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L_2 COHOMOLOGY OF THE BERGMAN METRIC FOR WEAKLY PSEUDOCONVEX DOMAINS

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1. Introduction

Suppose that Ω is a bounded pseudoconvex domain in C^n. The Bergman metric of Ω is a naturally defined Kaehler metric. We restrict our attention to those Ω whose Bergman metric is complete. By a theorem of Ohsawa [12], this includes all pseudoconvex domains with C^1 boundary. Every biholomorphic automorphism of Ω induces an isometry in the Bergman metric. The Hopf-Rinow theorem therefore implies that the Bergman metric of a homogeneous domain is complete.

Let $H_2^i(\Omega)$ denote the space of square integrable harmonic i-forms relative to the Bergman metric. The following result was proved in 1983 [6]:

THEOREM 1.1. *If Ω is strictly pseudoconvex, then $H_2^i(\Omega) = 0$, for $i \neq n$.*

Ohsawa and Takegoshi developed this work by giving both alternative proofs of Theorem 1.1 and applications to extension problems in several complex variables [13], [14]. Ideas of Gromov [8], were applied in [5] to give a conceptually clear proof of the results from [6].

In [5] the author developed a result of Gromov to give a criterion on the Bergman metric for the vanishing of $H_2^i(\Omega)$ for $i \neq n$ on domains in C^n for which the Bergman metric is complete. This criterion is the existence of a positive constant c_2 such that the estimate in Proposition 2.3 holds for all non-zero tangent vectors at all points. This approach led to a simpler proof of the earlier result in [6] for strictly pseudoconvex domains.

In the present paper we investigate this criterion more generally. We prove that it holds for pseudoconvex domains of finite type in C^2 and for locally convexifiable domains of finite type in C^n. We also verify it for homogeneous domains and for the domains with large automorphism groups given by $|z|^2 + |w|^{2p} < 1$, $p > 1$, in all dimensions. We provide an example of a bounded pseudoconvex Reinhardt domain in C^3 for which the criterion fails. The defining equation is $|w|^2 + |z_1 z_2|^2 + |z_1|^{10} + |z_2|^{10} < 1$. We do not determine however whether the cohomology vanishes for this domain.

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$|z|^2+|w|^{2p} < 1$, $p > 1$. Catlin suggested the domains $|w|^2+|z_1z_2|^2+|z_1|^{10}+|z_2|^{10} < 1$. The author also thanks the referee for improvements of the original manuscript.

2. Vanishing of L_2-cohomology for the Bergman metric

Our earlier paper [5] developed an idea by Gromov [8] for establishing vanishing theorems concerning the L_2-cohomology of complete Kaehler metrics. In the present section, we summarize the relevant results, of [5], pertaining to the Bergman metric of a pseudoconvex domain Ω in C^n. The Bergman metric is complete if either (i) Ω has C^1 boundary [12] or (ii) Ω admits a transitive group of biholomorphic automorphisms. In this paper, we always assume that the Bergman metric of Ω is complete.

Let M be a complete Kaehler manifold of complex dimension n. Suppose that $H_2^i(M)$ denotes the space of square integrable harmonic i-forms. If one takes the closure of the image of d, then $H_2^i(M)$ may be identified with the L_2 cohomology of the complex

$$\Lambda_2^0(M) \xrightarrow{d} \Lambda_2^1(M) \xrightarrow{d} \Lambda_2^2(M) \to \cdots \to \Lambda_2^{2n}(M).$$

That is, $H_2^i(M) = \ker \mathrm{d}/\overline{\mathrm{im}\, d}$. The following result is a brilliant observation by Gromov [8].

PROPOSITION 2.1. *Suppose that the Kaehler form ω of M can be written as $\omega = d\eta$, where η is bounded in supremum norm. Then $H_2^i M = 0$, for $i \neq n$.*

Suppose that Ω is a pseudoconvex domain in C^n. The Bergman kernel $K(z, w)$ is the integral kernel for projection from $L_2\Omega$ to the space of square integrable holomorphic functions. Here $L_2\Omega$ is the Hilbert space associated to the underlying Euclidean metric. The Kaehler form of the associated Bergman metric is $\omega = \sqrt{-1}\,\partial\bar\partial \log K(z, z) = d\eta$ with $\eta = -\sqrt{-1}\,\partial \log K = -\sqrt{-1}\,K^{-1}\partial K$. Suppose that g_ω is the Hermitian metric corresponding to the differential form ω. By applying Proposition 2.1 to the Bergman metric, with this natural choice of η, we deduce:

PROPOSITION 2.2. *Assume that for all non-zero $X \in T\Omega$, we have*

$$\frac{|\eta(X)|^2}{g_\omega(X, X)} < c_1$$

where c_1 is independent of both X and its basepoint $z \in \Omega$. Then $H_2^i\Omega = 0$, for $i \neq n$.

The criterion of Proposition 2.2 can be reformulated in terms of certain extremal problems involving holomorphic functions f defined on Ω. Let $\|f\|_2$ denote the L_2

norm of f with respect to the Lebesgue measure of Ω. If $z \in \Omega$, it follows from elliptic regularity that $\sup\{|f(z)|^2 \big| \|f\|_2 \leq 1\}$ is finite. Here the supremum is taken over all holomorphic functions defined on Ω. One has the following result [5]:

PROPOSITION 2.3. *Suppose that for all $X \in T\Omega$,*

$$\frac{\sup\{|f(z)|^2 \big| X_z f = 0 \text{ and } \|f\|_2 \leq 1\}}{\sup\{|f(z)|^2 \big| \|f\|_2 \leq 1\}} > c_2 > 0$$

with c_2 independent of both z and X. Then $H_2^i\Omega = 0$, for $i \neq n$.

For the unit ball in C^n, the Bergman kernel is given in closed form by $K(z, z) = c_n(1 - |z|^2)^{-n-1}$. The hypothesis of Proposition 2.2 is readily verified. Moreover, the ball is the model for all strictly pseudoconvex domains in C^n. Using the solution of $\overline{\partial} h = \beta$ in weighted L_2-spaces [9], we verified [5] that the criterion of Proposition 2.3 holds for all strictly pseudoconvex domains in C^n. The consequence that $H_2^i\Omega = 0$, for $i \neq n$, was proved much earlier in [6]. However, the main purpose of [5] was to give a conceptually simple proof that $H_2^i\Omega = 0$, when Ω is strictly pseudoconvex.

3. Homogeneous domains

Let Ω be a pseudoconvex domain in C^n. Assume that T: $\Omega \to \Omega$ is a holomorphic automorphism of Ω. If g: $\Omega \to C$ is a holomorphic function, and $J_T(w)$ is the Jacobian of T, then define f: $\Omega \to C$ by requiring that $g(w) = f(Tw)J_T(w)$. Clearly, f is holomorphic and $\|g\|_2 = \|f\|_2$, where $\|f\|_2$ is the L_2 norm with respect to Lebesgue measure. The Bergman kernel K has the characterization $K(z, z) = \sup\{|f(z)|^2 \big| \|f\|_2 \leq 1\}$, where the supremum is taken over holomorphic functions f: $\Omega \to C$. Setting $z = Tw$, one deduces that $K(z, z) = K(w, w)|J_T(w)|^{-2}$. If $\eta = -\partial \log K$, it follows that $\eta = T^*\eta - \partial \log J_T$. Moreover, the Bergman metric g_ω is invariant under T, $g_\omega = T^*g_\omega$. The crucial ratio of Proposition 2.2 thus satisfies the transformation rule

$$\frac{|T^*\eta(X)|^2}{T^*g_\omega(X, X)} = \frac{|\eta(X) + \partial \log J_T(X)|^2}{g_\omega(X, X)}. \tag{3.1}$$

One may employ (3.1) in the proof of:

PROPOSITION 3.2. *If Ω is a homogeneous pseudoconvex domain in C^n, then the L_2 cohomology of its Bergman metric satisfies $H_2^i\Omega = 0$, $i \neq n$.*

Proof. Let $p \in \Omega$ be any fixed basepoint. Since the group of biholomorphic automorphisms is transitive, we may pull back η and g_ω from the cotangent bundle at any $z \in \Omega$ to our chosen basepoint p. The key ratio of Proposition 2.2 is not invariant but its transformation property is characterized in formula (3.1). The criterion for vanishing of L_2-cohomology reduces to showing that $\sup |\partial \log J_T(p)| < \infty$, where the supremum is taken over all biholomorphic automorphisms $T \in \operatorname{Aut}\Omega$. Since p is fixed, the norm of $\partial \log J_T$ may be measured in either the Euclidean or the Bergman metric. By applying elliptic regularity theory to the holomorphic functions $J_T(w)$, one deduces that

$$|\partial \log J_T(p)| = \left|\frac{\partial J_T(p)}{J_T(p)}\right| \leq c_1 \sup_{w \in U} \left|\frac{J_T(w)}{J_T(p)}\right|$$

where U is any neighborhood of p and c_1 depends upon U. However, it follows from [7] that there exists a U so that the ratio $J_T(w)/J_T(p)$ is close to 1, uniformly in T. For the special case of bounded symmetric domains, one may cite [1] instead of [7]. □

The formula (3.1) is useful when Ω admits a large but not necessarily transitive group of automorphisms. To illustrate this point, we consider the domains Ω_p in $C^{\ell+m}$ given by $|z|^2 + |w|^{2p} < 1$, $p > 1$. The boundary of Ω_p is weakly pseudoconvex everywhere and strictly pseudoconvex except at points where $w = 0$. Here $z = (z_1, z_2, \ldots, z_\ell)$ and $w = (w_1, w_2, \ldots, w_m)$. By unitary rotation, which has constant Jacobian, one need only consider the ratio of Proposition 2.2 at points having the form $z = (|z|, 0, 0, \ldots, 0)$ and $w = (|w|, 0, 0, \ldots, 0)$. Also, Ω_p admits automorphisms S_a, $0 < a < 1$, given by

$$S_a(z, w) = \left(\frac{z_1 + a}{1 + az_1}, \frac{z_2\sqrt{1-a^2}}{1 + az_1}, \ldots, \frac{z_\ell\sqrt{1-a^2}}{1 + az_1}, \frac{w(1-a^2)^{1/2p}}{(1 + az_1)^{1/p}}\right).$$

The inverse of S_a is simply S_{-a}. Observe that $S_a(0, w) = (a, 0, \ldots, 0, w(1-a^2)^{1/2p})$. Since $z = 0$ only cuts Ω at strictly pseudoconvex boundary points, it suffices to show that $|\partial \log J_S(0, w)|$, with $S = S_a$, is uniformly bounded in a, because (3.1) holds and Proposition 2.2 has already been verified at strictly pseudoconvex boundary points. Recall that the Bergman metric is asymptotic to the model of the ball at strictly pseudoconvex boundary points. An elementary calculation gives $\partial \log J_S(0, w) = -a(1 + \ell + m/p)\partial z_1$. This last differential form is clearly bounded in both the Euclidean and Bergman metrics. Consequently, one has:

PROPOSITION 3.3. *If $p \geq 1$, the domain $|z|^2 + |w|^{2p} < 1$, in $C^{\ell+m}$, has $H_2^i\Omega = 0$ for $i \neq \ell + m$. Here the L_2 cohomology is taken with respect to the Bergman metric.*

An explicit formula for the Bergman kernel of the domains $|z|^2 + |w|^{2p} < 1$ was given by J. D'Angelo [4].

4. Domains of finite type in C^2

David Catlin [2] estimated the size of the Bergman kernel and the Bergman metric for domains Ω of finite type in C^2. The idea behind his method is to embed small polydiscs B near each point z_0 in the boundary of Ω. The dimensions of the polydiscs depend upon the type of $b\Omega$ at z_0. The Bergman kernels of the B are shown to be comparable at the center of the polydiscs. Rescaling the polydiscs gives the order of magnitude of the Bergman kernel. Catlin develops these ideas to estimate the Bergman, Caratheodory, and Kobayashi metrics.

This section will show that Catlin's approach can readily be adapted to establish the criterion of Proposition 2.3 for domains of finite type in C^2. We begin with the following extension of Theorem 6.1 of [2].

THEOREM 4.1. *Let Ω be a bounded pseudoconvex domain in C^n with smooth boundary. Assume that $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n)$ is a given point in D, $\beta_1, \beta_2, \dots, \beta_n$ are given positive numbers, and that there exists a function $\phi \in C^3(\overline{\Omega})$ such that:*

(*i*) $\phi(z) \le c_1$, $z \in \Omega$.
(*ii*) ϕ *is plurisubharmonic in* Ω.
(*iii*) Ω *contains the polydisc* $B = \{z \big| |z_i - \tilde{z}_i| < \beta_i, i = 1, 2, \dots, n\}$.
(*iv*) *In* Ω, ϕ *satisfies*

$$\sum_{i,j=1}^{n} \frac{\partial^2 \phi(z)}{\partial z_i \partial \overline{z}_j} t_i \overline{t}_j \ge c_2 \sum_{i=1}^{n} \beta_i^{-2} |t_i|^2, \quad z \in B.$$

(*v*) *If $D_i^{\alpha_i}$ denotes any mixed partial derivative in z_i and $\overline{z}_i$ of total order α_i, then $D^\alpha \phi = D_1^{\alpha_1} \dots D_n^{\alpha_n} \phi$ satisfies $|D^\alpha \phi(z)| \le c_\alpha \prod_{i=1}^{n} \beta_i^{-\alpha_i}$ for $z \in B$, $|\alpha| \le 3$.*

Then the hypothesis of Proposition 2.3 *holds. Specifically*

$$\frac{\sup\{|f(\tilde{z})|^2 \big| X_{\tilde{z}} f = 0 \text{ and } \|f\|_2 \le 1\}}{\sup\{|f(\tilde{z})|^2 \big| \|f\|_2 \le 1\}} > c_3 > 0$$

with c_3 independent of both $X \in T\Omega$ and its basepoint $\tilde{z} \in \Omega$.

Proof. Suppose we let $N(\Omega) = \sup\{|f(\tilde{z})|^2 \big| X_{\tilde{z}} f = 0 \text{ and } \|f\|_2 \le 1\}$ and $D(\Omega) = \sup\{|f(\tilde{z})|^2 \big| \|f\|_2 \le 1\}$ where the supremum is taken over holomorphic functions defined on Ω. Similarly, let $N(B)$ and $D(B)$ denote the analogous quantities where the supremum is taken over holomorphic functions defined on B. Since $B \subset \Omega$, it follows from the definitions that $N(\Omega) \le N(B)$ and $D(\Omega) \le D(B)$. It suffices to show that $N(\Omega) \ge c_4 N(B)$. If this inequality is established then

$$\frac{N(\Omega)}{D(\Omega)} \ge c_6 \frac{N(B)}{D(B)} > c_3 > 0.$$

The last inequality follows by rescaling the polydisc B to a product of unit discs. Both N and D scale in the same ratio, according to the L_2 norm.

The proof that $N(\Omega) \geq c_4 N(B)$ is a standard application of the solution of the inhomogeneous $\overline{\partial}$ equation in weighted L_2 spaces [10]. Suppose that f is a holomorphic function, defined on B, satisfying $X_{\tilde{z}} f = 0$, $\|f\|_2 = 1$, and $|f(\tilde{z})|^2 = N(B)$. The existence of such an optimizer f was shown in [5]. We follow the proof of Theorem 6.1 in [2], with $s = 0$, to extend f from B to Ω. One constructs a cut-off function $\psi \in C_0^\infty B$ with $\psi \equiv 1$ on a small neighborhood of the center $\tilde{z}$ of B. The theory of Hörmander [10] provides the solution of $\overline{\partial} u = \overline{\partial}(\psi f)$ in $L_2\Omega$ with weight $\exp(-\phi)\|(z - \tilde{z})/\beta\|^{-n-2}$. Here $(z - \tilde{z})/\beta = ((z_1 - \tilde{z}_1)/\beta_1, (z_2 - \tilde{z})/\beta_2, \ldots, (z_n - \tilde{z}_n/\beta_n)$. Because the weight function is bounded below on all of Ω and bounded above on the support of $\overline{\partial}(\psi f) = (\overline{\partial}\psi) f$, one has $\|u\|_2 \leq c_7$, in the standard L_2-norm of Ω. Let $v = \psi f - u$. Then v is holomorphic in Ω with $\|v\|_2 \leq c_8$. Because of the singularity of our weight function at $\tilde{z}$, $X_{\tilde{z}} v = X_{\tilde{z}} f = 0$ and $v(\tilde{z}) = f(\tilde{z})$. Since $c_8^{-1} v$ is an acceptable test function for the extremal problem determining $N(\Omega)$, one has $N(\Omega) \geq c_4 N(B)$. The proof that $D(\Omega) \geq c_5 D(B)$ is entirely similar, although this estimate is not needed here. □

In our next section we will give examples of domains of finite type in C^n, $n \geq 3$, where the criterion of Proposition 2.3 fails. In particular, the hypotheses of Theorem 4.1 cannot be satisfied in these examples. However, up to a locally defined biholomorphic equivalence with bounded Jacobian, Catlin [2] proved that the hypotheses of Theorem 4.1 hold for domains of finite type in C^2. Thus one has:

THEOREM 4.2. *Let Ω be a domain of finite type in C^2. Then $H_2^i \Omega = 0$, $i \neq 2$, where the symbol H_2^i denotes the L_2 cohomology group for the Bergman metric.*

McNeal [11] proved that the hypotheses of Theorem 4.1 hold for locally convexifiable domains of finite type in C^n. This gives:

THEOREM 4.3. *Let Ω be a locally convexifiable domain of finite type in C^n. Then $H_2^i \Omega = 0$, $i \neq n$, where the symbol H_2^i denotes the L_2 cohomology for the Bergman metric.*

5. Counterexamples in $C^n, n \geq 3$

We have proved that the criterion of Proposition 2.3 holds for domains of finite type in C^2. Our next result is that there exist bounded domains with real analytic boundary in C^n, for each $n \geq 3$, where the criterion of Proposition 2.3 fails. Recall that bounded domains, with real analytic boundary, necessarily are of finite type [3]. The existence of such counterexamples forecloses the natural strategy, developed in [5], to establish vanishing cohomology of the Bergman metric for pseudoconvex domains. However, we have not shown that $H_2^i\Omega \neq 0$, for some $i \neq n$, in these or any other examples of pseudoconvex domains.

Consider the bounded analytic domain Ω in C^3 given by $|w|^2 + |z_1z_2|^2 + |z_1|^{10} + |z_2|^{10} < 1$. Given $\epsilon > 0$, we want to establish the estimate

$$\frac{N}{D} = \frac{\sup\{|f(p)|^2 | X_p f = 0, \|f\|_2 \leq 1\}}{\sup\{|f(p)|^2 | \|f\|_2 \leq 1\}} < \epsilon \tag{5.1}$$

for some $p \in \Omega$ and $X \in T_p\Omega$. In fact $p = (w, z_1, z_2) = (1 - \delta, \gamma, 0)$, with small δ and γ, and $X = \frac{\partial}{\partial z_1}$. The quantity N/D is the crucial ratio of Proposition 2.3.

We first estimate the numerator N. Consider the product domain $D_\delta \times \Omega_\delta$. Here D_δ denotes a disc of radius $\frac{\delta}{2}$, centered at the origin in C, and Ω_δ denotes the domain $|z_1z_2|^2 + |z_1|^{10} + |z_2|^{10} < \frac{\delta}{2}$ in C^2. We translate $D_\delta \times \Omega_\delta$ so that its center moves from the origin to $(1 - \delta, 0, 0)$ in C^3. This embeds $D_\delta \times \Omega_\delta \subset \Omega$. The extremal property characterizing N immediately yields domain monotonicity $N = N(\Omega) \leq N(D_\delta \times \Omega_\delta)$. Since the Bergman kernel of D_δ is of order δ^{-2} at the origin, $N(D_\delta \times \Omega_\delta) \leq c_1\delta^{-2}N(\Omega_\delta)$. To establish this last upper bound, we use the characterization [5] of N in terms of an orthonormal basis for square integrable holomorphic functions. Since X will be tangent to the second factor Ω_δ, the orthonormal basis may be chosen to consist of products of functions defined on each factor.

Let $N_\delta = N(\Omega_\delta)$. To bound N_δ from above, at the point $(\gamma, 0)$, we normalize the standard basis, $z_1^i z_2^j$, for square integrable holomorphic functions on Ω_δ, and expand the extremal for N_δ in this orthonormal basis. Since $z_2 = 0$, at $(\gamma, 0)$, the extremal must depend only upon the z_1^i. It is elementary to estimate, for $i \geq 1$,

$$\int_{\Omega_\delta} |z_1|^{2i} \approx \left[\int_0^{\delta^{\frac{2}{5}}} \delta^{\frac{1}{5}} r^{2i+1}\, dr + \int_{\delta^{\frac{2}{5}}}^{\delta^{\frac{1}{10}}} r^{2i}\left(\frac{\delta}{r^2}\right) r\, dr\right] \approx \delta^{1+\frac{i}{5}} i^{-1}$$

and

$$\int_{\Omega_\delta} 1 \approx \int_0^{\delta^{\frac{2}{5}}} \delta^{\frac{1}{5}} r\, dr + \int_{\delta^{\frac{2}{5}}}^{\delta^{\frac{1}{10}}} \left(\frac{\delta}{r^2}\right) r\, dr \approx \delta|\log\delta|.$$

Here $\approx$ indicates commensurability; the ratio of the two sides is bounded above and below by absolute constants, independent of δ and i. To determine these elementary estimates, we use the observation that Ω_δ is commensurable to the domains $\{|z_1z_2|^2 < c_2\delta\} \cap \{|z_1|^{10} < c_2\delta\} \cap \{|z_2|^{10} < c_2\delta\}$. It follows that there are L_2-normalized orthogonal holomorphic functions commensurable to $\phi_0 = (\delta|\log\delta|)^{-\frac{1}{2}}$ and $\phi_i = \delta^{-\frac{1}{2}} i^{\frac{1}{2}} (z_1/\delta^{\frac{1}{10}})^i$. We expand the extremal for N_δ as $f = \sum_{i=0}^{\infty} a_i\phi_i$ with $\sum_{i=0}^{\infty} |a_i|^2 < c_3$.

We now take $X = \frac{\partial}{\partial z_1}$ and $\gamma = \delta^{\frac{1}{10}}|\log\delta|^{-\frac{1}{4}}$. One computes

$$\begin{aligned} Xf &= a_1 X\phi_1 + \sum_{i=2}^{\infty} a_i X\phi_i \\ &= a_1\delta^{-\frac{1}{2}}\delta^{-\frac{1}{10}} + \sum_{i=2}^{\infty} a_i\delta^{-\frac{1}{2}} i^{\frac{1}{2}} i\, \delta^{-\frac{1}{10}} \left(\frac{z_1}{\delta^{\frac{1}{10}}}\right)^{i-1}. \end{aligned}$$

So $Xf(\gamma, 0) = 0$ gives

$$\begin{aligned} a_1 &= -\sum_{i=2}^{\infty} a_i i^{\frac{3}{2}} |\log\delta|^{-\frac{(i-1)}{4}} \\ &= -|\log\delta|^{-\frac{1}{4}} \sum_{i=2}^{\infty} a_i i^{\frac{3}{2}} |\log\delta|^{-\frac{(i-2)}{4}}. \end{aligned}$$

Thus $|a_1| \le c_4 |\log\delta|^{-\frac{1}{4}}$. Consequently,

$$\begin{aligned} |f(\gamma, 0)| &\le |a_0||\phi_0| + |a_1||\phi_1| + \sum_{i=2}^{\infty} |a_i||\phi_i| \\ &\le c_5(\delta^{-\frac{1}{2}}|\log\delta|^{-\frac{1}{2}} + |\log\delta|^{-\frac{1}{4}}\delta^{-\frac{1}{2}}|\log\delta|^{-\frac{1}{4}} \\ &\quad + \sum_{i=2}^{\infty} \delta^{-\frac{1}{2}}|a_i| i^{\frac{1}{2}} |\log\delta|^{-\frac{i}{4}}) \\ &\le c_6(\delta^{-\frac{1}{2}}|\log\delta|^{-\frac{1}{2}} + \delta^{-\frac{1}{2}}|\log\delta|^{-\frac{1}{2}} \sum_{i=2}^{\infty} |a_i| i^{\frac{1}{2}} |\log\delta|^{-\frac{(i-2)}{4}}) \\ &\le c_7\delta^{-\frac{1}{2}}|\log\delta|^{-\frac{1}{2}}. \end{aligned}$$

Since f is the extremal for N_δ, we have $N_\delta \le c_8\delta^{-1}|\log\delta|^{-1}$ at $(\gamma, 0)$. Thus $N \le c_9\delta^{-3}|\log\delta|^{-1}$ at $(1-\delta, \gamma, 0)$, with $X = \frac{\partial}{\partial z_1}$.

We now turn to the denominator D of (5.1). A lower bound for D will be given by providing a specific test function. The fractional linear transformation $T_a w = \frac{w-a}{1-aw}$, $0 < a < 1$, is a biholomorphic self map of the unit disc in C. One calculates $\frac{\partial}{\partial w}(T_a w) = (1-a^2)(1-aw)^{-2}$. The change of variable formula for double integrals gives

$$\int_{|w|\le 1} \left|\frac{1-a^2}{(1-aw)^2}\right|^2 = \pi.$$

Moreover, for $0 < s < 1$,

$$(1-s)^{-2}\int_{|w|\le 1-s} \left|\frac{1-a^2}{(1-aw)^2}\right|^2 = \int_{|z|\le 1} \left|\frac{1-a^2}{(1-a(1-s)z)^2}\right|^2 = \frac{|1-a^2|^2\pi}{|1-a^2(1-s)^2|^2}.$$

Differentiation yields, for small s and $a = 1 - \delta$,

$$\int_{|w|=1-s} \left| \frac{1-a^2}{(1-aw)^2} \right|^2 \approx \frac{(1-a)^2}{(1-a^2(1-s)^2)^3}.$$

Our test function will be the L_2-normalization of $z_1(1-a^2)(1-aw)^{-2}$.

Using Fubini's theorem,

$$I = \int_{\Omega} \left| \frac{1-a^2}{(1-aw)^2} \right|^2 |z_1|^2 \approx \int_0^1 \int_{|w|=1-s} \left| \frac{1-a^2}{(1-aw)^2} \right|^2 \int_{\Omega_s} |z_1|^2 \, ds.$$

Earlier in this section we found that $\int_{\Omega_s} |z_1|^2 \approx s^{1+\frac{1}{5}}$. Therefore

$$I \approx \int_0^1 \frac{(1-a)^2}{(1-a^2(1-s)^2)^3} \, s^{1+\frac{1}{5}} \, ds.$$

Note that $a = 1 - \delta$. Thus $a(1-s) = (1-\delta)(1-s) = 1 - \delta - s + \delta s$. So

$$I \approx \int_0^{\delta} \frac{\delta^2}{\delta^3} \, s^{1+\frac{1}{5}} \, ds + \int_{\delta}^1 \frac{\delta^2}{s^3} \, s^{1+\frac{1}{5}} \, ds \approx \delta^{1+\frac{1}{5}}.$$

Our L_2-normalized test function for D is commensurable to $f = (1-a^2)z_1(1-aw)^{-2}\delta^{-\frac{1}{2}}\delta^{-\frac{1}{10}}$. Evaluating at $w = 1-\delta$, $z_1 = \gamma = \delta^{\frac{1}{10}}|\log\delta|^{-\frac{1}{4}}$, gives $f \approx \delta^{-1}\delta^{-\frac{1}{2}}|\log\delta|^{-\frac{1}{4}}$. So $D \geq c_9\delta^{-3}|\log\delta|^{-\frac{1}{2}}$.

Finally, the ratio in (5.1), at $(1-\delta, \gamma, 0)$ satisfies

$$\frac{N}{D} \leq c_{10} \frac{\delta^{-3}|\log\delta|^{-1}}{\delta^{-3}|\log\delta|^{-\frac{1}{2}}} = c_{10}|\log\delta|^{-\frac{1}{2}} < \epsilon.$$

for δ sufficiently small. Therefore, the criterion of Proposition 2.3 fails for the domain in C^3 given by

$$|w|^2 + |z_1 z_2|^2 + |z_1|^{10} + |z_2|^{10} < 1. \tag{5.2}$$

Similar counterexamples exist in C^n, $n \geq 4$. One uses the defining equation (5.2) where $w = (1-\delta, 0)$, with 0 the origin in C^{n-3}. Small modifications of the above argument yield (5.1) in this more general situation.

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