = ST\alpha = 1 - 1/\alpha$, or $\gamma - 1 = -1/\alpha$.

Thus we have $\rho(\hat{R}) = R_2 - R_3$ and $\rho^2(\hat{R}) = R_3 - R_1$, so that $\hat{R} + \rho(\hat{R}) + \rho^2(\hat{R}) = 0$. □

We can now extend the second relation to the entire remainder term.

COROLLARY 9. *Suppose that $R(s)$ is the remainder term for a Dirichlet series which corresponds to an entire modular integral F with rational period function q on $\Gamma(1)$. Then $R(s)$ satisfies*

$$R + \rho(R) + \rho^2(R) = 0, \tag{60}$$

where ρ is the mapping (58).

Proof. Any rational period function on $\Gamma(1)$ can be written as in (21) so that $q - q_0$ is a linear combination of terms of the form $\hat{q} = q_1 - q_2$ with q_1 and q_2 given by (54). Then, since q_0 makes no contribution to the remainder term, $R(s)$ is a linear combination of terms of the form $\hat{R} = R_1 - R_2$ where R_1 and R_2 are given by (55), hence by (57). Theorem 8 implies that $\hat{R}$ satisfies (60). The corollary follows from the fact that ρ is a linear map. □

6. The converse Hecke theorem

We now prove the following converse to Theorem 6.

THEOREM 10. *Suppose the Dirichlet series*

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{61}$$

converges absolutely in the half-plane $\sigma > \gamma$. Suppose that the function $\Phi(s)$ defined by

$$\Phi(s) = (2\pi)^{-s}\Gamma(s)\phi(s) \tag{62}$$

satisfies:

(*a*) *$\Phi(s)$ has a meromorphic continuation to the whole s-plane with, at worst, simple poles at integer points s*;

(*b*) *$\Phi(s)$ is bounded in every lacunary vertical strip of the form*

$$S(\sigma_1, \sigma_2; t_0)\colon\ \sigma_1 \le \sigma \le \sigma_2,\ |t| \ge t_0 > 0; \quad \textit{and} \tag{63}$$

(*c*) *$\Phi(s)$ satisfies the functional equation*

$$\Phi(2k - s) - i^{2k}\Phi(s) = R(s), \tag{64}$$

where $R(s)$ is given by (53).

Then $\phi(s)$ is the Dirichlet series associated with an entire modular integral F of weight $2k$ on $\Gamma(1)$ with rational period function $q(z)$, where $q(z)$ is given by (21).

Proof. Following Hecke [5], [6] we take the inverse Mellin transform of Φ and interchange the sum and integral to get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Phi(s)y^{-s}ds=\sum_{n=1}^{\infty}a_n e^{-2\pi ny}=F(iy), \tag{65}$$

for any $c>0$. Let L be a positive integer with $L>\gamma$ and $L\geq 2k$, and fix c between L and $L+1$. We move the line of integration to $\sigma=2k-c$, use the functional equation (64) and make a change of variables to get

$$\begin{aligned}(F|T)(iy)-F(iy) &= \frac{i^{-2k}}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty}R(s)y^{-s}ds\\ &\quad -\frac{1}{2\pi i}\sum_{m=2k-L}^{L-1}\operatorname*{Res}_{s=m}\{\Phi(s)y^{-s}\}.\end{aligned} \tag{66}$$

We will be done if we can show that the right hand side of (66) is $q(iy)$, with $q(z)$ given by (21). The modular relation (for $z\in\mathcal{H}$) then follows by the identity theorem.

In order to evaluate the integral in (66) we must evaluate

$$\frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty}i^{s+2k-2l}\delta^{s-l}B(s,l-s)y^{-s}ds,$$

and

$$\frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty}i^{s-2k}\delta^{s-2k+l}B(s-2k+l,2k-s)y^{-s}ds,$$

for $l\in\mathbf{Z}$, $1\leq l\leq k$, and $\delta\in\mathbf{R}$, $\delta\neq 0$. If we let $a=s$ and $b=l$ in (48) we have

$$\int_0^{\infty}\frac{y^s}{(iy-\delta)^l}\frac{dy}{y}=i^{s-2l}\delta^{s-l}B(s,l-s),$$

for $0<\sigma<l$ and $\delta\in\mathbf{R}$, $\delta\neq 0$. Since $\frac{1}{(iy-\delta)^l}$ is of bounded variation, we have the inverse Mellin transform [12, Theorem 9a]

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}i^{s-2l}\delta^{s-l}B(s,l-s)y^{-s}ds=\frac{1}{(iy-\delta)^l},$$

for $0<c<l$ and $y>0$. Letting $c=1/2$ we have

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}i^{s+2k-2l}\delta^{s-l}B(s,l-s)y^{-s}ds=\frac{i^{2k}}{(iy-\delta)^l}, \tag{67}$$

for any $y>0$. We move the line of integration from $\sigma=2k-c$ to $\sigma=1/2$, and pick up the negatives of the residues at $s=2k-L,2k-L+1,\ldots,0$. Then we apply (67) to get

$$\frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty}i^{s+2k-2l}\delta^{s-l}B(s,l-s)y^{-s}ds=\frac{i^{2k}}{(iy-\delta)^l}-\sum_{m=2k-L}^{0}\frac{a_m(\delta,l)}{(iy)^m}. \tag{68}$$

In a similar way we have that

$$\frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} i^{s-2k}\delta^{s-2k+l}B(s-2k+l,2k-s)y^{-s}ds$$
$$= \frac{i^{-2k}\delta^{l}}{(iy)^{2k-l}(iy-\delta)^{l}} - \sum_{m=2k-L}^{2k-l}\frac{b_m(\delta,l)}{(iy)^m}. \quad (69)$$

The sum in (66) is a rational function with at worst a pole at zero. This, along with (68) and (69), implies that the right hand side of (66) is $q(iy)$, with $q(z)$ a rational function. Thus q is a rational period function of weight $2k$. As a result, any pole of q at zero has order at most $2k$. Substituting (68) and (69) into (66), shows that $q(z)$ has the form (21). □

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