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TWISTOR SPINORS ON CONFORMALLY FLAT MANIFOLDS

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1. Introduction

Twistor spinors were introduced by R. Penrose as solutions of a conformally invariant field equations in general relativity. In this paper we consider Riemannian spin manifolds carrying twistor spinors. Outside their zero set they can be seen as conformal analogues of *parallel spinors*. As an example, a twistor spinor with a zero exists on the standard sphere. Moreover, A. Lichnerowicz proved in [Li, Thm. 7] that the sphere with its standard conformal structure is the only *compact* Riemannian spin manifold carrying twistor spinors with zeros.

To a spinor field ϕ one can canonically associate a vector field V_ϕ as the dual of the 1-form $X \mapsto \sqrt{-1}\langle \phi, X \cdot \phi\rangle$, where the dot refers to the Clifford multiplication and the bracket is the canonical hermitian inner product on the space of spinors. The associated vector field of a twistor spinor is *conformal*; i.e., its local flow consists of conformal transformations. There are twistor spinors for which the associated conformal field is trivial as well as twistor spinors with non-trivial conformal field, for example on the standard sphere.

In a previous paper the authors showed the following:

THEOREM 1.1 [KR1, THM. A]. *If the Riemannian spin manifold (M, g) carries a twistor spinor with zero and with non-trivial conformal field then the manifold is conformally flat.*

In this paper we obtain a converse statement, more precisely we describe the global types of conformally flat manifolds, which carry twistor spinors with zero and with non-trivial conformal field.

Let (M, g) be a *conformally flat* Riemannian manifold; i.e., every point p has an open neighbourhood U, such that (U, g) is conformally equivalent to an open subset of Euclidean space. A conformally flat manifold is called *developable* if there is a conformal map $\delta\colon M^n \longrightarrow S^n$ into the standard sphere; δ is uniquely determined up to a conformal diffeomorphism of the sphere; i.e., up to a *Möbius transformation*. It follows that a developable conformally flat manifold carries twistor spinors with zero and with non-trivial assciated conformal fields. The universal covering $(\widetilde{M}, g)$ of a

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conformally flat manifold (M, g) is developable. Now we assume in addition that M is *orientable*. Since the fundamental group $\pi_1(M) = \pi_1$ acts by orientable deck transformations on $\widetilde{M}$, there is an induced homomorphism $\rho\colon \pi_1 \longrightarrow \mathcal{M}^+(\mathrm{n})$ in the group of orientable Möbius transformations which is uniquely determined up to conjugation in $\mathcal{M}^+(\mathrm{n})$. The homomorphism ρ is called the *holonomy representation*; cf. [Ku1, Ch.7]. The manifold is developable if and only if ρ is trivial.

We say that the holonomy representation $\rho\colon \pi_1 \to \mathcal{M}^+(\mathrm{n})$ *fixes a point* $p \in S^n$ if for all $\gamma \in \pi_1$ the point $p \in \mathrm{Im}(\delta)$ is a fixed point of $\delta_\gamma = \rho(\gamma)$. In this case there is the induced representation

$$\rho_* = \rho_{p,*}\colon \pi_1 \longrightarrow \mathbb{CO}^+(\mathrm{n}),$$

in the *special conformal group* $\mathbb{CO}^+(\mathrm{n}) = \mathbb{R}^+ \times \mathbb{SO}(\mathrm{n})$ of Euclidean space $\mathbb{R}^n = T_pM$, given by $\rho_*(\gamma) = (d\delta_\gamma)_p\colon T_pS^n \to T_pS^n$. Here is the main result of this paper.

THEOREM 1.2. *Let (M, g) be an orientable conformally flat manifold with development $\delta\colon \widetilde{M} \to S^n$ and holonomy representation ρ. Then (M, g) carries a spin structure and a twistor spinor with zero if and only if the following conditions are satisfied*:

(a) *The holonomy representation ρ fixes a point $p \in \mathrm{Im}(\delta) \subset S^n$, the induced representation is orthogonal, i.e.,*

$$\rho_{*,p}\colon \pi_1 \longrightarrow \mathbb{SO}(\mathrm{n}),$$

and admits a lift

$$\widetilde{\rho_{*,p}}\colon \pi_1 \longrightarrow \mathrm{Spin(n)}$$

with respect to the two–fold covering Spin(n) $\to \mathbb{SO}(\mathrm{n})$.

(b) *If μ*: Spin(n) $\to \mathbb{U}(\Delta_n)$ *is the* unitary spin representation *then the representation*

$$\mu \circ \widetilde{\rho_{*,p}}\colon \pi_1 \longrightarrow \mathbb{U}(\Delta_n)$$

has a trivial subrepresentation; *i.e., there is an invariant subspace $V \subset \Delta_n$ of dimension at least one on which π_1 acts as identity.*

There is a twistor spinor ϕ with non-trivial associated conformal field V_ϕ if and only if in addition to conditions a) and b) also the following condition holds:

(c) *The induced holonomy representation $\rho_{*,p}$ on T_pS^n has a trivial subrepresentation; i.e.,*

$$\rho_{*,p}\colon \pi_1 \longrightarrow \mathbb{SO}(n-1).$$

The main idea of the proof of part (b) is that a twistor spinor ψ with zero p on S^n is uniquely determined by the value $D\psi(p)$ of the Dirac operator at the zero and that ψ induces a twistor spinor with zero on M if and only if the spinor $D\psi(p)$ is fixed under the action of the fundamental group $\pi_1 = \pi_1(M)$.

It follows immediately that a *Kleinian manifold* in the sense of [Ku1, (7.8)] which carries a twistor spinor with zero is already developable. We show in Example 3.6 that there are conformally flat manifolds for which the assumptions of Theorem 1.2 are satisfied and neither $\rho_{*,p}$ nor $\mu \circ \widetilde{\rho_{*,p}}$ is trivial.

In [KR2], the authors construct a complete Riemannian metric on $\mathbb{R}^4$, which is half-conformally flat but not conformally flat and which carries a 2-dimensional space of twistor spinors with common zero point. In this case the associated conformal fields vanish identically. See also [KR3] for this construction in all even dimensions > 4.

2. Conformal invariance of twistor spinors

Let $f\colon (\overline{M}, \overline{g}) \longrightarrow (M, g)$ be a conformal map between oriented n–dimensional Riemannian manifolds with $f^*g = \sigma^{-2}\overline{g}$ for a smooth function $\sigma\colon \overline{M} \longrightarrow \mathbb{R}^+$. Let M carry a spin structure $\xi\colon P_{spin}(M, g) \longrightarrow P_{so}(M, g)$, which is a Spin(n)–principal bundle with a twofold covering ξ of the S$\mathbb{O}$(n)–principal bundle $P_{so}(M, g)$ of positively oriented orthonormal frames of (M, g); cf. [LM]. Via the conformal map f, this induces a spin structure $P_{spin}(\overline{M}, \overline{g}) \longrightarrow P_{so}(\overline{M}, \overline{g})$ on $(\overline{M}, \overline{g})$ and the induced map

$$f_{so}\colon\ P_{so}(\overline{M}, \overline{g}) \to P_{so}(M, g);\ (e_1, \dots, e_n) \mapsto \left(\sigma^{-1} f_*(e_1), \dots, \sigma^{-1} f_*(e_n)\right)$$

lifts to a bundle map $f_{spin}\colon\ P_{spin}(\overline{M}, \overline{g}) \to P_{spin}(M, g)$, which is uniquely determined up to a sign. This induces a *bundle isometry* $f_*\colon\ \Sigma(\overline{M}, \overline{g}) \to \Sigma(M, g)$ of the associated *spinor bundles* $\Sigma(M, g) = P_{spin}(M, g) \times_\mu \Delta_n$ resp., $\Sigma(\overline{M}, \overline{g}) = P_{spin}(\overline{M}, \overline{g}) \times_\mu \Delta_n$. Here μ: Spin(n) $\longrightarrow U(\Delta_n)$ is the unitary *spin representation* on the complex vector space Δ_n and $\dim \Delta_{2m} = \dim \Delta_{2m+1} = 2^m$. We denote by $\mathcal{D}$, resp. $\overline{\mathcal{D}}$, the *twistor operator* of (M, g), resp. $(\overline{M}, \overline{g})$; it is given by $\mathcal{D}\colon\ \Gamma(\Sigma M) \to \Gamma(TM \otimes \Sigma M)$ with

$$\mathcal{D}\phi = \sum_{i=1}^{n} e_i \otimes \left(\nabla^s_{e_i}\phi + \frac{1}{n} e_i \cdot D\phi \right),$$

where $\nabla^s_X\phi$ is the *spinor derivative* of the spinor field $\phi \in \Gamma(\Sigma M)$ in the direction of the vector field X, $\cdot$ is the *Clifford product* and D the *Dirac operator*. Twistor spinors are the elements of $\ker \mathcal{D}$, resp. $\ker \overline{\mathcal{D}}$; i.e., a twistor spinor on M satisfies the *twistor equation*

$$\nabla^s_X\phi + \frac{1}{n} X \cdot D\phi = 0, \tag{1}$$

for all vector fields X on the manifold. Via the induced map f_* we can pull back spinors; i.e., for a spinor field ϕ on M the equation $f_*(\overline{\phi}) = \phi$ defines a spinor field $\overline{\phi}$ on $\overline{M}$. If ϕ is a twistor spinor on (M, g) then $\sigma^{1/2}\overline{\phi}$ is a twistor spinor on $(\overline{M}, \overline{g})$; cf. [BFGK, ch.1.4].

Hence $f^{\mathcal{D}}\colon \ker\mathcal{D} \to \ker\overline{\mathcal{D}};\ \phi \mapsto \sigma^{1/2}\overline{\phi}$ maps twistor spinors of (M, g) on twistor spinors of $(\overline{M}, \overline{g})$.

Now fix a point $q \in M$. Then a twistor spinor ϕ is uniquely determined by the values $(\phi(q), D\phi(q)) \in \Sigma_q M \oplus \Sigma_q M \cong \Delta_n \oplus \Delta_n$. Hence $\ker\mathcal{D}$ can be identified with a subspace of $\Delta_n \oplus \Delta_n$. Under this identification we can express

$$f^{D}\colon \ker\mathcal{D} \subset \Sigma_q M \oplus \Sigma_q M \longrightarrow \ker\overline{\mathcal{D}} \subset \Sigma_p\overline{M} \oplus \Sigma_p\overline{M}$$

(where $q = f(p)$) by the homomorphism

$$(\phi(q), \psi(q)) \longmapsto \left(\sigma^{1/2}(p)\overline{\phi}(p), \sigma^{3/2}(p)\overline{\psi}(p) - \frac{n-2}{2}\sigma^{1/2}(p)\,\mathrm{grad}\,\sigma(p)\cdot\overline{\phi}(p)\right)$$

for $\phi \in \ker\mathcal{D}$ and $\psi(q) = D\phi(q)$. This follows from the formula

$$\overline{D}(\sigma^{1/2}\overline{\phi}) = \sigma^{(n+1)/2}\overline{D(\sigma^{-(n-2)/2}\phi)} = \sigma^{3/2}\overline{D\phi} - \frac{n-2}{2}\sigma^{1/2}\,\mathrm{grad}\,\sigma\cdot\overline{\phi};$$

cf. [LM, Thm. 5.24].

Now we consider twistor spinors with zeros. Let $\mathcal{T}_q := \{\phi \in \ker\mathcal{D} \mid \phi(q) = 0\}$, resp. $\overline{\mathcal{T}}_p := \{\phi \in \ker\overline{\mathcal{D}} \mid \phi(p) = 0\}$. Then we identify $\mathcal{T}_q$ with the subspace $\{D\phi(q) \mid \phi \in \mathcal{T}_q\} \subset \Sigma_q M \cong \Delta_n$ and $f^{\mathcal{D}}\colon \mathcal{T}_q \to \overline{\mathcal{T}}_p$ with the *isomorphism* $\psi(q) \mapsto \sigma^{3/2}\overline{\psi(p)}$ (here $\psi = D\phi$ for some $\phi \in \mathcal{T}_p(M, g)$).

3. Conformally flat manifolds and twistor spinors

In this section we prove our main result, Theorem 1.2 and construct examples. On our way we prove the following results:

LEMMA 3.1. *Let (M, g) be an orientable conformally flat spin manifold with development $\delta\colon \widetilde{M} \to S^n$ and holonomy representation ρ. If (M, g) carries a twistor spinor with zero q then there is a point $p = \mathrm{Im}(\delta) \subset S^n$, which is fixed by the holonomy representation ρ, such that the induced representation $\rho_{*,p}$ is orthogonal, i.e.,*

$$\rho_{*,p}\colon \pi_1 \longrightarrow \mathrm{SO(n)}.$$

Proof. If $\phi \in \mathcal{T}_q M$, then the universal covering space $\widetilde{M}$ carries a twistor spinor $\widetilde{\phi}$ with zero $\widetilde{q} \in \pi^{-1}(q)$. Here $\pi\colon \widetilde{M} \to M$ denotes the universal covering map. For $p = \delta(\widetilde{q})$ consider the isomorphism

$$\delta^{\mathcal{D}};\ \mathcal{T}_p(S^n) \longrightarrow \mathcal{T}_{\widetilde{q}}(M),$$

and denote by $\psi \in \mathcal{T}_p(S^n)$ the twistor spinor with zero at p and with $\delta^{\mathcal{D}}(\psi) = \widetilde{\phi}$. Let $Z_{\widetilde{\phi}} := \{\widetilde{q} \in \widetilde{M} \mid \widetilde{\phi}_{\widetilde{q}} = 0\}$ be the zero set of $\widetilde{\phi}$. Since a twistor spinor on S^n has at most one zero, the zero set $Z_{\widetilde{\phi}}$ is mapped onto p under δ.

Since the deck transformations act on $\widetilde{M}$ by isometries the zero set $Z_{\widetilde{\phi}}$ is invariant under the action of π_1. It follows that the holonomy representation ρ fixes $p \in S^n$. Hence $\rho_{*,p}\colon \pi_1 \to \mathbb{CO}^+(\mathrm{n})$. Let $\gamma \in \pi_1$ and consider

$$(\delta_\gamma)_*\colon T_pS^n \longrightarrow T_pS^n.$$

Since $\delta^{\mathcal{D}}\psi = \phi$, for $q \in \delta^{-1}(p)$ we obtain

$$D\psi(p) = \sigma^{3/2}(q)\,\delta_*(D\widetilde{\phi}(q))$$

and

$$\begin{aligned} D\psi(\delta_\gamma p) &= D\psi(p) = \sigma^{3/2}(\gamma q)\,\delta_*(D\widetilde{\phi}(\gamma q)) = \sigma^{3/2}(\gamma q)\,\delta_*(\gamma_*\left(D\widetilde{\phi}(q)\right) \\ &= \sigma^{3/2}(\gamma q)(\delta_\gamma)_*\delta_*\left(D\widetilde{\phi}(q)\right). \end{aligned}$$

Since the maps δ_* and $(\delta_\gamma)_*$ are bundle isometries we have $\sigma(q) = \sigma(\gamma q)$ for all $\gamma \in \pi_1$, hence $\rho_*(\gamma) \in \mathbb{SO}(\mathrm{n})$. We also have

$$D\psi(p) = (\delta_\gamma)_*\,(D\psi(p)) \tag{2}$$

for all $\gamma \in \pi_1$. □

LEMMA 3.2. *Let (M, g) be an orientable conformally flat manifold with development $\delta\colon \widetilde{M} \to S^n$ and holonomy representation ρ which fixes a point $p \in \mathrm{Im}(\delta) \subset S^n$ and which is orthogonal, i.e., $(\rho_*)_p\colon \pi_1 \to \mathbb{SO}(\mathrm{n})$. Then (M, g) carries a spin structure if and only if $\rho_{*,p}$ can be lifted to a representation*

$$\widetilde{\rho_{*,p}}\colon \pi_1 \to \mathrm{Spin(n)}$$

with $\rho_{*,p} = \xi \circ \widetilde{\rho_{*,p}}$,, where $\xi\colon \mathrm{Spin(n)} \to \mathbb{SO}(\mathrm{n})$ is the twofold covering.

Proof. (M, g) is spin if the representation $\pi_1 \to \mathrm{Aut}(P_{so}(M, g))$ of the fundamental group in the automorphism group $\mathrm{Aut}(P_{so}(M, g))$ of $P_{so}(M, g)$, the *group of orientation preserving isometries* of (M, g), has a lift $\pi_1 \to \mathrm{Aut}(P_{spin}(M, g))$ into the automorphism group of the spin structure with respect to the canonical projection $\mathrm{Aut}(P_{spin}(M, g)) \to \mathrm{Aut}(P_{so}(M, g))$; see [Fr, Ch.9]. Since ρ fixes p and since an element $\alpha \in \mathrm{Aut}(P_{so}(M, g))$ is uniquely determined if we know the action of α on a fibre we conclude that $\rho_{*,p}\colon \pi_1 \to \mathbb{SO}(\mathrm{n})$ has a lift $\widetilde{\rho_{*,p}}\colon \pi_1 \to \mathrm{Spin(n)}$ with respect to ξ. □

Proof of Theorem 1.2. If M carries a twistor spinor ϕ with zero then (a) holds by Lemma 3.1 and Lemma 3.2. It follows from Equation 2 in the Proof of Lemma 3.1 that (b) holds.

Let $V = V_\psi$ be the associated conformal field to the twistor spinor ψ with zero p on S^n constructed in the proof of Lemma 3.1; i.e., $\langle V, X\rangle = \sqrt{-1}\langle\psi, X \cdot \psi\rangle$ for all tangent vectors X. Let $W_\psi := \operatorname{grad}\operatorname{div} V(p)$. Then the vector field V_ψ is not identically zero if and only if $W \neq 0$; cf. [KR1, Prop.3.2]. Since ψ is invariant under the holonomy representation, it follows that $\rho_{*,p}(W) = W$ which implies (c).

On the other hand, if (a) holds it follows from Lemma 3.2 that (M, g) carries a spin structure. Now it follows from (b) that the twistor spinor ψ on S^n such that $\psi(p) \in \Delta_n$ is fixed by $\mu \circ (\rho_*)_p$ induces a twistor spinor $\tilde{\phi}$ on $\tilde{M}$ by the equation

$$D\psi(p) = \sigma^{3/2}(q)\delta_* \left(D\tilde{\phi}(q)\right)$$

which satisfies

$$\tilde{\phi}(\gamma q) = \gamma_* \tilde{\phi}(q)$$

and which therefore induces a twistor spinor on (M, g) with zero.

Now assume that, in addition, (c) holds, i.e. there is a unit vector $V \in T_pS^n$ such that $(\delta_\gamma)_*(V) = V$ for all $\gamma \in \pi_1$. Let $\xi \in \Delta_n$ be a spinor which is fixed under the holonomy representation $\rho_{*,p}$; i.e., $(\delta_\gamma)_*\xi = \xi$ for all $\gamma \in \pi_1$.

Furthermore the spinor $V \cdot \xi$ is fixed under $\mu \circ \rho_{*,p}$.

Assume first that the dimension n is even. Hence we can assume that ξ is an *even* spinor, i.e., $\xi \in \Delta_n^+$. Let ψ be the twistor spinor on S^n with zero in p and with $D\psi(p) = \xi + iV \cdot \xi$. Then we obtain for the associated vector field $V_{D\psi}(p) = 2|\xi|^2 V(p)$ and it follows from [KR1, Prop.3.2] that the associated conformal field V_ψ does not vanish identically. Hence there is a twistor spinor ϕ on M with a zero and with non-trivial associated conformal field.

If the dimension n is odd, we use the fact that the spin representation on Δ_n can be described as the spin representation $\Delta_{n-1} = \Delta_{n-1}^+ \oplus \Delta_{n-1}^-$, where $\xi \in \Delta_{n-1}^+$ and $n = 2m + 1$. Let $(e_1, \ldots, e_n)$ with $V = e_n$ be an orthonormal basis. Then $e_1, \ldots, e_{n-1}$ act on both components and e_n acts on $u^+ + u^- \in \Delta_{n-1}^+ \oplus \Delta_{n-1}^-$ as $e_n \cdot (u^+ + u^-) = (-1)^m\sqrt{-1}(u^+ - u^-)$. Let ψ be the twistor spinor on S^n defined by $D\psi(p) = (-1)^{m+1}\xi$. Then $V_{D\psi}(p) = 2|\xi|^2 V$, hence the associated conformal field $V_{D\psi}$ does not vanish identically. This finishes the proof.

Now we construct examples. Let $A \in \mathbb{SO}(n-1)$. Then we denote by $F_A \in \mathbb{CO}^+(\mathrm{n})$ the following conformal transformation: Let $N = (1, 0, \ldots, 0) \in S^n = \{x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^n x_i^2 = 1\}$ and let $\sigma\colon S^n - \{N\} \to \mathbb{R}^n$ be the stereographic projection, for which $\sigma((0, x_1, x_2, \ldots, x_n)) = (x_1, x_2, \ldots, x_n)$. Then F_A is determined by

$$\sigma \circ F_A \circ \sigma^{-1}((x_1, x_2, \ldots, x_n)) = (x_1, A(x_2, \ldots, x_n)).$$

It follows that $F_A(N) = N$ and the linearization of F_A at N is given by

$$(df_A)_N\,(x_1, \ldots, x_n) = (x_1, A(x_2, \ldots, x_n)),$$

resp. $(dF_A)_N = A \in \mathbb{SO}(n-1) \subset \mathbb{SO}(\mathrm{n})$.

LEMMA 3.3. *For $A \in \mathbb{SO}(n-1), n \geq 3$ there is an orientable conformally flat manifold $M = M_A^n$ whose holonomy representation ρ fixes a point $p \in \mathrm{Im}(\delta)$ and is given by*

$$\rho\colon \pi_1(M) \cong \mathbb{Z} \longrightarrow \mathbb{CO}^+(\mathrm{n});\ 1 \longmapsto F_A.$$

Hence the induced representation of π_1 on T_pS^n is given by

$$\rho_{*,p}\colon \pi_1(M) \cong \mathbb{Z} \longrightarrow \mathbb{SO}(n-1);\ 1 \longmapsto A.$$

Proof. Let $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$. We denote by $x = (x_1, \overline{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ the components. We set

$$\begin{aligned} X_1 &:= [-1,1] \times B^{n-1}\left(\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \frac{1}{2}\right) \times \mathbb{R}^{n-1} \\ &= \left\{ (x_1, \overline{x}) \in [-1,1] \times \mathbb{R}^{n-1} \;\middle|\; \|\overline{x}\| < \frac{1}{2} \text{ or } |x_1| < \frac{1}{2} \right\}, \end{aligned}$$

where $B^{n-1}(r) := \left\{ x \in \mathbb{R}^{n-1} \mid, |x\| < r \right\}$. We denote by

$$S^n = \{y = (y_0, \ldots, y_n) \in \mathbb{R}^{n+1} \mid \|y\| = 1\}$$

the unit n–sphere with *closed upper hemisphere* $H_+^n = \{y \in S^n \mid y_0 \geq 0\}$. The boundary $\partial H_+^n = \{y \in S^n \mid y_0 = 0\}$ is then the equator. Then for $(0, y_1, \ldots, y_n) \in \partial H_+^n$ we obtain $\sigma((0, y_1, \ldots, y_n)) = (y_1, \ldots, y_n)$.

Let X_2 be the disjoint union of X_1 and H_+^n where we identify $x = (x_1, \overline{x}) \in X_1 \subset \mathbb{R}^n$ with $y \in H_+^n$ if and only if $x = \sigma(y)$ and $x_1 \in (-1/2, 1/2)$.

Let $\widetilde{M} = \mathbb{Z} \times X_2/\sim$ where we identify

$$(m, (1, \overline{x})) \sim (m+1, (-1, \overline{x}))$$

for all $\overline{x} \in B^{n-1}(1/2)$ and all $m \in \mathbb{Z}$. For $n \geq 3$ $\widetilde{M}$ is simply-connected. The map $f\colon \widetilde{M} \to \widetilde{M}$ defined by: $f(m, (x_1, \overline{x})) = (m+1, (x_1, A\overline{x}))$ for $x = (x_1, \overline{x}) \in X_1$, and $f(m, \sigma^{-1}(x_1, \overline{x})) = (m+1, \sigma^{-1}(x_1, A\overline{x}))$ for $(x_1, \overline{x}) \in \mathbb{R}^n - B^n(1)$, $x_1 \in \mathbb{R}$, $\overline{x} \in \mathbb{R}^{n-1}$ as well as $f(m, N) = (m+1, N)$ generates a $\mathbb{Z}$–action by isometries which act freely and properly discontinuous. Hence $M = M_A = \widetilde{M}/_f\mathbb{Z}$ is a conformally flat manifold and we can choose the development map $\delta\colon \widetilde{M} \to S^n$ such that $\delta|\{1\} \times H_+^n$ is the identity (i.e., $\delta((1, y)) = y$, $y \in H_+^n$).

It follows that the holonomy representation

$$\rho\colon \pi_1(M_A) \cong \mathbb{Z} \longrightarrow \mathcal{M}^+(n)$$

is given by $\rho(1) = F_A$ and it fixes $N \in S^n$. □

Remark 3.4. M is not *Kleinian* since for a Kleinian manifold the holonomy representation does not have fixed points; cf. [Ku1, (7.8)].

The subgroup $\rho(\pi_1(M)) \subset \mathcal{M}^+(\mathrm{n})$ consists of *parabolic elements*; cf. [Ku2, (2.2)].

Example 3.5. For $n = 2m > 4, m \in \mathbb{N}, \alpha \in \mathbb{R}, \alpha \notin 2\pi\mathbb{Z}$ define the following element in Spin(n):

$$A_\alpha := ((\sin\alpha)e_1 + (\cos\alpha)e_2) \cdot e_2 \cdot ((\sin\alpha)e_3 + (\cos\alpha)e_4) \cdot e_4.$$

Here $(e_1, \ldots, e_n)$ denotes a positiviely oriented orthonormal basis. Using an explicit spin representation μ: Spin(n) $\longrightarrow \mathbb{U}(\Delta_n)$ (for example [BFGK, Chapter 1.1]) we obtain

$$\mu(A_\alpha) = -E \otimes \cdots \otimes R_{-\alpha} \otimes R_{-\alpha},$$

where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \; R_\alpha := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

and $\mu(A_\alpha)$ acts on $\Delta_n = \mathbb{C}^{2^m} \cong \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. It follows that the eigenvalue 1 of $\mu(A_\alpha)$ has multiplicity 2^{m-1} and the eigenvalues $\pm\exp(2\pi\alpha)$ have multiplicity 2^{m-2}. If ξ: Spin(n) $\longrightarrow$ $\mathbb{SO}$(n) we conclude that $\xi(A_\alpha)$ is a rotation of the form

$$-R_{2\alpha} \oplus R_{-2\alpha} \oplus \mathrm{Id}$$

on $\mathbb{R}^n = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^{n-4}$; in particular $\xi(A_\alpha) \in \mathbb{SO}(4) \subset \mathbb{SO}(\mathrm{n})$.

Example 3.6. Combining Lemma 3.3 and Example 3.6, for $n = 2m > 6$ and $\alpha \in \mathbb{R} - 2\pi\mathbb{Z}$ we obtain an element $A_\alpha \in$ Spin(n) and a conformally flat manifold M_α such that the holonomy representation

$$\rho\colon \pi_1(M) \cong \mathbb{Z} \longrightarrow \mathcal{M}^+(n)$$

is given by $\rho(1) = F_{A_\alpha}$ and such that ρ fixes a point $N \in S^n$ with induced representation

$$\rho_{*,N}\colon \pi_1(M) \longrightarrow \mathbb{SO}(4) \subset \mathbb{SO}(\mathrm{n}),$$

$$\rho_{*,N}(1) = -R_{2\alpha} \oplus R_{-2\alpha} \oplus \mathrm{Id}.$$

Hence the manifold M_α, which is not developable, satisfies all assumptions of Theorem 1.2. We conclude that on M_α there exist twistor spinors with zero and with non-trivial associated conformal field. It follows that for $p \in \delta^{-1}(N)$ we have $\dim \mathcal{T}_p(M_\alpha) = \dim\ker(\mu(A_\alpha) - \mathrm{Id}) = 2^{m-1}$, hence $\dim \mathcal{T}_p(M_\alpha) = n2^{m-1} = m2^m$.

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