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UMBILIC FOLIATIONS AND CURVATURE

GERARD WALSCHAP

Recall that a submanifold L of a Riemannian manifold is totally umbilic if it curves equally in all directions, i.e., if there is some vector $N \perp L$ such that the second fundamental tensor S_X of L in any direction $X \perp L$ is given by $S_X = \langle X, N \rangle \,\mathrm{Id}$. In this note, we investigate some properties of k-dimensional Riemannian foliations $\mathcal{F}^k$ with totally umbilic leaves, which we call umbilic foliations for short. Notice that a Riemannian flow ($k = 1$) is always umbilic. It is to be expected that restrictions increase with k. In fact, we show that on manifolds M^n of positive sectional curvature, there are no umbilic foliations if $k > (n - 1)/2$. This is a best possible estimate, since for example a Euclidean 3-sphere admits an abundance of Riemannian flows. Similarly, it turns out that on spaces of nonpositive curvature, an umbilic foliation of dimension $n - 1$ is, when lifted to the universal cover, 'almost always' a foliation by horospheres or by hypersurfaces equidistant from a totally geodesic one.

Although Riemannian foliations $\mathcal{F}^k$ seem to arise naturally in geometry (see for instance [12] for a recent spectacular example), our understanding of them is still quite limited, even in the simplest case, that of constant curvature: Gromoll and Grove have shown that on Euclidean spheres they are homogeneous—i.e., orbit foliations of groups of isometries—if $k \leq 3$ [8], and that Riemannian flows are always flat (the orthogonal complement is an integrable totally geodesic distribution) or homogeneous in any space of constant curvature [7]. We show that the latter result extends to all umbilic foliations on space forms. The starting point is the observation that on manifolds with curvature bounded from below, the mean curvature form of an umbilic foliation is closed as soon as it is basic (this also generalizes some results in [5], [6], [7]). As a consequence, on such manifolds, Riemannian flows with basic mean curvature are necessarily homogeneous; i.e., they are locally spanned by a Killing field.

1. The local geometry

For notational purposes, we begin by recalling some elementary facts about Riemannian foliations. The reader is referred to [2], [8], [11] for further details and other

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results that will be used here. The leaves of the foliation are always assumed to be complete. $\mathcal{F}$ induces an orthogonal splitting $TM = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of M into horizontal and vertical subbundles, with $\mathcal{V}$ tangent to the leaves. We write $e = e^h + e^v$ for the corresponding decomposition of $e \in TM$. The local geometry of $\mathcal{F}$ is determined by two tensor fields: the O'Neill tensor A is the (2,1) tensor field on $\mathcal{H}$ with values in $\mathcal{V}$ given by

$$A_X Y = \overset{v}{\nabla}_X Y = \tfrac{1}{2}[X, Y]^v. \tag{1-1}$$

$\mathcal{F}$ is said to be *flat* if A identically vanishes, or equivalently, if the distribution $\mathcal{H}$ is integrable. The second fundamental tensor of $\mathcal{F}$ is the horizontal 1-form S with values in self-adjoint transformations of $\mathcal{V}$ given by

$$S_X V = -\overset{v}{\nabla}_V X. \tag{1-2}$$

A vector field X is said to be *basic* if it is the horizontal lift of some vector field on the local quotient manifold determined by $\mathcal{F}$. $[X, V]$ is always vertical for basic X and vertical V, and if $A_X^* : \mathcal{V} \to \mathcal{H}$ denotes the adjoint of A_X, then

$$\overset{h}{\nabla}_V X = \overset{h}{\nabla}_X V = -A_X^* V. \tag{1-3}$$

A geodesic γ that is horizontal at one point is everywhere horizontal, and induces local diffeomorphisms f^t between neighborhoods of $\gamma(0)$ in the leaf and corresponding neighborhoods of $\gamma(t)$ by horizontally lifting local projections of γ. The derivative of these "holonomy displacements" is given by $f_*^t u = J(t)$, where J is a nowhere zero Jacobi field along γ with $J(0) = u$, [8]. One can now give a short proof of [10, Proposition 2.2], which is stated here in a slightly different way:

LEMMA 1.4. *Along a horizontal geodesic γ, one has the Riccati-type equation:*

$$S_{\dot\gamma}'^{\,v} = S_{\dot\gamma}^2 - A_{\dot\gamma} A_{\dot\gamma}^* + R_{\dot\gamma}^v, \quad \textit{where } R_{\dot\gamma}^v := R^v(\cdot, \dot\gamma)\dot\gamma.$$

Proof. Consider a holonomy Jacobi field J as above. By (1-2) and (1-3),

$$J' = -A_{\dot\gamma}^* J - S_{\dot\gamma} J.$$

Thus, if T is a vertical vector field along γ, then

$$\begin{aligned}
\langle R(T, \dot\gamma)\dot\gamma, J\rangle &= -\langle T, J''\rangle = \langle T, (A_{\dot\gamma}^* J)'^v\rangle + \langle T, (S_{\dot\gamma} J)'^v\rangle \\
&= \langle T, A_{\dot\gamma} A_{\dot\gamma}^* J\rangle + \langle T, S_{\dot\gamma} J\rangle' - \langle T'^v, S_{\dot\gamma} J\rangle \\
&= \langle A_{\dot\gamma} A_{\dot\gamma}^* T, J\rangle + \langle S_{\dot\gamma} T, J\rangle' - \langle S_{\dot\gamma}(T'^v), J\rangle \\
&= \langle A_{\dot\gamma} A_{\dot\gamma}^* T, J\rangle + \langle (S_{\dot\gamma} T)'^v, J\rangle - \langle S_{\dot\gamma}^2 T, J\rangle - \langle S_{\dot\gamma}(T'^v), J\rangle.
\end{aligned}$$

Rearranging terms,

$$\langle (S_{\dot\gamma}T)'^{v} - S_{\dot\gamma}(T'^{v}), J\rangle = \langle (S_{\dot\gamma}^2 - A_{\dot\gamma}A_{\dot\gamma}^* + R_{\dot\gamma}^v)T, J\rangle.$$

Since for any t_0, there exist holonomy fields J_i that form an orthonormal basis of the vertical space at $\gamma(t_0)$, the lemma follows. □

1.4 is only one of several identities relating the A and S tensors. We mention one other that will be used in the next section: For horizontal X, Y, and vertical U, V, one always has

$$\text{(1-5)} \qquad \langle d^\nabla S_{X,Y}U, V\rangle = -(\langle \nabla_U(A_XY), V\rangle + \langle \nabla_V(A_XY), U\rangle),$$

where d^∇ is the exterior covariant derivative operator on the End TM bundle-valued forms on M, [13]. (1-5) can be found in a different but equivalent formulation in [2].

2. Umbilic foliations

The *mean curvature form* of $\mathcal{F}$ is the horizontal 1-form κ on M given by $\kappa(X) = \operatorname{tr} S_X$. Following [8], we say $\mathcal{F}$ is *isoparametric* if κ is basic, that is, if $\kappa(X)$ is constant along a leaf for basic X. When $\mathcal{F}$ is umbilic, there is a horizontal vector field N such that $S_XV = \langle X, N\rangle V$, and κ is, up to a constant, just the metric dual of N. In this case, as observed in [6], (1-5) immediately yields

$$\text{(2-1)} \qquad -\tfrac{1}{k}\langle U, V\rangle\, d\kappa\,(X, Y) = \langle \nabla_U(A_XY), V\rangle + \langle \nabla_V(A_XY), U\rangle.$$

PROPOSITION 2.2. *Let $\mathcal{F}$ be an umbilic foliation on a manifold with sectional curvature bounded from below. If $\mathcal{F}$ is isoparametric, then the mean curvature form is closed, and for basic X, Y, A_XY is a Killing field when restricted to any leaf.*

Proof. It should be noted that when M is compact, the assertion is true even if the foliation is not umbilic; see [15]. In the present context, since $\mathcal{F}$ is isoparametric, $d\kappa(X, Y)$ is constant along leaves for basic X, Y, and by (2-1), so is $\langle \nabla_U(A_XY), U\rangle$ for unit U. Thus, if c is a geodesic in the induced metric on the leaf, then $\langle A_XY \circ c, \dot c\rangle'$ is constant, so that $\langle A_XY \circ c, \dot c\rangle$ is a linear function. This function must then be constant by the curvature assumption and O'Neill's formula [11, Corollary 1, #3]. Consequently, $d\kappa(X, Y) = 0$ and A_XY is Killing. But $d\kappa(X, U)$ is always zero for vertical U, so that κ is closed. □

As an immediate consequence, we have the following generalization of some results in [5] and [7]:

COROLLARY 2.3. *Let M be a manifold whose curvature is bounded from below. A Riemannian flow $\mathcal{F}$ on M is isoparametric iff it is homogeneous, i.e., iff $\mathcal{F}$ is locally generated by a Killing field of M.*

Proof. If $\mathcal{F}$ is isoparametric, then by 2.2, the mean curvature form is closed, and locally equals $d\phi$ for some function ϕ which is constant along leaves. The argument in [7] for space forms actually goes through without changes in the more general setting of nonconstant curvature: If T is a (local) unit vector field spanning $\mathcal{F}$, it is straightforward to check that $e^{-\phi}T$ is Killing. The converse is clear. □

Remark. Let $\mathcal{F}^k$ be as in Proposition 2.2. If $k \leq n-2$, then for each $p \in M$, either the leaf through p is totally geodesic, or else there exists a hypersurface $\mathcal{N}$ through p for which the restriction of $\mathcal{F}$ to $\mathcal{N}$ is totally geodesic (and Riemannian). In particular, if $k = n-2$ then $\mathcal{N}$, with the induced metric, splits locally isometrically as $L^k \times I$, where I is an interval. To see this, let $\kappa = d\phi$ in some neighborhood of p. If $\kappa(p) \neq 0$ (that is, if the leaf through p is not totally geodesic), let $\mathcal{N}$ denote the hypersurface $\{q \mid \phi(q) = \phi(p)\}$. Since ϕ is constant along leaves, $\mathcal{F}$ restricts to $\mathcal{N}$, and this restriction is totally geodesic because N is orthogonal to $\mathcal{N}$.

THEOREM 2.4. *Let $\mathcal{F}^k$ be an umbilic foliation on a manifold M^n of nonnegative sectional curvature. Then $\mathcal{F}$ is totally geodesic if $k > \frac{n-1}{2}$.*

Proof. Let γ denote a horizontal geodesic in M, V a vertical vector field along γ. Then $S_{\dot\gamma}'^{\,v}V = (S_{\dot\gamma}V)'^{v} - S_{\dot\gamma}(V'^{v}) = \langle N\circ\gamma, \dot\gamma\rangle' V$, and the Riccatti equation 1.4 becomes

$$\langle N\circ\gamma, \dot\gamma\rangle' V = \langle N\circ\gamma, \dot\gamma\rangle^2 V - A_{\dot\gamma}A_{\dot\gamma}^* V + R_{\dot\gamma}^v V.$$

In particular, $\langle N\circ\gamma, \dot\gamma\rangle' = \langle N\circ\gamma, \dot\gamma\rangle^2 - |A_{\dot\gamma}^* V|^2 + K_{V,\dot\gamma}$ for any unit vertical V, with $K_{V,\dot\gamma}$ denoting the sectional curvature of the plane spanned by V and $\dot\gamma$. Since $k > (n-1)/2$ however, $A_{\dot\gamma}^* : \mathcal{V}^k \to \mathcal{H}^{n-k} \bigcap \dot\gamma^{\perp}$ has nontrivial kernel, and by choosing V above in that kernel, we have $\langle N\circ\gamma, \dot\gamma\rangle' \geq \langle N\circ\gamma, \dot\gamma\rangle^2$. This forces $\langle N\circ\gamma, \dot\gamma\rangle$ to be identically zero. γ being arbitrary, $\mathcal{F}$ is totally geodesic. □

Notice that the estimate for k in 2.4 is optimal, since on a Euclidean 3-sphere, there exist (necessarily umbilic) Riemannian flows that are not totally geodesic. On the other hand, when the curvature of M is strictly positive, M cannot admit equidistant totally geodesic submanifolds of dimension greater than $(n-1)/2$ by the second Rauch comparison theorem [9]. 2.4 then implies

COROLLARY 2.5. *An umbilic foliation $\mathcal{F}^k$ on a manifold M^n of positive curvature must have dimension $k \leq \frac{n-1}{2}$.*

For positive curvature, the argument in 2.4 actually shows that $\mathcal{F}$ must be substantial along all leaves in the terminology of [8], and thus highly nonflat. This contrasts sharply with the nonpositive curvature case, where $k = n-1$ is possible (and $\mathcal{F}$ is then necessarily flat). For instance, in hyperbolic space, families of concentric horospheres, or families of hypersurfaces equidistant from a totally geodesic one are all examples of codimension one umbilic foliations. But these are essentially the only 'nice' ones:

THEOREM 2.6. *Let $\mathcal{F}$ be an umbilic foliation of codimension one on a manifold M of nonpositive curvature.*

(1) *If the mean curvature of $\mathcal{F}$ is never zero, then the induced foliation in the universal cover $\tilde{M}$ of M is by horospheres.*
(2) *If $\mathcal{F}$ is isoparametric, then the induced foliation $\tilde{\mathcal{F}}$ on $\tilde{M}$ is either by horospheres or by hypersurfaces equidistant from a totally geodesic one. Moreover, $\tilde{M}$ is then a warped product $\tilde{M} = \mathbb{R} \times_h F^{n-1}$, and $\tilde{\mathcal{F}}$ is the collection of all $\{t\} \times F^{n-1}$, $t \in \mathbb{R}$.*

Proof. Denote by X one of the two unit vector fields orthogonal to the lifted foliation in the universal cover. Because X is both totally geodesic and irrotational (the latter meaning that ∇X is symmetric when restricted to the orthogonal complement $X^\perp$ of X), the one-form metrically dual to X is closed, and thus equals df for some $f : \tilde{M} \to \mathbb{R}$. Equivalently, $X = \nabla f$, $|\nabla f| = 1$. If c is any—not necessarily horizontal—geodesic of $\tilde{M}$, one has $(f \circ c)' = \langle \nabla f \circ c, \dot{c} \rangle$, and

$$(f \circ c)'' = \langle \nabla_D(\nabla f \circ c), \dot{c} \rangle = \langle \nabla_{\dot{c}} \nabla f, \dot{c} \rangle = \langle \nabla_{\dot{c}^v} \nabla f, \dot{c} \rangle,$$

since ∇f is totally geodesic. Moreover $\nabla_{\dot{c}^v} \nabla f$ is vertical because ∇f has unit length, so that

$$(f \circ c)'' = \langle \nabla_{\dot{c}^v} \nabla f, \dot{c}^v \rangle = -|\dot{c}^v|^2 \langle \nabla f, N \rangle \circ c = \pm |\dot{c}^v|^2 \cdot |N| \circ c.$$

Thus, $\pm f$ is a convex function in (1). By [1], $\pm f$ is then a Busemann function and the foliation is by concentric horospheres.

In (2), if N is zero at some point, then it vanishes along the leaf L through that point, and L is totally geodesic. The other leaves are then obtained by exponentionating the flat normal bundle of L at constant distances. If N is never zero, the argument for (1) above shows that the foliation is by horospheres. In any case, since $A \equiv 0$ and κ is basic, κ is closed by 2.2, and equals dh for some function h. h is actually constant along leaves, and may thus be viewed as a function $\mathbb{R} \to \mathbb{R}$ (that is, we identify h with the function $\tilde{h}$ defined by $\tilde{h} \circ f = h$). Holonomy displacements are then homotheties, and it is straightforward to check that $\tilde{M}$ is isometric to $\mathbb{R} \times_\phi F$, where F is the fiber $f^{-1}(0)$, and $\phi(t) := e^{h(0)} e^{-h(t)}$, see also [3]. □

3. Umbilic foliations in constant curvature

In [7], Gromoll and Grove found that any Riemannian flow on a space of constant curvature is flat or isoparametric, and deduced that such a flow is necessarily flat or homogeneous, the latter meaning that it is locally generated by a subgroup of isometries. We generalize this to all umbilic foliations on spaceforms. It should be noted, though, that Riemannian foliations on space forms are in general isoparametric only when the curvature is positive or zero. Since we are primarily interested in the local geometry, we may assume, without loss of generality, that the ambient space is simply connected.

THEOREM 3.1. *An umbilic foliation $\mathcal{F}^k$, $k > 1$, on the complete simply connected space M^n_c of constant curvature c is flat if $c \leq 0$, and homogeneous—in fact totally geodesic—if $c \geq 0$.*

Proof. A flat Riemannian foliation is always locally congruent to a metric product foliation when the ambient space has nonnegative sectional curvature [10], [16]. Thus, for $c \geq 0$, we need only consider the case when $\mathcal{F}$ is not flat. It is straightforward to check that if the A-tensor is zero at some point p, then it vanishes everywhere: In fact, $A \equiv 0$ on the leaf through p, since the ambient space has constant curvature. Moreover, A satisfies a linear differential equation [11, {3}] along geodesics orthogonal to the leaf, which becomes homogeneous in the constant curvature case. Thus, A vanishes everywhere. With no assumption as yet on c, suppose that $\mathcal{F}$ is not flat, so that at each $p \in M$, there is some horizontal $x \in M_p$ with $A_x \neq 0$. If $\gamma(t) = \exp(tx)$, then

$$A_{\dot\gamma} A^*_{\dot\gamma} V = -(\langle N \circ \gamma, \dot\gamma \rangle' - \langle N \circ \gamma, \dot\gamma \rangle^2 - c)V, \tag{3-2}$$

for vertical V along γ by 1.4. It follows that A_x is onto $\mathcal{V}$, and $\mathcal{F}$ is substantial along all leaves in the terminology of [8]. By [8, Remark 2.11], $\mathcal{F}$ is then isoparametric, and N is basic.

First, consider the case $c \geq 0$. We claim that $N \equiv 0$; i.e., $\mathcal{F}$ is totally geodesic. Indeed, by a classical result [14, Chapter 7, Theorem 29], if some leaf is not totally geodesic, then it lies inside a totally geodesic M^{k+1}_c. N must then be tangent to this submanifold, so that $N \restriction_L$ spans the normal bundle of L in M^{k+1}_c, and A_N is zero along L. With γ now denoting a geodesic integral curve of $N/|N|$ starting at some point of L, (3-2) becomes $\langle N \circ \gamma, \dot\gamma \rangle' = \langle N \circ \gamma, \dot\gamma \rangle^2 + c$. Since $c \geq 0$, $\langle N \circ \gamma, \dot\gamma \rangle \equiv 0$, a contradiction. Thus $\mathcal{F}$ is totally geodesic, and in particular, every $A_X Y$ is Killing on M for basic X, Y, so that $\mathcal{F}$ is homogeneous.

Next, consider the case $c < 0$, or $c = -1$ after normalization. Now, for any Riemannian submersion $\pi : M \to B$, one always has, for horizontal $x \in TM$,

$$A^*_x A_x = \tfrac{1}{3}(R^M(\cdot, x)x - \pi^* R^B(\pi_* \cdot, \pi_* x)\pi_* x), \tag{3-3}$$

where $\pi^* e$ denotes the horizontal lift to TM of $e \in TB$. Thus, in constant curvature, $A^*_X A_X$ preserves basic fields whenever X itself is basic; cf. also [8]. Recall that along any leaf L, there is a basic X with A_X onto $\mathcal{V}$. It follows from 2.2 and (3-3) that there exist basic X_i such that $A_X X_i$ form an orthonormal basis of Killing fields on

L. L is then necessarily flat in the induced metric, with $A_X X_i$ parallel on L, and being umbilic, must be a horosphere inside some totally geodesic M_c^{k+1}. Viewing $M_c^n = \mathbb{H}^n$ as the upper half-space $\{p \in \mathbb{R}^n \mid p^n > 0\}$, the totally geodesic M_c^{k+1} containing L is, up to congruence, $0 \times \mathbb{H}^{k+1} \subset \mathbb{H}^n$, and $L = \{(0, p) \mid p^{k+1} = 1\}$. The remainder of the argument now follows from elementary hyperbolic geometry: Identifying the underlying space of $\mathbb{H}^n$ metrically with the subset $\{p \in \mathbb{R}^n \mid p^n > 0\}$ of flat Euclidean space, we first observe that any other leaf L', being equidistant from L in the hyperbolic metric, must remain within bounded Euclidean distance from L because L is at constant height $x^n = 1$. But L' is a horosphere in some totally geodesic $(k+1)$-submanifold, and the description of these in the upper half-space model then implies that L' is a k-plane parallel to L in the Euclidean sense. It is straightforward to check that sections in the normal bundle $\nu(L)$ of L that are parallel in the hyperbolic metric are also parallel in the Euclidean metric (and vice versa). It follows that the leaves of $\mathcal{F}$ are obtained by exponentiating the parallel sections of $\nu(L)$ at constant distance. But then $\mathcal{F}$ is flat, contrary to assumption. $\square$

References

1. W. Ballmann, M. Gromov, and V. Schroeder, *Manifolds of nonpositive curvature*, Birkhäuser, Boston, Mass., 1985.
2. A. Besse, *Einstein manifolds*, Springer-Verlag, New York, 1987.
3. R. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1–49.
4. G. Cairns, *Totally umbilic Riemannian foliations*, Michigan Math. J. **37** (1990), 145–159.
5. G. Cairns and R. Escobales, *Note on a theorem of Gromoll-Grove*, Bull. Australian Math. Soc. (to appear).
6. R. Escobales and P. Parker, *Geometric consequences of the normal curvature cohomology class in umbilic foliations*, Indiana J. Math. **37** (1988), 389–408.
7. D. Gromoll and K. Grove, *One-dimensional metric foliations in constant curvature spaces*, Differential geometry and complex analysis, Springer-Verlag, New York, 1985, pp. 165–168.
8. ———, *The low-dimensional metric foliations of Euclidean spheres*, J. Differential Geometry **28** (1988), 143–156.
9. D. Gromoll, W. Klingenberg and W. Meyer, *Riemannsche geometrie im grossen*, Springer-Verlag, New York, 1987.
10. H. Kim and Ph. Tondeur, *Riemannian foliations on manifolds with nonnegative curvature*, Manuscripta Math. **74** (1992), 39–45.
11. B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.
12. G. Perelman, *Proof of the soul conjecture of Cheeger and Gromoll G. Perelman*, J. Differential Geometry **40** (1994), 209–212.
13. W. Poor, *Differential geometric structures*, McGraw-Hill, New York, 1981.
14. M. Spivak, *A comprehensive introduction to differential geometry*, vol. IV, Publish or Perish, Houston, Texas, 1979.
15. P. Tondeur, *Foliations on Riemannian manifolds*, Springer-Verlag, New York, 1988.
16. G. Walschap, *Metric foliations and curvature*, J. Geom. Anal. **2** (1992), 373–381.

Department of Mathematics, University of Oklahoma, Norman, OK 73019
gwalschap@uoknor.edu