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NORM INEQUALITIES IN THE CORACH-PORTA-RECHT THEORY AND OPERATOR MEANS

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1. Introduction

Throughout this note, an operator means a bounded linear operator acting on a Hilbert space. In particular, an operator A on H is positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$.

In [2], Corach-Porta-Recht gave a norm inequality as a key of their theory on differential geometry. Afterwards, we pointed out that it is equivalent to the Heinz inequality [6]. On the other hand, Furuta [10] showed that the Cordes inequality

$$\|A^t B^t\| \leq \|AB\|^t \quad \text{for } A, B \geq 0 \text{ and } 0 \leq t \leq 1 \tag{1}$$

is equivalent to the Löwner-Heinz inequality (cf. [16])

$$A \geq B \geq 0 \quad \text{implies} \quad A^t \geq B^t \quad \text{for } 0 \leq t \leq 1. \tag{2}$$

Under such situation, we developed Furuta's argument on the equivalence of (1) and (2) in [8]. However the Jensen inequality [12]

$$(X^* A X)^t \geq X^* A^t X \quad \text{for } A \geq 0 \text{ and contractions } X \tag{3}$$

is not discussed there.

Very recently, Corach-Porta-Recht [3] proposed the norm inequality, denoted the CPR inequality,

$$\|(A \sharp_t B)^{1/2}(C \sharp_t D)^{1/2}\| \leq \|A^{1/2}C^{1/2}\|^{1-t}\|B^{1/2}D^{1/2}\|^t \tag{4}$$

for positive operators A, B, C and D, where $\sharp_t$ is the t-power mean defined by

$$A \sharp_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2} \tag{5}$$

for invertible $A, B \geq 0$ and $t \in [0, 1]$; see [15]. As stated in [3], (1) is the special case of (4), i.e., take $A = C = 1$ in (4). For the sake of convenience, the t-power mean defined by (5) is extended as in [11]: For $t \in \mathbb{R}$,

$$A \natural_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2} \tag{5'}$$

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for invertible $A, B \geq 0$,

In this note, we show that the CPR inequality (4) is implied by the Jensen inequality (3). Moreover we consider the reverse inequality of the CPR inequality:

$$(4') \qquad \|(A \natural_t B)^{1/2}(C \natural_t D)^{1/2}\| \geq \|A^{1/2}C^{1/2}\|^{1-t}\|B^{1/2}D^{1/2}\|^t$$

for $A, B, C, D \geq 0$ with invertible A, C and $1 \leq t \leq 2$. Similarly we do those of the Cordes and Jensen inequalities (1) and (3):

$$(1') \qquad \|A^t B^t\| \geq \|AB\|^t \quad \text{for} \quad 1 \leq t \leq 2,$$

$$(3') \qquad (X^*AX)^t \leq X^*A^tX \quad \text{for contractions } X \text{ and } 1 \leq t \leq 2.$$

Thus we prove that the inequalities (1)–(4), (1′), (3′) and (4′) are mutually equivalent; the proof is done in an elementary way and clarifies the importance of the Löwner-Heinz inequality (2). Next one of them is discussed in a general setting (cf. [7]): A nonnegative continuous function f on $[0, \infty)$ is operator monotone if and only if the Jensen inequality holds for $f^*(x) = xf(x^{-1})$; i.e.,

$$f^*(X^*AX) \geq X^*f^*(A)X$$

for $A \geq 0$ and contractions X. Here we remark that if f is the operator monotone function corresponding to an operator mean σ, i.e.,

$$A \,\sigma\, B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2},$$

then f^* corresponds to the transpose ${}^t\sigma$ of σ, defined by $A \,{}^t\sigma\, B = B \,\sigma\, A$; see [15]. Finally we give a simple proof of the fact that a real-valued continuous function f on $[0, \infty)$ is operator monotone if and only if $\tilde{f}(x) = xf(x)$ is operator convex.

2. The CPR inequality

First of all, we state our result.

THEOREM 1. *The following inequalities hold and follow from each other, where A, B, C and D are positive operators*:

- (I_1) $\|(A \sharp_t B)^{1/2}(C \sharp_t D)^{1/2}\| \leq \|A^{1/2}C^{1/2}\|^{1-t}\|B^{1/2}D^{1/2}\|^t$ *for* $0 \leq t \leq 1$.
- (I_2) $\|(A \natural_t B)^{1/2}(C \natural_t D)^{1/2}\| \geq \|A^{1/2}C^{1/2}\|^{t-1}\|B^{1/2}D^{1/2}\|^t$ *for invertible* A, C *and* $1 \leq t \leq 2$.
- (II_1) $\|A^tB^t\| \leq \|AB\|^t$ *for* $0 \leq t \leq 1$.
- (II_2) $\|A^tB^t\| \geq \|AB\|^t$ *for* $1 \leq t \leq 2$.
- (II_3) $\|A^{1/2}B^{1/2}\| \leq \|AB\|^{1/2}$.

(II_4) $\|A^2B^2\| \geq \|AB\|^2$.
(II_5) $\|A^tB^tA^t\| \leq \|ABA\|^t$ *for* $0 \leq t \leq 1$.
(II_6) $\|A^tB^tA^t\| \geq \|ABA\|^t$ *for* $1 \leq t \leq 2$.
(III) $A \geq B \geq 0$ *implies* $A^t \geq B^t$ *for* $0 \leq t \leq 1$.
(IV_1) $(X^*AX)^t \geq X^*A^tX$ *for contractions* X *and* $0 \leq t \leq 1$.
(IV_2) $(X^*AX)^t \leq X^*A^tX$ *for contractions* X *and* $1 \leq t \leq 2$.

We remark that $(I_1) \Rightarrow (II_1) \Rightarrow (II_5)$ is stated in [3], $(II_5) \Rightarrow (II_1)$ is easily checked, and the equivalence of (II_1) and (III) is proved by Furuta [10]. To prove the others, we prepare the following lemmas, one of which is made for the proof of an extension of the Furuta inequality in [11] and is quite usefull for such a discussion.

LEMMA 2. *For invertible operators* X *and* $A \geq 0$,

$$(X^*AX)^t = X^*A^{1/2}(A^{1/2}XX^*A^{1/2})^{t-1}A^{1/2}X$$

for all $t \in \mathbb{R}$. *In particular,*

$$A \sharp_t B = B \sharp_{1-t} A$$

for $0 \leq t \leq 1$.

Proof. For the sake of convenience, we give a simple proof via the polar decomposition: Let $X^*A^{1/2} = UH$ be the polar decomposition of $X^*A^{1/2}$. Then, for $s \in \mathbb{R}$, we have

$$\begin{aligned}(X^*AX)^{1+s} &= (UH^2U^*)^{1+s} = UH^{2+2s}U^* \\ &= UHH^{2s}HU^* = X^*A^{1/2}(A^{1/2}XX^*A^{1/2})^sA^{1/2}X.\end{aligned}$$ □

Next we reformulate the Jensen inequality (3) as follows:

LEMMA 3. *If* $A \geq 0$, *then, for any operator* X, *(i)* $X^*A^tX \leq \|X\|^{2-2t}(X^*AX)^t$ *for* $0 \leq t \leq 1$ *and (ii)* $(X^*AX)^t \leq \|X\|^{2t-2}X^*A^tX$ *for* $1 \leq t \leq 2$.

Note that (ii) is easily obtained using Lemma 2 and (i) is just a reformulation of the Jensen inequality (3).

The following lemma is a simple application of Lemma 3.

LEMMA 4. The following inequalities hold for invertible $A, B, C, D \geq 0$:

(i_1) $A \sharp_t B \leq \|A^{1/2}C^{1/2}\|^{2-2t}(C^{-1} \sharp_t B)$ for $0 \leq t \leq 1$.
(i_2) $C \sharp_t D \leq \|B^{1/2}D^{1/2}\|^{2t}(C \sharp_t B^{-1})$ for $0 \leq t \leq 1$.
(ii_1) $A \natural_t B \geq \|A^{1/2}C^{1/2}\|^{2t-2}(C^{-1} \natural_t B)$ for $1 \leq t \leq 2$.
(ii_2) $C \natural_t D \geq \|B^{1/2}D^{1/2}\|^{2t}(C \natural_t B^{-1})$ for $1 \leq t \leq 2$.

Proof. (i_1) It follows from Lemma 3 (i) that

$$\begin{aligned} A \sharp_t B &= C^{-1/2}C^{1/2}A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}C^{1/2}C^{-1/2} \\ &\le C^{-1/2}\|C^{1/2}A^{1/2}\|^{2-2t}(C^{1/2}A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2}C^{1/2})^t C^{-1/2} \\ &= \|A^{1/2}C^{1/2}\|^{2-2t}C^{-1/2}(C^{1/2}BC^{1/2})^t C^{-1/2} \\ &= \|A^{1/2}C^{1/2}\|^{2-2t}C^{-1} \sharp_t B. \end{aligned}$$

(i_2) It follows from Lemma 2 and the above (i_1) that

$$\begin{aligned} C \sharp_t D &= D \sharp_{1-t} C \\ &\le \|B^{1/2}D^{1/2}\|^{2-2(1-t)}(B^{-1} \sharp_{1-t} C) \\ &= \|B^{1/2}D^{1/2}\|^{2t}(C \sharp_t B^{-1}). \end{aligned}$$

The proofs of (ii_1) and (ii_2) are similar to those of (i_1) and (i_2). □

Now we prove Theorem 1 based on Lemmas 2 and 4.

PROOF OF THEOREM 1. The proof is divided into two parts, namely the equivalence $(I_1) \Rightarrow (II_1) \Rightarrow (II_3) \Rightarrow (III) \Rightarrow (IV_1) \Rightarrow (I_1)$ and the implication $(III) \Rightarrow (IV_2) \Rightarrow (I_2) \Rightarrow (II_2) \Rightarrow (II_4)$. Since (II_3) and (II_4) are clearly equivalent and proved in [8], it suffices to show the equivalence and implication stated above.

$(III) \Rightarrow (IV_1)$. It suffices to show that

$$(CAC)^t \ge CA^tC$$

for invertible positive operators A and $C \le 1$. It follows from Lemma 2 that

$$\begin{aligned} (CAC)^t &= CA^{1/2}(A^{1/2}C^2A^{1/2})^{t-1}A^{1/2}C \\ &= CA^{1/2}(A^{-1/2}C^{-2}A^{-1/2})^{1-t}A^{1/2}C \\ &\ge CA^{1/2}(A^{-1})^{1-t}A^{1/2}C \quad \text{by (III)} \\ &= CA^tC. \end{aligned}$$

$(IV_1) \Rightarrow (I_1)$. It follows from Lemma 4 (i_1) and (i_2) that

$$\begin{aligned} &\|(A \sharp_t B)^{1/2}(C \sharp_t D)^{1/2}\|^2 \\ &= \|(C \sharp_t D)^{1/2}(A \sharp_t B)(C \sharp_t D)^{1/2}\| \\ &\le \|A^{1/2}C^{1/2}\|^{2-2t}\|(C \sharp_t D)^{1/2}(C^{-1} \sharp_t B)(C \sharp_t D)^{1/2}\| \\ &= \|A^{1/2}C^{1/2}\|^{2-2t}\|(C^{-1} \sharp_t B)^{1/2}(C \sharp_t D)(C^{-1} \sharp_t B)^{1/2}\| \\ &\le \|A^{1/2}C^{1/2}\|^{2-2t}\|B^{1/2}D^{1/2}\|^{2t}\|(C^{-1} \sharp_t B)^{1/2}(C \sharp_t B^{-1})(C^{-1} \sharp_t B)^{1/2}\| \\ &= \|A^{1/2}C^{1/2}\|^{2-2t}\|B^{1/2}D^{1/2}\|^{2t}, \end{aligned}$$

because $C \sharp_t B^{-1} = (C^{-1} \sharp_t B)^{-1}$.

Since $(I_1) \Rightarrow (II_1)$ by taking $A = C = 1$ and $(II_1) \Leftrightarrow (III)$ and $(II_1) \Leftrightarrow (II_3)$ are shown in [10] and [8] respectively, the first half is proved.

For the latter half, we prove $(III) \Rightarrow (IV_2) \Rightarrow (I_2)$ because $(I_2) \Rightarrow (II_2) \Rightarrow (II_4)$ is easily seen.

$(III) \Rightarrow (IV_2)$. Let $s = t + 1$; then $0 \le t \le 1$. Then it follows from Lemma 2 that for a contraction X and $A \ge 0$,

$$\begin{aligned}(X^*AX)^s &= X^*A^{1/2}(A^{1/2}XX^*A^{1/2})^tA^{1/2}X \\ &\le X^*A^{1/2}(A^{1/2}A^{1/2})^tA^{1/2}X \quad \text{by (III)} \\ &= X^*A^sX.\end{aligned}$$

$(IV_2) \Rightarrow (I_2)$: The proof is quite similar to that of $(IV_1) \Rightarrow (I_1)$. As a matter of fact, it follows from Lemma 4 (ii_1) and (ii_2) that

$$\begin{aligned}&\|(A \natural_t B)^{1/2}(C \natural_t D)^{1/2}\|^2 \\ &= \|(C \natural_t D)^{1/2}(A \natural_t B)(C \natural_t D)^{1/2}\| \\ &\ge \|A^{1/2}C^{1/2}\|^{2t-2}\|(C \natural_t D)^{1/2}(C^{-1} \natural_t B)(C \natural_t D)^{1/2}\| \\ &= \|A^{1/2}C^{1/2}\|^{2t-2}\|(C^{-1} \natural_t B)^{1/2}(C \natural_t D)(C^{-1} \natural_t B)^{1/2}\| \\ &\ge \|A^{1/2}C^{1/2}\|^{2t-2}\|B^{1/2}D^{1/2}\|^{2t}\|(C^{-1} \natural_t B)^{1/2}(C \natural_t B^{-1})(C^{-1} \natural_t B)^{1/2}\| \\ &= \|A^{1/2}C^{1/2}\|^{2t-2}\|B^{1/2}D^{1/2}\|^{2t}.\end{aligned}$$

So the proof is complete.

3. Operator monotone functions

A binary operation m among positive operators is called a mean if m is upper-semicontinuous and satisfies

$$A \le C \text{ and } B \le D \text{ implies } A\ m\ B \le C\ m\ D$$

and the transformer inequality

$$T^*(A\ m\ B)T \le T^*AT\ m\ T^*BT$$

for all T. We note that if T is invertible, then it is replaced by the equality

$$T^*(A\ m\ B)T = T^*AT\ m\ T^*BT.$$

Now the Kubo-Ando theory on operator means says that there is an affine-isomorphism of the operator means σ onto the nonnegative operator monotone functions f on $[0, \infty)$ such that

$$A\ \sigma\ B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}$$

for invertible $A, B \geq 0$, or simply

$$f(x) = 1 \, \sigma \, x \quad \text{for} \quad x \geq 0,$$

which is called the representing function of σ. Clearly a binary operation ${}^t\sigma$ defined by

$$A \, {}^t\sigma \, B = B \, \sigma \, A$$

is an operator mean if so is σ. If f is the representing function of σ, then that of ${}^t\sigma$ is given by

$$f^*(x) = 1 \, {}^t\sigma \, x = x(x^{-1} \, {}^t\sigma \, 1) = x(1 \, \sigma \, x^{-1}) = xf(x^{-1}).$$

Since ${}^t(\sharp_t) = \sharp_{1-t}$ by Lemma 2, (*III*) $\Rightarrow$ (*IV_1*) in Theorem 1 suggests the following generalization:

THEOREM 5. *Let f be a nonnegative continuous function on $[0, \infty)$ such that $\lim_{x\to 0} f^*(x)$ exists. Then f is operator monotone if and only if the Jensen inequality holds for f^*, i.e.,*

$$f^*(X^*AX) \geq X^* f^*(A) X$$

for $A \geq 0$ and contractions X.

Proof. We use the following formula of Furuta's type instead of Lemma 2: for invertible operators $A \geq 0$ and X,

$$f^*(X^*AX) = X^*A^{1/2} f(A^{-1/2}(XX^*)^{-1}A^{-1/2})A^{1/2}X.$$

This can be checked via the Weierstrass approximation theorem.

Now we assume that f is operator monotone. For invertible positive operators A and $C \leq 1$, we have

$$\begin{aligned} f^*(CAC) &= CA^{1/2} f(A^{-1/2}C^{-2}A^{1/2})A^{1/2}C \\ &\geq CA^{1/2} f(A^{-1})A^{1/2}C \quad \text{by} \quad C^{-2} \geq 1 \\ &= Cf^*(A)C. \end{aligned}$$

Conversely we take invertible $B \geq A \geq 0$. Since $B^{-1} \leq A^{-1}$, there exists an invertible contraction X such that $B^{-1/2} = A^{-1/2}X$. Since $B^{1/2} = X^{-1}A^{1/2} = A^{1/2}X^{*-1}$, we have

$$\begin{aligned} X^*A^{-1/2} f(B) A^{-1/2}X &= X^*A^{-1/2} f(A^{1/2}(XX^*)^{-1}A^{1/2})A^{-1/2}X \\ &= f^*(X^*A^{-1}X) \\ &\geq X^* f^*(A^{-1})X \\ &= X^*A^{-1/2} f(A) A^{-1/2}X, \end{aligned}$$

so that $f(B) \geq f(A)$, as desired. □

A real-valued function g on $[0, \infty)$ is operator convex if it satisfies

$$g(sA + (1-s)B) \leq sg(A) + (1-s)g(B)$$

for $A, B \geq 0$ and $0 \leq s \leq 1$. Hansen-Pedersen [13] proved that for a real-valued function g on $[0, \infty)$, g is operator convex and $g(0) \leq 0$ if and only if

$$g(X^*AX) \leq X^*g(A)X$$

for $A \geq 0$ and contractions X; see also Davis [4], [5] and [7], [9], [14].

COROLLARY 6. *A real-valued continuous function f on $[0, \infty)$ is operator monotone if and only if f is operator concave, i.e., $-f$ is operator convex.*

Proof. It is known in [7, Theorem 2] that the operation $f \longrightarrow f^*$ preserves the operator concavity. By the theorem of Hansen-Pedersen stated above, the operator concavity of f is equivalent to f^* satisfying the Jensen inequality. Therefore Theorem 5 leads us to the conclusion. □

Finally we give an elementary proof of the characterization of operator monotone functions by the operator convexity, see [1, Theorem III.2].

THEOREM 7. *A real-valued continuous function f on $[0, \infty)$ is operator monotone if and only if $\tilde{f}(x) = xf(x)$ is operator convex.*

Proof. Suppose that f is operator monotone. Take $A \geq 0$ and a contraction X. Then we have

$$\begin{aligned} \tilde{f}(X^*AX) &= X^*A^{1/2}f(A^{1/2}XX^*A^{1/2})A^{1/2}X \\ &\leq X^*A^{1/2}f(A)A^{1/2}X \quad (\text{since} \quad XX^* \leq 1) \\ &= X^*\tilde{f}(A)X. \end{aligned}$$

Conversely suppose that $\tilde{f}$ is operator convex and $A \geq B \geq 0$. Since we may assume that they are invertible, we have $B^{1/2} = A^{1/2}X$ for some invertible contraction X. Hence it follows that

$$\begin{aligned} X^*A^{1/2}f(A)A^{1/2}X &= X^*\tilde{f}(A)X \\ &\geq \tilde{f}(X^*AX) \\ &= X^*A^{1/2}f(A^{1/2}XX^*A^{1/2})A^{1/2}X \\ &= X^*A^{1/2}f(B)A^{1/2}X, \end{aligned}$$

so that $f(A) \geq f(B)$. This completes the proof. □

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