ILLINOIS JOURNAL OF MATHEMATICS
Volume 42, Number 1, Spring 1998

VARIATION IN PROBABILITY, ERGODIC THEORY AND ANALYSIS

MUSTAFA A. AKCOGLU, ROGER L. JONES AND PETER O. SCHWARTZ

1. Introduction

In many areas of analysis, ergodic theory and probability, square functions have proven to be one of the most useful tools to study convergence properties. (See the paper by Stein [17] for a very informative historical discussion of the importance of various square functions in several areas.) For example, the martingale square function was used by Burkholder, Gundy and Silverstein [7] to give the first real variable characterization of H_p. An ergodic square function was used by Bourgain [3] in his proof that the ergodic averages along the sequence of squares converge a.e. In this paper we consider operators that are closely related to the square functions, but have very different properties.

Let $(\mathcal{F}_k)$ denote an increasing sequence of σ-fields. Then the martingale square function is defined by

$$Sf(x) = \left(\sum_{k=1}^{\infty} |E_k f(x) - E_{k-1} f(x)|^2 \right)^{\frac{1}{2}},$$

where E_k denotes the conditional expectation operator with respect to the σ-field $\mathcal{F}_k$. This operator, which maps L^p to L^p for each p, $1 < p < \infty$, gives a measure of the square variation of the martingale sequence $(E_k f)$. It is natural to ask about the L^p boundedness properties of the q-variation operator

$$V_q f(x) = \left(\sum_{k=1}^{\infty} |E_k f(x) - E_{k-1} f(x)|^q \right)^{\frac{1}{q}},$$

for $1 \le q < 2$. In Section 2 we show that if $q < 2$ then the operator V_q is very badly behaved. In particular, we show that it is possible to have $V_q f(x) = \infty$ a.e. even for bounded functions, f. The arguments provide a revealing contrast to the Hilbert space techniques that come into play when $q \ge 2$. The martingale result is

Received May 14, 1997.
1991 Mathematics Subject Classification. Primary 60G46, 28D05; Secondary 42B25, 40A05.
M. Akcoglu and P. Schwartz were partially supported by an NSERC grant.
R. Jones was partially supported by grants fron the National Science Foundation.
The authors would like to acknowledge several helpful conversations with Jeffrey Rosenthal.

known [6], but we are unable to find a reference. We therefore supply a proof, which, moreover, is a prototype of our arguments in the more complex situations that follow.

More generally, we consider pointwise convergent averaging operators of several types: differentiaton operators, ergodic averages, and integration against the Poisson kernel. Our interest is in demonstrating that even for nice functions the convergence in each context is slow as measured by variation operators. Each new operator raises new difficulties, but at the core of all our arguments is the concept of independence and the Strong Law of Large Numbers. The ideas are most easily understood in the case of martingales and so that case is presented first.

On $L^p([0,1))$ one may consider differentiation operators, $D_\ell f(x) = \frac{1}{\ell}\int_0^\ell f(x+t)dt$. For a decreasing sequence (ℓ_k) we define the associated q-variation operator, $V_{q,D}f(x) = (\sum_{k=1}^\infty |D_{\ell_k}f(x) - D_{\ell_{k-1}}f(x)|^q)^{\frac{1}{q}}$. It was shown in [12, 13] that $V_{2,D}$ is a bounded operator on L^p, $1 < p < \infty$, but we will see in Section 3 that quite different behavior can occur if $q \in [1,2)$. We also consider, in Section 2, the q-variation operator $V_{q,C}$ that compares the dyadic martingale and the associated dyadic differentiation operator. This operator,

$$V_{q,C}f(x) = \left(\sum_{k=1}^{\infty} |D_{2^{-k}}f(x) - E_k f(x)|^q\right)^{\frac{1}{q}}$$

is shown in [13] to be bounded in L^p, $1 < p < \infty$, if $q = 2$. We show that if $q \in [1,2)$, this operator can diverge a.e. even for bounded functions.

There are similar questions in ergodic theory. Let $A_n f$ denote the usual averages; $A_n f(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k x)$. Given an increasing sequence (n_k), we can form the q-variation operator

$$V_{q,E}f(x) = \left(\sum_{k=1}^{\infty} |A_{n_k}f(x) - A_{n_{k-1}}f(x)|^q\right)^{\frac{1}{q}}.$$

As in the differentiation case, it is known (see [11], [12], [13]) that for $q = 2$, $V_{q,E}$ is a bounded operator on all $L^p(X)$, $1 < p < \infty$, and that it is weak type (1,1). (See [3], [9], [10] for other related square functions.) However, we will show in Section 4 that for $q \in [1,2)$, the properties of the operator depend on the sequence. This gives perspective to the well known fact that there is no rate at which ergodic averages converge.

In harmonic analysis other operators have also played an important role in measuring the variation of a sequence of functions [3], [12], [13], [16]. In 1955, W. Rudin [15] looked at functions F which are analytic in the interior of the unit circle, and studied the operator $V(F,\theta) = \int_0^1 |F'(re^{i\theta})|dr$. He showed that there exists an analytic function on the disc with continuous boundary values such that $V(F,\theta) = \infty$ for a.e. θ. Standard calculus techniques show that this is equivalent to the existence of a sequence (r_k) such that $\sum_{k=1}^\infty |F(r_k e^{2\pi i\theta}) - F(r_{k+1}e^{2\pi i\theta})| = \infty$ for a.e. θ. It is not hard to show (see Section 5) that for such functions, the 2-variation $V_2(F,\theta) = (\sum_{k=1}^{\infty} |F(r_k e^{2\pi i\theta}) - F(r_{k+1}e^{2\pi i\theta})|^2)^{\frac{1}{2}} < \infty$ for a.e. θ. It is natural to ask about the properties of the operator $V_q(F,\theta) = (\sum_{k=1}^{\infty} |F(r_k e^{2\pi i\theta}) - F(r_{k+1}e^{2\pi i\theta})|^q)^{\frac{1}{q}}$ for $q \in [1,2)$ which measures the q-variation of the analytic function F. This will be the subject of Section 5, where we show that $V_q(F,\theta)$ may be infinite for a.e. θ even if F is an analytic function with continuous boundary values. Taking q to be 1, this gives a new proof of Rudin's theorem. (See also [2] where it is shown that there is, nonetheless, a dense set of θ where $V_q(F,\theta)$ is finite.)

The reader will note that each of the constructions share common features. Taking advantage of this, we present the cases in increasing order of difficulty, with each case presenting a new difficulty. The martingale case is the easiest to understand. It makes straightforward use of Rademacher functions and stopping times. The differentiation case also makes use of the Rademacher functions, but we need to select the subsequence of Rademacher functions more carefully and the sets where bad behavior occurs are harder to control. The ergodic case is similar to the differentiation case, but makes use of a reverse martingale, and an analog of the Rademacher functions on $\mathbb{Z}$. The analytic function case requires replacing the Rademacher functions, which are independent, by exponentials, $(e^{2\pi i n_k \theta})$, with (n_k) rapidly increasing, hence approximating the independence. Many of the subtleties in the following theorems are associated with the fact that we want to construct bounded functions for which the q-variation is infinite. If we were willing to settle for an L^2 function for which the q-variation is infinite, the arguments would be much simpler.

2. The martingale case

Consider the unit interval $X = [0,1)$ with Lebesgue measure μ. The sequence of Rademacher functions, (r_k), are defined by $r_k(x) = \operatorname{sgn}\sin(2\pi 2^{k-1}x)$ for $k \geq 1$. Let $\mathcal{F}_k$ denote the dyadic σ-field with 2^k atoms, and let E_k denote conditional expectation with respect to $\mathcal{F}_k$.

Fix an increasing sequence of positive integers, (n_k), and define the q-variation operator associated with this sequence by

$$V_q f(x) = \left(\sum_{k=1}^{\infty} |E_{n_k} f(x) - E_{n_{k-1}} f(x)|^q \right)^{\frac{1}{q}}.$$

We are interested in showing that the q-variation operator, for $q < 2$, can be made to diverge. If we only wanted to show divergence for $f \in L^2(X)$, the problem is trivial. We just consider $f_n(x) = \sum_{k=1}^{n} a_k r_{n_k}(x)$ where $\sum_{k=1}^{\infty} |a_k|^2 < \infty$ but $\sum_{k=1}^{\infty} |a_k|^q = \infty$ for $q < 2$. For example $a_k = \frac{1}{\sqrt{k}\log k}$ will do. However, to find an L^∞ bounded martingale with the same property requires a more complicated argument. To obtain a bounded martingale we introduce a stopping time, $\tau(x) = \inf\{n : |f_n(x)| \geq 1\}$. We find a martingale so that the above properties hold, but such that the stopped martingale and the original martinagle are not too different. The fact that we can do this is the content of the following theorem.

THEOREM 2.1. *Let (n_k) be an increasing sequence of positive integers. There is a function $f \in L^\infty(X)$, with $\|f\|_\infty \leq 1$, but such that the associated dyadic martingale satisfies $V_q f(x) = \infty$ a.e. for all $q \in [1,2)$.*

Proof. We will first prove the theorem in the case $n_k = k$ for each k. The key to proving the theorem is the following lemma.

LEMMA 2.2. *Let N_0, L, ϵ and $q \in [1,2)$ be given. Then there is a function f and an integer N such that*

1. $\|f\|_\infty \leq 1$,
2. $E_k(f) = 0$ *for all* $k \leq N_0$,
3. f *is measurable with respect to the* σ*-field* $\mathcal{F}_{N_0+N}$,
4. $V_q f(x) \geq L$ *except possibly on a set of measure less than* ϵ.

Proof. Fix a large integer M so that $L^q M^{q-2} < \epsilon$. Define N to be the smallest integer such that $N \geq (LM)^q$. We now define a martingale $f_n = \sum_{k=1}^n \frac{1}{M} r_k(x)$. We can associate with this martingale a stopping time defined by $\tau(x) = \inf\{n : |f_n(x)| \geq 1\}$. Note that since the martingale takes steps on size $\pm\frac{1}{M}$, we actually have $\tau(x) = \inf\{n : |f_n(x)| = 1\}$. We now consider the stopped martingale $f_n^\tau = f_{n\wedge\tau}$. Let (d_k) denote the martingale difference sequence associated with the stopped martingale. That is, $d_k = f_k^\tau - f_{k-1}^\tau$. We have

$$V_q(f_N^\tau)(x) = \left(\sum_{k=1}^N |d_k(x)|^q\right)^{\frac{1}{q}} = \frac{1}{M}(\tau(x) \wedge N)^{\frac{1}{q}}.$$

Since $f_N^\tau(x) = 1$ if $\tau < N$, we have

$$\mu\{\tau < N\} \leq \|f_N^\tau\|_2^2 \leq \sum_{k=1}^N \|d_k\|_2^2 \leq N\frac{1}{M^2} = \frac{(LM)^q}{M^2} < \epsilon.$$

On the set where $\tau \geq N$, a set with measure at least $1-\epsilon$, we have $V_q f_N^\tau(x) = \frac{1}{M} N^{\frac{1}{q}} \geq L$, as required.

In the above construction f_N^τ is clearly measureable with respect to $\mathcal{F}_N$. If we had started the construction with $r_{N_0+1}, r_{N_0+2}, \ldots$, rather than $r_1, r_2, \ldots$, then we would have $E_k(f_N^\tau) = 0$ for all $k \leq N_0$ and f_N^τ would be measurable with respect to $\mathcal{F}_{N_0+N}$. □

We now continue with the proof of Theorem 2.1. First assume that $N_0 = 0$. Let $L = 100 \times 2^1$ and $\epsilon = \frac{1}{2^1}$. Let f be the function obtained by Lemma 2.2, and define

$g_1 = \frac{1}{2}f$. Denote by N_1 the integer N obtained by Lemma 2.2. We then have

$$\left(\sum_{n=1}^{N_1} |E_{n+1}g_1(x) - E_n g_1(x)|^q\right)^{\frac{1}{q}} \geq \frac{1}{2}L = 100$$

for x in a set $X_1 \subset X$ of measure greater than $1 - \frac{1}{2}$.

We now take N_0 in Lemma 2.2 to be N_1. We take $L = 100^2 \times 2^2$ and $\epsilon = \frac{1}{2^2}$. Let f be the function obtained by Lemma 2.2 and define $g_2 = \frac{1}{2^2}f$. Since $E_n f = 0$ for $0 \leq n \leq N_0$, and $E_n g_1 = g_1$ for all $n \geq N_0$, if we define $b_2 = g_1 + g_2$, we see that $V_q(b_2) \geq 100^2$ for $x \in X_2 \subset X$ with measure at least $1 - \frac{1}{2^2}$.

We repeat the construction, so that at the kth stage we have $N_0 = N_1 + N_2 + \cdots + N_{k-1}$. We have b_{k-1} measurable with respcet to $\mathcal{F}_{N_0}$, and $V_q b_{k-1} > 100^{k-1}$ on $X_{k-1} \subset X$ with $\mu(X_{k-1}) > 1 - \frac{1}{2^{k-1}}$. We take $L = 100^k 2^k$ and $\epsilon = \frac{1}{2^k}$. We construct the function f using Lemma 2.2, and define g_k to be $\frac{1}{2^k}f$. We let $b_k = b_{k-1} + g_k$. Hence $V_q b_k > 100^k$ on a set $X_k \subset X$ of measure greater than $1 - \frac{1}{2^k}$.

At each stage $\|g_k\|_\infty \leq \frac{1}{2^k}$ so $\sum_k g_k$ converges. Let $b = \sum_k g_k$, then $\|b\|_\infty \leq 1$ and for all k $V_q b(x) \geq V_q(b_k) \geq 100^k$ on a set of measure at least $1 - \frac{1}{2^k}$. Since k was arbitrary, we are done for fixed q.

By taking a sequence (q_k) which converge to 2 from below, and by using q_k at the kth stage of the construction, we get a single function that works for all $q < 2$. If (n_k) is an arbitrary increasing sequence of positive integers, we just use r_{n_k} instead of r_k in the construction of the example. There is no other change in the proof. □

For f defined on $[0, 1)$ and extended periodically with period 1, let D_k denote the differentiation operator defined by

$$D_k f(x) = 2^k \int_0^{\frac{1}{2^k}} f(x+t)dt,$$

and as before, let $E_k f$ denote conditional expectation with respect to the dyadic σ-field. Define the q-variation operator $V_{q,C}$, which compares the differentiation operator and the dyadic martingale, by

$$V_{q,C} = \left(\sum_{k=1}^{\infty} |D_k f(x) - E_k f|^q\right)^{\frac{1}{q}}.$$

In the case $q = 2$ it is shown in [13] that this is a bounded operator on all L^p, $1 < p < \infty$ and is even weak type (1,1). However for $q < 2$ we have the following theorem.

THEOREM 2.3. *There is a function* $f \in L^\infty[0, 1)$ *with* $\|f\|_\infty \leq 1$ *such that* $V_{q,C} f = \infty$ *a.e. for all* $q < 2$.

Proof. The main tool in proving this theorem will be the following analog of Lemma 2.2.

LEMMA 2.4. *Let N_0, L_0, ϵ_0 and $q \in [1, 2)$ be given. There is a function f and an integer N such that*

1. $\|f\|_\infty \leq 1$,
2. *the smallest index of the Rademacher functions used in the construction of f is at least N_0,*
3. *f depends on only N Rademacher functions,*
4. *$V_{q,C} f(x) > L_0$ except possibly on a set of size ϵ_0.*

Proof. Assume $\epsilon_0 < \frac{1}{8}$. Let $L > 16(L_0 + 2)$ and $\epsilon < \frac{\epsilon_0}{6}$. Fix J so large that if $N \geq J$ and if $B_1, B_2, \dots, B_N$ are independent sets, each with measure at least $\frac{1}{8}$ then

$$\frac{1}{N}\sum_{k=1}^{N} \chi_{B_k}(x) > \frac{1}{16}$$

except possibly on a set of measure at most ϵ. Such a J exists by the Strong Law of Large Numbers. As in the proof of Lemma 2.2 let M be chosen so that $L^q M^{q-2} < \epsilon$. Define N to be the smallest integer such that $N \geq (LM)^q$. If $N < J$, increase L (and hence M) so that we can assume $N \geq J$. Fix a large integer d so that $\frac{N}{2^d} < \epsilon$. We will form a martingale using the Rademacher functions as we did in the proof of Lemma 2.2. However, we will use only a subsequence (r_{kd}), of the Rademacher functions. This does not change the arguments given there. We again form the martingale $f_n = \frac{1}{M}\sum_{k=1}^{n} r_{dk}$ and define the stopping time $\tau = \inf\{n : |f_n| \geq 1\}$. We again consider the stopped martingale defined by $f_n^\tau = f_{n\wedge\tau}$, and work with the function $f = f_N^\tau$. By the same arguments as in the proof of Lemma 2.2 we have $m\{\tau < N\} < \epsilon$.

If we let $\Delta_k f = |D_{dk} f - E_{dk} f|$, then we first need to estimate $\Delta_k r_{dk}$. We note that the intervals associated with the averages D_{dk} are the same length as the lengths of intervals where r_{dk} is constant. We now observe that $\Delta_k r_{dk} \geq 1$ on an interval of length at least $\frac{1}{2^{dk+1}}$ on the right hand side of each dyadic interval where r_{dk} is constant. Thus $\Delta_k r_{dk} \geq 1$ on a set of measure at least $\frac{1}{2}$. If we use only these dyadic intervals where $r_{dk} > 0$, we have a set of measure at least $\frac{1}{4}$. (We only use the intervals where $r_{dk} = 1$ since later when we introduce the stopped version, and consider $\Delta_k(r_{dk}\chi_{\tau\geq k})$ this will be the same as $\Delta_k r_{dk}$ for $x \in \{\tau \geq k\} \cap \{r_{dk} = 1\}$, but may change if $x \in \{r_{kd} = -1\}$.) For each k, denote the union of these intervals by B_k. Note that the sets B_k are independent.

We now estimate $\Delta_k r_{dj}$ for $j < k$. We have averages that are much shorter than the lengths of intervals where r_{dj} is constant. Hence only intervals of length $\frac{1}{2^{dk}}$ located at the right hand side of each dyadic interval of length $\frac{1}{2^{dj}}$ can contribute non-zero values. There are only 2^{dj} such intervals, and hence the total measure of such intervals is at most $\frac{2^{dj}}{2^{dk}}$. Denote this set by $E(k, j)$ and note that on this set we have $\Delta_k r_{dj} < 2$.

We will also need estimates for $\Delta_k r_{dj}$ where $j > k$. In this case the averages are long compared to the lengths of the intervals were r_{dj} is constant. Since all lengths are dyadic, we see that both $D_{dk} r_{dj}$ and $E_{dk} r_{dj}$ will be exactly zero. Hence $\Delta_k r_{dj} = 0$. However this may not be the case for the stopped version. If we consider $\Delta_k (r_{dj} \chi_{\tau \geq j})$, we see that if $x \in \{\tau \geq j\}$ but $x + \frac{1}{2^{dk}} \in \{\tau < j\}$ then we can get as much as $\frac{2^{dk}}{2^{dj}}$.

We are now ready to estimate $V_{q,C} f$. We have

$$\begin{aligned} V_{q,C} f(x) &\geq \left(|\Delta_k \left(\frac{1}{M} \sum_{j=1}^{N} r_{dj} \chi_{\tau \geq j} \right) |^q \right)^{\frac{1}{q}} \\ &\geq \frac{1}{M} \left(|\Delta_k (r_{dk} \chi_{\tau \geq k})|^q \right)^{\frac{1}{q}} - \left(|\Delta_k \left(\sum_{j=1}^{k-1} r_{dj} \chi_{\tau \geq j} \right) |^q \right)^{\frac{1}{q}} \\ &\quad - \left(|\Delta_k \left(\sum_{j=k+1}^{N} r_{dj} \chi_{\tau \geq j} \right) |^q \right)^{\frac{1}{q}} \\ &= A - B - C. \end{aligned}$$

We first estimate A. Note that $\Delta_k (r_{dk} \chi_{\tau > k}) = \Delta_k (r_{dk})$ on the set $B_k \cap \{\tau \geq k\}$. Hence, if $x \in \{\tau \geq N\}$, we have $\Delta_k (r_{dk} \chi_{\tau \geq k}) \geq 1$. Recalling that $\mu\{\tau < N\} < \epsilon < \epsilon_0 < \frac{1}{8}$, we see that $\mu(B_k \cap \{\tau \geq N\}) \geq \mu(B_k) - \mu(\{\tau < N\}) \geq \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$. Hence if $x \in \{\tau \geq N\}$, we have

$$A > \frac{1}{M} \left(\sum_{k=1}^{N} |\chi_{B_k}|^q \right)^{\frac{1}{q}}.$$

On the set where $\tau \geq N$ the sets B_k are independent. Since we have $m(B_k \cap \{\tau \geq N\}) > \frac{1}{8}$, we see, by the Strong Law of Large Numbers and our assumption on N, that we have $A > \frac{1}{16} \frac{1}{M} N^{\frac{1}{q}}$ except on a set of measure less than ϵ where $\tau \leq N$, and a set of measure ϵ where the strong law has not caused the average to be at least $\frac{1}{16}$.

We must now show that the other two pieces are small. For B we have

$$\begin{aligned} \|B\|_1 &\leq \int_0^1 \sum_{k=1}^{N} \sum_{j=1}^{k-1} 2\chi_{E(k,j)} dx \\ &\leq \sum_{k=1}^{N} \sum_{j=1}^{k-1} 2m(E(k, j)) \\ &\leq \sum_{k=1}^{N} \sum_{j=1}^{k-1} 2\frac{2^{dj}}{2^{dk}} \\ &\leq 2\frac{N}{2^d} < 2\epsilon. \end{aligned}$$

For C we have

$$\begin{aligned} \|C\|_1 &\leq \int_0^1 \sum_{k=1}^{N} \sum_{j=k+1}^{N} \frac{2^{dk}}{2^{dj}} dx \\ &\leq 2\frac{N}{2^d} < 2\epsilon. \end{aligned}$$

We now note that $m(B > 1) \leq \|B\|_1 \leq 2\epsilon$ and $m(C > 1) \leq \|C\|_1 < 2\epsilon$. Hence $V_{q,C} f > \frac{1}{16}L - 2$ except on a union of the sets $\{\tau < N\}$, $\{B > 1\}$, $\{C > 1\}$, and on the set where the strong law did not get us close enough to the average of the measures of B_k. The union of these sets is less than $6\epsilon = \epsilon_0$. Recalling the definition of L, and that $L_0 \geq 2$ we see that $V_{q,C} f > \frac{1}{16}L - 2 \geq L_0$. □

To complete the proof of Theorem 2.3 we use the above lemma in the same way that we used Lemma 2.2 to obtain Theorem 2.1. The details are very similar, and we only sketch them.

We first note that we can repeat the construction, starting the next block with N_0 so large that the new function constructed has as little interaction with the earlier fucntions as we desire. Further, we can make the L^∞ norms so small that they add up to 1. By taking a sequence of ϵ's which go to zero, we can get the exceptional sets to be as small as we want. We make the construction with a sequence of q_ks that converge to 2 from below. Hence we can apply the operator $V_{q,C}$ to the sum of the constructed functions, and get a value of infinity a.e. for each $q < 2$. □

3. The differentiation case

Let (ℓ_k) denote a decreasing sequence of numbers from the interval $[0, 1)$. For f defined on $[0, 1)$ and extended periodically with period 1, let D_{ℓ_k} denote the differentiation operator defined by $D_{\ell_k} f(x) = \frac{1}{\ell_k} \int_0^{\ell_k} f(x + t)dt$. Define the q-variation operator $V_{q,D}$ by

$$V_{q,D} = \left(\sum_{k=1}^{\infty} |D_{\ell_k} f(x) - D_{\ell_{k-1}}|^q \right)^{\frac{1}{q}}.$$

We then have the following theorem.

THEOREM 3.1. *Assume that* $\liminf_{k\to\infty} \frac{\ell_k}{\ell_{k-1}} = \lambda_0 < 1$. *Then there is a function* $f \in L^\infty[0, 1)$ *with* $\|f\|_\infty \leq 1$ *such that* $V_{q,D} f = \infty$ *a.e. for all* $q < 2$.

Proof. The main tool in proving this theorem will be the following analog of Lemma 2.2.

LEMMA 3.2. *Let N_0, L_0, ϵ_0 and $q \in [1, 2)$ be given. Then for any λ, $\lambda_0 < \lambda < 1$, there is a function f and an integer N such that*

1. $\|f\|_\infty \leq 1$,
2. *the smallest index of the Rademacher functions used in the construction of f is at least N_0,*
3. *f depends on only N Rademacher functions,*
4. *$V_{q,D}f(x) > L_0$ except possibly on a set of size ϵ_0.*

Proof. Assume without loss of generality that $\epsilon_0 < \frac{1-\lambda}{32}$. Take $\epsilon < \frac{\epsilon_0}{6}$ and $L > \frac{64}{1-\lambda}(L_0+2)$. Fix J so large that if $N \geq J$ and if $B_1, B_2, \ldots, B_N$ are independent sets, each with measure at least $\frac{1-\lambda}{32}$ then

$$\frac{1}{N}\sum_{k=1}^{N} \chi_{B_k}(x) > \frac{1-\lambda}{64}$$

except possibly on a set of measure at most ϵ. Such a J exists by the Strong Law of Large Numbers. As in the proof of Lemma 2.2, let M be chosen so that $L^q M^{q-2} < \epsilon$. Define N to be the smallest integer such that $N \geq (LM)^q$. If $N < J$, increase L (and hence M) so that we can assume $N \geq J$. Fix a large integer d so that $\frac{N}{2^d} < \epsilon$. We will form a martingale using the Rademacher functions, as we did in the previous section. However, we will need to select a subsequence of the Rademacher functions to use in the construction. Once the sequence of Rademacher functions is selected, the proof is almost the same as the proof of Lemma 2.4. We again form the martingale $f_n = \frac{1}{M}\sum_{k=1}^{n} r_{m_k}$ and define the stopping time $\tau = \inf\{n : |f_n| \geq 1\}$. We again consider the stopped martingale defined by $f_n^\tau = f_{n\wedge\tau}$, and work with the function $f = f_N^\tau$. By the same arguments as in the proof of Lemma 2.2 we have $m\{\tau < N\} < \epsilon$.

By hypothesis, we know there are infinitely many k such that $\frac{\ell_k}{\ell_{k-1}} < \lambda$. Let G denote the set of k with that property. Let n_1 denote the first $k \in G$ and let m_1 denote the largest integer such that $\frac{1}{2^{m_1}} > \ell_{n_1-1}$. If $m_1 < N_0$, select a larger integer $k \in G$ for n_1, so that $m_1 > N_0$. If we let $\Delta_k f = |D_{\ell_{n_k-1}} f - D_{\ell_{n_k}} f|$ then we first need to estimate $\Delta_1 r_{m_1}$. We see that the intervals associated with both averages $D_{\ell_{n_1}}$ and $D_{\ell_{n_1-1}}$ are no longer than the intervals where r_{m_1} is constant. Using the fact that the ratio between ℓ_{n_1} and ℓ_{n_1-1} is at most λ, a simple computation shows that $\Delta_1 r_{m_1} > 1-\lambda$ on an interval of length at least $\frac{1}{2}(\ell_{n_1-1} - \ell_{n_1})$ in each dyadic interval where r_{m_1} is constant. Thus $\Delta_1 r_{m_1} > 1-\lambda$ on a set of measure at least $2^{m_1}(\frac{1}{2})(\ell_{n_1-1} - \ell_{n_1}) > \frac{1}{4}(1-\lambda)$. If we use only the dyadic intervals where $r_{m_1} > 0$, we have a set of measure at least $\frac{1}{8}(1-\lambda)$. Denote this set by B_1. We can replace B_1 by a possibly smaller set of measure at least $\frac{1}{16}(1-\lambda)$ so that B_1 will consist of a union of 2^{m_1-1} dyadic intervals, and is periodic with period $\frac{1}{2^{m_1-1}}$.

Assume that $n_1, n_2, \ldots, n_{k-1}$, and $m_1, m_2, \ldots, m_{k-1}$, have been selected, and sets $B_1, B_2, \ldots, B_{k-1}$, have been determined.

We now select n_k from G so large that the associated integer m_k, defined to be the largest integer such that $\frac{1}{2^{m_k}} > \ell_{n_k-1}$, satisfies $m_k - m_{k-1} > d$. Further, we want m_k to be so large that χ_{B_j} and r_{m_k} are independent for each $j < k$. By selecting n_k large enough, we will be able to obtain this independence.

As before, we will have $\Delta_k r_{m_k} > 1-\lambda$ on a set of measure at least $2^{m_k}(\frac{1}{2})(\ell_{n_k-1} - \ell_{n_k}) > \frac{1}{4}(1-\lambda)$. As before, if we use only the dyadic intervals where $r_{m_k} > 0$, we have a set of measure at least $\frac{1}{8}(1-\lambda)$. Denote this set by B_k. We can replace B_k by a possilby smaller set of measure at least $\frac{1}{16}(1-\lambda)$ so that B_k will consist of a union of 2^{m_k-1} dyadic intervals, and is periodic with period $\frac{1}{2^{m_k-1}}$.

We now need to estimate $\Delta_k r_{m_j}$ for $j < k$. We now have averages that are much shorter than the lengths of intervals where r_{m_j} is constant. Hence only intervals of length ℓ_{n_k-1} located at the right hand side of each dyadic interval of length $\frac{1}{2^{m_k}}$ can contribute non-zero values. There are only 2^{m_j} such intervals, and hence the total measure of such intervals is at most $2^{m_j}\ell_{n_k-1} < \frac{2^{m_j}}{2^{m_k}}$. Denote this set by $E(k,j)$ and note that on this set we have $\Delta_k r_{m_k} < 2$.

We continue the construction until we reach n_N. We will also need estimates for $\Delta_k r_{m_j}$ where $j > k$. In this case the averages are long compared to the lengths of the intervals were r_{m_j} is constant. Hence both operators will be close to zero. In particular, we see that $D_{m_k-1} r_{m_j} \le \frac{\ell_{n_k-1}}{2^{m_j}}$ for all x, and the same estimate holds for D_{m_k}. Hence $\Delta_k r_{m_j} \le 2\frac{\ell_{n_k-1}}{2^{m_j}} < 2\frac{2^{m_k}}{2^{m_j}}$ for all x.

We are now ready to estimate $V_{q,D}f$. We have estimates very similar to those in the proof of Lemma 2.4:

$$\begin{aligned}
V_{q,D}f(x) &\ge \left(|\Delta_k\left(\frac{1}{M}\sum_{j=1}^{N} r_{m_j}\chi_{\tau\ge j}\right)|^q\right)^{\frac{1}{q}} \\
&\ge \frac{1}{M}\left(|\Delta_k\left(r_{m_k}\chi_{\tau\ge k}\right)|^q\right)^{\frac{1}{q}} - \left(|\Delta_k\left(\sum_{j=1}^{k-1} r_{m_j}\chi_{\tau\ge j}\right)|^q\right)^{\frac{1}{q}} \\
&\quad - \left(|\Delta_k\left(\sum_{j=k+1}^{N} r_{m_j}\chi_{\tau\ge j}\right)|^q\right)^{\frac{1}{q}} \\
&= A - B - C.
\end{aligned}$$

We first estimate A. Note that $\Delta_k(r_{m_k}\chi_{\tau>k}) = \Delta_k(r_{m_k})$ on the set $B_k \cap \{\tau \ge k\}$. Hence, if $x \in \{\tau \ge N\}$, we have $\Delta_k(r_{m_k}\chi_{\tau>k}) \ge 1-\lambda$. Recalling that $\mu\{\tau < N\} < \epsilon < \epsilon_0 < \frac{1-\lambda}{32}$, we see that $\mu(B_k \cap \{\tau \ge N\}) \ge \mu(B_k) - \mu(\{\tau < N\}) \ge \frac{1-\lambda}{16} - \frac{1-\lambda}{32}$. Hence if $x \in \{\tau \ge N\}$, we have

$$A > \frac{1}{M}(1-\lambda)\left(\sum_{k=1}^{N} |\chi_{B_k}|^q\right)^{\frac{1}{q}}.$$

On the set where $\tau \geq N$ the sets B_k are independent. Since we have $m(B_k \cap \{\tau \geq N\}) > \frac{1}{32}(1-\lambda)$, we see that by the Strong Law of Large Numbers, and our assumption on N, we have $A > \frac{1-\lambda}{64}\frac{1}{M}N^{\frac{1}{q}}$ except on a set of measure less than ϵ where $\tau < N$, and a set of measure ϵ where the strong law has not caused the average to be at least $\frac{1-\lambda}{64}$.

We must now show that the other two pieces are small, but these estimates are exactly the same as those for the corresponding pieces in the proof of Lemma 2.4.

We now note that $m(B > 1) \leq \|B\|_1 \leq 2\epsilon$ and $m(C > 1) \leq \|C\|_1 < 2\epsilon$. Hence $V_{q,D}f > \frac{1-\lambda}{64}L - 2$ except on a union of the sets $\{\tau < N\}$, $\{B > 1\}$, $\{C > 1\}$ and on the set where the strong law did not get us close enough to the average of the measures of B_k. The union of these sets is less than $6\epsilon < \epsilon_0$. Recalling the definition of L, we see that $V_{q,D}f > \frac{1-\lambda}{64}L - 2 \geq L_0$. □

To complete the proof of Theorem 3.1 we use the above lemma in the same way that we used Lemma 2.2 to obtain Theorem 2.1. The details are the same and we omit them. □

While it is clear that the restriction on the sequnce (ℓ_k) in Theorem 3.1 can be weakened, some condition that implies rapid growth is essential. To see this we note the following theorem.

THEOREM 3.3. *Assume that* $\sum_{k=2}^{\infty}(1 - \frac{\ell_k}{\ell_{k-1}})^q < \infty$. *Then the q-variation operator $V_{q,D}f(x)$ is finite a.e. for all bounded f. In fact $V_{q,D}$ is a bounded operator from $L^q[0,1)$ to itself.*

Proof. First we write

$$|D_{\ell_k}f(x) - D_{\ell_{k-1}}f(x)| = \left|\left(\frac{1}{\ell_k} - \frac{1}{\ell_{k-1}}\right)\int_0^{\ell_k} f(x+t)dt - \frac{1}{\ell_{k-1}}\int_{\ell_k}^{\ell_{k-1}} f(x+t)dt\right|.$$

Using the triangle inequality we see that

$$\begin{aligned} V_{q,D}f(x) &\leq \left(\sum_{k=1}^{\infty}\left|\left(\frac{1}{\ell_k} - \frac{1}{\ell_{k-1}}\right)\int_0^{\ell_k} f(x+t)dt\right|^q\right)^{\frac{1}{q}} \\ &\quad + \left(\sum_{k=1}^{\infty}\left|\frac{1}{\ell_{k-1}}\int_{\ell_k}^{\ell_{k-1}} f(x+t)dt\right|^q\right)^{\frac{1}{q}} \\ &= Af(x) + Bf(x). \end{aligned}$$

In the following, Mf denotes the standard (one-sided) Hardy-Littlewood maximal function defined by $Mf(x) = \sup_{y>0}|\frac{1}{y}\int_0^y f(x+t)dt|$. We know this operator is bounded on all L^p, $1 < p < \infty$.

We first show that $\|Af\|_q \le C_q \|f\|_q$. To see this we just write

$$\begin{aligned}
\|Af\|_q^q &\le \int_0^1 \sum_{k=1}^{\infty} \left| \left(\frac{1}{\ell_k} - \frac{1}{\ell_{k-1}} \right) \int_0^{\ell_k} f(x+t)dt \right|^q dx \\
&\le \sum_{k=1}^{\infty} \left(\frac{1}{\ell_k} - \frac{1}{\ell_{k-1}} \right)^q \ell_k^q \int_0^1 \left| \frac{1}{\ell_k} \int_0^{\ell_k} f(x+t)dt \right|^q dx \\
&\le \sum_{k=1}^{\infty} \left(1 - \frac{\ell_k}{\ell_{k-1}} \right)^q \int_0^1 (Mf(x))^q \, dx \\
&\le \sum_{k=1}^{\infty} \left(1 - \frac{\ell_k}{\ell_{k-1}} \right)^q \|Mf\|_q^q \\
&\le \sum_{k=1}^{\infty} \left(1 - \frac{\ell_k}{\ell_{k-1}} \right)^q c_q \|f\|_q^q \\
&\le C_q^q \|f\|_q^q.
\end{aligned}$$

For B we have a similar argument. We write

$$\begin{aligned}
\|B\|_q^q &\le \int_0^1 \sum_{k=1}^{\infty} \left| \frac{1}{\ell_{k-1}} \int_{\ell_k}^{\ell_{k-1}} f(x+t)dt \right|^q dx \\
&\le \sum_{k=1}^{\infty} \left(\frac{1}{\ell_{k-1}} \right)^q \int_0^1 \left| \int_{\ell_k}^{\ell_{k-1}} f(x+t)dt \right|^q dx \\
&\le \sum_{k=1}^{\infty} \left(\frac{1}{\ell_{k-1}} \right)^q \int_0^1 \left| \int_0^{\ell_{k-1}-\ell_k} f(x+t)dt \right|^q dx \\
&\le \sum_{k=1}^{\infty} \left(1 - \frac{\ell_k}{\ell_{k-1}} \right)^q \int_0^1 \left| \frac{1}{\ell_{k-1}-\ell_k} \int_0^{\ell_{k-1}-\ell_k} f(x+t)dt \right|^q dx \\
&\le \sum_{k=1}^{\infty} \left(1 - \frac{\ell_k}{\ell_{k-1}} \right)^q \int_0^1 |Mf(x)|^q dx \\
&\le \sum_{k=1}^{\infty} \left(1 - \frac{\ell_k}{\ell_{k-1}} \right)^q c_q \int_0^1 |f(x)|^q dx \\
&\le C_q^q \|f\|_q^q.
\end{aligned}$$

□

4. The ergodic case

We can also establish an ergodic theory analog to the above results. Let (X, Σ, m, τ) denote a dynamical system, with (X, Σ, m) a probability space, and τ an ergodic measurable, measure preserving point transformation from X to itself. Let $A_n f(x)$ denote the ergodic average of length n. That is, $A_n f(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k x)$. For an increasing sequence of positive integers (ℓ_k), $q \geq 1$, and each positive integer L, we can define the q-variation operator:

$$V_{q,E}^L f(x) = \left(\sum_{k=1}^{L} |A_{\ell_k} f(x) - A_{\ell_{k-1}} f(x)|^q\right)^{\frac{1}{q}},$$

and $V_{q,E} f(x) = \lim_{L\to\infty} V_{q,E}^L f(x)$. For $q = 2$ it is shown in [12] and [13] that this operator is bounded on L^p, $1 < p < \infty$, and is even weak(1,1). However for $q < 2$ we have the following theorem.

THEOREM 4.1. *Let (ℓ_k) denote an increasing sequence of positive integers such that* $\liminf_{k\to\infty} \frac{\ell_{k-1}}{\ell_k} < \lambda_0 < 1$. *Then there is an $f \in L^\infty(X)$ such that $\|f\|_\infty \leq 1$, but $V_{q,E} f = \infty$ a.e.*

Proof. To prove this we first prove a result on $\mathbb{Z}$. We will then transfer this to the dynamical system in the standard way. See [8]. For ϕ a function on $\mathbb{Z}$ we introduce the operator D defined by $D(\phi) = \lim_{R\to\infty} \frac{1}{2R}\sum_{r=-R}^{R} \phi(r)$ if the limit exists. We note that the limit will exist if ϕ is a periodic function. See [1] for a discussion of related issues.

For each n, let $\phi_n : \mathbb{Z} \to \{1, -1\}$ denote the "Rademacher function on $\mathbb{Z}$" defined by $\phi_n(k) = 1$ if $0 \leq k < 2^n$, $\phi_n(k) = -1$ if $2^n \leq k < 2^{n+1}$, and ϕ_n is periodic with period 2^{n+1}. We see that these functions are independent, and in particular, $D(\phi_n\phi_m) = 0$ for $m \neq n$. On $\mathbb{Z}$ we define the q-variation operator $V_{q,Z}^L$ by

$$V_{q,Z}^L \phi(r) = \left(\sum_{k=1}^{L} |A_{\ell_k}\phi(r) - A_{\ell_{k-1}}\phi(r)|^q\right)^{\frac{1}{q}},$$

where $A_n\phi(r) = \frac{1}{n}\sum_{k=0}^{n-1}\phi(r+k)$. With this notation, we can now state the following lemma.

LEMMA 4.2. *If* $\liminf_{k\to\infty} \frac{\ell_{k-1}}{\ell_k} = \lambda_0 < 1$ *then given L_0, ϵ_0, and λ such that $\lambda_0 < \lambda < 1$, there is an N_0 and ϕ such that $\|\phi\|_{\ell^\infty} \leq 1$ and such that if $B = \{r | V_{q,Z}^{N_0}\phi(r) > L_0\}$ then $D(\chi_B) > 1 - \epsilon_0$.*

Proof. This proof is similar to the proof of Lemma 3.2 in the differentiation case. Take $\epsilon < \frac{\epsilon_0}{6}$ and $L > \frac{64}{1-\lambda}(L_0 - 2)$. Select J as in the proof of Lemma 3.2. As in the proof of Lemma 3.2, select N and M. We also select an integer d so that $\frac{N}{2^d} < \epsilon$. By the hypothesis of the lemma, there is an infinte set of k's such that $\frac{\ell_{k-1}}{\ell_k} < \lambda$. Let G denote this set. Select $n_1 \in G$. Let m_1 be such that $2^{m_1-1} < \ell_{n_1} \leq 2^{m_1}$. In the following we will inductively select $n_1 < n_2 < n_3 < \ldots < n_N$ from G and corresponding m_k where $2^{m_k-1} < \ell_{n_k} \le 2^{m_k}$. Arguing as in the differentiation case, we define $\Delta_k\phi(r) = |A_{\ell_{n_k}}\phi(r) - A_{\ell_{n_k-1}}\phi(r)|$, and show that $\Delta_1\phi_{m_1} > \frac{1}{4}(1-\lambda)$ on a set B_1, where $D(B_1) > \frac{1}{8}(1-\lambda)$ and B_1 consists of a union of dyadic "intervals" in $\mathbb{Z}$, and is periodic with period 2^{m_1}. We then select $n_2 \in G$ so large that we have independence of B_1 and ϕ_{m_2}. We can also assume that $m_2 - m_1 > d$. We then continue the process as in the differentiation case, and show very similar estimates. As in the differentiation case we need to construct a function with small enough ℓ^∞ norm. In the differentiation case we used a stoppping time. Here we need to reverse time; that is, after we construct our sequence $(n_k)_{k=1}^N$, we consider the "reverse stopping time", τ, defined by $\tau(r) = \inf\{n \,|\, |\frac{1}{M}\sum_{j=0}^{n}\phi_{n_{N-j}}(r)| \ge 1\}$. (This reversal is necessary since we have the large intervals where ϕ_n is constant when n is large.) Using this reversed time, we construct ϕ_N^τ as in the martingale and differentiation case. Since the details are now nearly the same, we omit them. □

We now establish the ergodic theory version of the above lemma.

LEMMA 4.3. *Given a large number L and a small number $\epsilon > 0$, there is an N and a function f such that $\|f\|_\infty < 1$ and such that $V_{q,E}^N f(x) > L$ for $x \in X_0 \subset X$, with $m(X_0) > 1-\epsilon$.*

Proof. We use Lemma 4.2. Let ϕ be the function given by Lemma 4.2 so that $D(B) > 1 - \frac{\epsilon}{3}$. Find R_0 so that if $R > R_0$ then $\frac{1}{2R}\sum_{k=-R}^{R}\chi_B(k) > 1-\frac{\epsilon}{3}$. Note that the operator $V_{q,Z}^N$ only looks a finite distance into the future. Denote this distance by J. Select $R > R_0$ so that $\frac{J}{2R} < \frac{\epsilon}{3}$. Make a Rohlin tower of height $2R+J$ such that the tower has measure at least $1-\frac{\epsilon}{3}$. Note that we have $\sum_{k=-R}^{R}\chi_B(k) > 2R(1-\frac{\epsilon}{3}$. For x in the base of the tower, define $f(\tau^n x) = \phi(n)$ for $n = -R, -(R-1), \dots, R+J$, then we see that for x in the bottom $2R$ steps of the tower, with the exception of a bad collection of $2R\frac{\epsilon}{3}$ steps, we have $V_{q,E}f(x) > L$. Hence we have $m\{x\colon V_q f(x) > L\} \ge \frac{2R(1-\frac{\epsilon}{3})}{2R+J} - \frac{\epsilon}{3} > 1-\epsilon$. □

We can now complete the proof of Theorem 4.1. Note that $\{x\colon V_{q,E}f(x) = \infty\}$ is invariant, so if for every $q < 2$ we can show $m\{x : V_{q,E}f(x) = \infty\} > 0$ then we are done.

Fix a sequence (q_k) so that $q_k \to 2$, and $q_k < 2$ for each k. We will use the lemma inductively to construct the desired function. Let $V_{q,E}^N f(x) = (\sum_{k=1}^N |A_{\ell_k}f(x) - A_{\ell_{k+1}}f(x)|^q)^{\frac{1}{q}}$. First let $q = q_1$ and $L_1 = 100 \times 2^1$. Use the lemma to find a function f_1 with $\|f_1\|_\infty \le 1$ and $V_{q,E}f_1(x) > L_1$ on at least $\frac{7}{8}$ of the space. We can find an integer N_1 so that in fact $V_{q,E}^{N_1}f_1(x) > L_1$ on at least $\frac{3}{4}$ of the space. Let $b_1 = g_1 = \frac{1}{2}f_1$. Then $\|g_1\|_\infty \le \frac{1}{2}$ and $V_{q,E}^{N_1}g_1 > 100$ on a set of size at least $\frac{3}{4}$.

If for all $q < 2$, $V_{q,E}g_1 = \infty$ on a set of positive measure, we are done, so we can assume for some $q < 2$, that $V_{q,E}g_1(x) < \infty$ a.e. Thus we can find a number B_1 and $q' < 2$ such that $m\{x \,|\, V_{q',E}g_1 < B_1\} > \frac{1}{2}$.

Next let $q = \max(q', q_2)$ and take $L_2 = 100 \times 2^2 \times B_1 \times N_1$. We find f_2 with $\|f_2\|_\infty \leq 1$ and with $V_{q,E} f_2 > L_2$ on a set of measure at least $\frac{7}{8}$.

Define $g_2 = \frac{1}{2^2 N_1} f_2$. Then $V_q g_2(x) > 100 B_1$ on a set of size at least $\frac{7}{8}$ and $\|g_2\|_\infty \leq \frac{1}{2^2 N_1}$. Define $b_2 = b_1 + g_2$. We have $V_{q,E}(g_2) = V_{q,E}(b_2 - b_1) \leq V_{q,E}(b_2) + V_{q,E}(b_1)$. Hence $V_{q,E}(b_2) \geq V_q(g_2) - V_q(b_1) \geq 100 B_1 - B_1 = 99 B_1 \geq 99^2$ on at least $\frac{7}{8} - \frac{1}{2}$ of the space. Further, we can find N_2 so large that $V_{q,E}^{N_2} b_2 > 99^2$ on a set of measure at least $\frac{1}{4}$.

In the same way we construct a sequence $b_1, b_2, \ldots, b_k$, and an increasing sequence $N_1 < N_2 < \cdots < N_k$ with $V_{q,E}^{N_j} b_j > 99^j$ on a set of measure at least $\frac{1}{4}$ for $j = 1, 2, \ldots, k$. We now construct b_{k+1}. Find $q \geq q_k$ and B_k so large that $m\{x : V_{q,E} b_k < B_k\} > \frac{7}{8}$. (As before, such a q and B_k exists, since otherwise for all $q < 2$, $V_{q,E} b_k = \infty$ on a set of positive measure and we are done.) Let $L_{k+1} = 100 B_k \times 2^{k+1} \times N_k$. Find f_{k+1} by the lemma. Let $g_{k+1} = \frac{1}{2^{k+1} N_k} f_{k+1}$. Then $\|g_{k+1}\|_\infty \leq \frac{1}{2^{k+1} N_k}$ and $V_{q,E} g_{k+1} > 100 B_k$ on a set of measure at least $\frac{7}{8}$. Define $b_{k+1} = b_k + g_{k+1}$. We have $V_{q,E}(g_{k+1}) = V_{q,E}(b_k + g_{k+1} - b_k) \leq V_{q,E}(b_{k+1}) + V_{q,E}(b_k)$. Hence $V_{q,E}(b_{k+1}) \geq V_{q,E}(g_{k+1}) - V_{q,E}(b_k) \geq 100 B_k - B_k = 99 B_k \geq 99^{k+1}$ on a set of measure at least $\frac{3}{8}$. Then there is an N_{k+1} such that $V_{q,E}^{N_{k+1}}(b_{k+1}) > 99^{k+1}$ on at least $\frac{1}{4}$ of the space.

We continue the construction, and define a function $b = \lim_{k\to\infty} b_k$. The limit exists since we have uniform convergence. We will be done if we can show that for all $q < 2$ we have $V_q b = \infty$ on a set of positive measure.

We note that for each $k \geq 1$ we have for all $q \leq q_k$, that $V_{q,E}^{N_k} b_k = V_{q,E}^{N_k}((b_k - b) + b) \leq V_{q,E}^{N_k}(b - b_k) + V_{q,E}^{N_k}(b)$. Hence $V_{q,E}^{N_k} b \geq V_{q,E}^{N_k}(b_k) - V_{q,E}^{N_k}(b - b_k) \geq 99^k - N_k \|b - b_k\|_\infty$. We note that $\|b - b_k\|_\infty = \|g_{k+1} + g_{k+2} + \ldots\|_\infty \leq \sum_{j=k+1}^\infty \|g_j\|_\infty \leq \sum_{j=k+1}^\infty \frac{1}{2^j N_{j-1}} \leq \frac{1}{N_k}$. Hence for each $q \leq q_k$ we have $V_{q,E}^{N_k} b \geq 99^k - N_k \frac{1}{N_k} \geq 99^k - 1$ on a set of size at least $\frac{1}{4}$.

From this we see that for each $q < 2$, $V_{q,E}(b) = \infty$ on a set of positive measure, and we are done. □

Theorem 4.1 has the following corollary.

COROLLARY 4.4. *Given any increasing sequence (n_k), of positive integers, there is an $f \in L^\infty(X)$ such that $\|f\|_\infty \leq 1$, but $V_{1,E} f = \infty$ a.e.*

Proof. Let (ℓ_k) denote a subsequence of the (n_k) such that (ℓ_k) satisfies the hypothesis of Theorem 4.1. Let $I_k = \{n_j | \ell_{k-1} < n_j \leq \ell_k\}$. We know that we can find a function so that $\|f\|_\infty \leq 1$, but such that the 1-variation associated with the sequence (ℓ_k) is infinite. Now just note that by the triangle inequality, $|A_{\ell_k} f(x) - A_{\ell_{k-1}} f(x)| \leq \sum_{j \in I_k} |A_{n_j} f(x) - A_{n_{j-1}} f(x)|$. □

We also have the following analog to Theorem 3.3. Since the proof is almost the same as the proof of Theorem 3.3, we omit the proof.

THEOREM 4.5. *Let (X, Σ, m, τ) denote a dynamical system. Assume that $\sum_{k=2}^{\infty}(1 - \frac{\ell_{k-1}}{\ell_k})^q < \infty$. Then the q-variation operator $V_{q,E} f(x)$ is finite a.e. for all bounded f. In fact $V_{q,E}$ is always a bounded operator from $L^q(X)$ to itself.*

As we saw in Corollary 4.4, the case $q = 1$ yields a divergent operator for all subsequences. The triangle inequality shows that the 1-variation along a subsequence is less than the 1-variation along the full sequence, $n_k = k$. The following theorem shows that the 1-variation operator for the full sequence is only finite on constant functions.

THEOREM 4.6. *Let τ is any measure preserving ergodic transformation on a non-atomic probability space. Consider the 1-variation operator with $n_k = k$. If f is any non-constant function in $L^1(X)$ then $V_{1,E} f(x) = +\infty$ for a.e. x.*

Proof. Let $B(\epsilon) = \{x \| f(x) - \int_X f(t)dt| > \epsilon\}$. Since f is not a constant, there is some $\epsilon_0 > 0$ such that $m(B(\epsilon_0) > 0$. Let $B = B(\epsilon_0)$. Write

$$\begin{aligned} |A_k f(x) - A_{k+1} f(x)| &= \left| \frac{1}{k+1} A_k - f(\tau^k x) \frac{1}{k+1} \right| \\ &= \left| A_k f(x) - f(\tau^k x) \right| \frac{1}{k+1}. \end{aligned}$$

For a.e. x there is an integer $n(x)$ such that for $k > n(x)$ we have

$$\left| A_k f(x) - \int_X f(t)dt \right| < \frac{\epsilon_0}{2}.$$

For such x, if $k > n(x)$ we have

$$\begin{aligned} |A_k f(x) - A_{k+1} f(x)| &= \left| A_k f(x) - \int f(t)dt + \int f(t)dt - f(\tau^k x) \right| \frac{1}{k+1} \\ &\geq \left| \int f(t)dt - f(\tau^k x) \right| \frac{1}{k+1} - \frac{\epsilon_0}{2(k+1)}. \end{aligned}$$

Hence if in addition, $\tau^k x \in B$ then we have

$$\begin{aligned} |A_k f(x) - A_{k+1} f(x)| &\geq \epsilon_0 \frac{1}{k+1} - \frac{\epsilon_0}{2(k+1)} \\ &= \frac{\epsilon_0}{2(k+1)}. \end{aligned}$$

Hence

$$V_{q,E}f(x) \geq \sum_{k=n_0}^{\infty} \frac{\epsilon_0}{2(k+1)} \chi_B(\tau^k x).$$

However we see

$$\begin{aligned}\sum_{k=n_0}^{\infty} \frac{\epsilon_0}{2(k+1)} \chi_B(\tau^k x) &= \frac{\epsilon_0}{2} \sum_{k=n_0}^{\infty} \chi_B(\tau^k x) \sum_{j=k}^{\infty} \frac{1}{j+1} - \frac{1}{j+2} \\ &= \frac{\epsilon_0}{2} \sum_{j=n_0}^{\infty} \frac{1}{(j+1)(j+2)} \sum_{k=n_0}^{j} \chi_B(\tau^k x).\end{aligned}$$

For some N_0 large enough, we have

$$\frac{1}{j+1} \sum_{k=n_0}^{j} \chi_B(\tau^k x) > \frac{m(B)}{2} \quad \text{for all } j > N_0.$$

Hence $V_{q,E}f(x) \geq \frac{\epsilon_0}{2} \frac{m(B)}{2} \sum_{j=N_0}^{\infty} \frac{1}{j+2} = \infty$. □

5. The analytic case

Our aim is to construct a function F which is analytic in the open unit disc $D = \{ z \mid |z| < 1 \}$, continuous in the closed unit disc, and a sequence of numbers $\{ r_k \}$, $0 < r_k < r_{k+1} < 1$, such that

$$\sum_{k=1}^{\infty} |F(r_{k+1}e^{2\pi it}) - F(r_k e^{2\pi it})|^q = \infty$$

for each $q \in [1, 2)$ and for a.e. t in the unit circle $\mathsf{T} = [0, 1)$, considered as a measure space with Lebesgue measure μ. The construction is similar to the one in the martingale case. Recall that in the martingale case we considered a function $F(x) = \sum_{k=1}^{\infty} a_k r_k(x)$ such that $\sum_k a_k^2 \leq 1$ but $\sum_k a_k^q = \infty$ for each q, $1 \leq q < 2$. To keep the functions bounded we modified the function on a small set and actually considered $\sum_k a_k r_k(x) \chi_{\tau \geq k}(x)$. To obtain our analytic function we will do much the same kind of construction. We will consider

$$F(re^{2\pi it}) = \sum_{k=1}^{\infty} a_k r^{n_k} e^{2\pi i n_k t},$$

where n_ks are selected so that $e^{2\pi i n_k t}$ behave much like a sequence of independent identically distributed random variables. Then $\sum_k a_k e^{2\pi i n_k t}$ will behave much like the martingale example. We will have $\sum_k a_k^q = \infty$. We will also select a sequence r_m so that $r_m^{n_k} \cong 1$ for $k \leq m$ and $r_m^{n_k} \cong 0$ for $k > m$. If we can achieve this, then

$$|F(r_{k+1}e^{2\pi it}) - F(r_k e^{2\pi it})| \cong |a_{k+1}|$$

and hence

$$\sum_k |F(r_{k+1}e^{2\pi it}) - F(r_k e^{2\pi it})|^q \cong \sum_k |a_{k+1}|^q = \infty .$$

As in the martingale case we will need to introduce a stopping time to keep the function bounded. Since we will want our final function to be analytic, we will need to approximate $\chi_{\tau \geq n}$ by the boundary values of an analytic function. The introduction of this "analytic stopping time" will result in a change on only a very small set.

The series

$$F(re^{2\pi it}) = \sum_{n=0}^{\infty} s_n r^n e^{2\pi int}$$

will be defined in blocks. We will now explain the basic construction used to obtain these polynomials. In the course of the main proof, however, there will be additional requirements on this basic construction. By an analytic polynomial on T we mean the boundary function corresponding to a polynomial in z.

The basic construction. Let $\beta > 0$ and $\xi > 0$ be two numbers, $L \geq 2$ an integer, and $\alpha = \beta/L$. Let $\psi_k(t) = e^{2\pi i b_k t}$, where the b_k' s are positive integers to be specified later. We will define a sequence of continuous functions g_k: $\mathsf{T} \to \mathsf{C}$ and a sequence analytic polynomials h_k. Let $g_0 = h_0 = 0$. If $g_0, \ldots, g_k$ are already defined, $k \geq 0$, let

$$H_k = \{ t \mid t \in \mathsf{T},\ |(g_0 + \cdots + g_k)(t)| < \beta \} .$$

Find a set $E_k \subset H_k$ of measure $\mu(E_k) < 2^{-(k+3)}\xi$ and a continuous function $\varphi_k : \mathsf{T} \to [0, 1]$ such that $\varphi_k(t) = 0$ if and only if $t \in G_k = \mathsf{T} - H_k$ and $\varphi_k(t) = 1$ if $t \in H_k - E_k$. Also, find an integer $v_k \geq 1$ and a Fejer polynomial $\theta_k(t) = \sum_{|j|<v_k} c_j e^{2\pi ijt}$, such that $|\varphi_k(t) - \theta_k(t)| < 2^{-(k+3)}$ for all $t \in \mathsf{T}$. Then define $g_{k+1} = \alpha\varphi_k\psi_{k+1}$ and $h_{k+1} = \alpha\theta_k\psi_{k+1}$. The positive integer b_{k+1} appearing in ψ_{k+1} will be chosen sufficiently large such that h_{k+1} is an analytic polynomial and such that all the inner products (g_j, g_{k+1}), $0 \leq j \leq k$, have moduli less than $2^{-2(k+1)}\alpha^2/(k + 1)$. Later, in the main proof, there will be another condition to be satisfied that will require that b_{k+1} must also be larger than another lower bound.

LEMMA 5.1. *Let $1 \leq p < 2$. Let $\beta > 0$ and $\xi > 0$ be two numbers. Then there are integers $L \geq 2$ and $K \geq 1$ such that if the functions g_k and h_k, $1 \leq k \leq K$, are constructed as in* (2), *corresponding to the numbers β, ξ, and to this integer L, then they satisfy the following conditions.*

1. $|g_1 + \cdots + g_k| < 2\beta$ *and* $|h_1 + \cdots + h_k| < 3\beta$ *for* $1 \leq k \leq K$.

2. *There is a set $B \subset \mathsf{T}$ such that $\mu(B) < \xi$ and, for all $t \in \mathsf{T} - B$,*

$$|g_1(t)|^q + \cdots + |g_K(t)|^q > 2,$$

$$|h_1(t)|^q + \cdots + |h_K(t)|^q > 1,$$

and $|h_k(t)| \geq (7/8)|g_k(t)| = (7/8)\alpha$ for $1 \leq k \leq K$.

Proof. An easy induction shows that $|g_1 + \cdots + g_k| < 2\beta$ for all $1 \leq k \leq K$. Since

$$|g_k - h_k| < 2^{-(k+3)}|g_k| = 2^{-(k+3)}\alpha = 2^{-(k+3)}\beta/L,$$

we also have $|h_1 + \cdots + h_k| < 3\beta$ for $1 \leq k \leq K$. To prove (2) define $B = \cup_{k=1}^K (G_k \cup E_k)$. Since G_ks form an increasing sequence of sets, B is also equal to the union of G_K with $\cup_{k=1}^K E_k$. Note that if $t \in \mathsf{T} - B$ then $|g_k(t)| = \alpha$ for each k, $1 \leq k \leq K$. Hence

$$|g_1(t)|^q + \cdots + |g_K(t)|^q = K\alpha^p$$

for $t \in \mathsf{T} - B$. Now,

$$\mu(B) \leq \mu(G_K) + \sum_{k=1}^K \mu(E_k) \leq \mu(G_K) + \sum_{k=1}^K 2^{-(k+3)}\xi < \mu(G_K) + \xi/2.$$

Hence the proof will be completed by showing that there is a choice for K and $L = \beta/\alpha$ such that $K\alpha^p = K\beta^q/L^q > 2$ and $\mu(G_K) < \xi/2$. Since $|(g_0 + \cdots + g_K)(t)| \geq \beta$ on G_K,

$$\begin{aligned} \beta^2\mu(G_K) &\leq \|g_0 + \cdots + g_K\|_2^2 \\ &\leq \sum_{k=1}^K \|g_k\|_2^2 + 2\sum_{k=1}^K \sum_{j=1}^{k-1} |(g_j,\, g_k)| \\ &\leq K\alpha^2 + 2\sum_{k=2}^K \sum_{j=1}^{k-1} 2^{-2(k+1)}\alpha^2/(k+1) \\ &\leq K\alpha^2 + \alpha^2 = (K+1)\alpha^2 = (K+1)\beta^2/L^2. \end{aligned}$$

Since $1 \leq q < 2$, it is clear that there are integers L and K such that $(K+1)/L^2 < \xi/2$ and $K\beta^q/L^q > 2$. $\square$

THEOREM 5.2. *There is a function F which is analytic in the open unit disc $D = \{z \mid |z| < 1\}$ and continuous on the closed unit disc, and a sequence of numbers $\{r_k\}$, $0 < r_k < r_{k+1} < 1$, such that*

$$\sum_{k=1}^{\infty} |F(r_{k+1}e^{2\pi it}) - F(r_k e^{2\pi it})|^q = \infty$$

for each $q \in [1,\ 2)$ and for a.e. t in the unit circle T.

Proof. Choose $\beta^{(m)} > 0, \xi^{(m)} > 0$, and $1 \le q^{(m)} < 2$ such that $\beta = \sum_{m=1}^{\infty} \beta^{(m)} < 1$, $\xi = \sum_{m=1}^{\infty} \xi^{(m)} < 1$, and such that $q^{(m)}$ is an increasing sequence converging to 2. Use Lemma 5.1 to find the integers $L^{(m)} \ge 1$ and $K^{(m)} \ge 1$ for each m, such that if the functions $g_1^{(m)}, \dots, g_{K^{(m)}}^{(m)}$ and $h_1^{(m)}, \dots, h_{K^{(m)}}^{(m)}$ are constructed as in part 2 of Lemma 5.1, using the parameters $\beta^{(m)}$, $L^{(m)}$, $\alpha^{(m)} = \beta^{(m)}/L^{(m)}$, and $\xi^{(m)}$, then they satisfy the following conditions.

(5.1) $|h_1^{(m)} + \cdots + h_k^{(m)}| < 3\beta^{(m)}$ for $1 \le k \le K^{(m)}$.

(5.2) There is a set $B^{(m)} \subset \mathsf{T}$ such that $\mu(B^{(m)}) < \xi^{(m)}$ and, for $t \in \mathsf{T} - B^{(m)}$,

$$|h_1^{(m)}(t)|^{q^{(m)}} + \cdots + |h_{K^{(m)}}^{(m)}(t)|^{q^{(m)}} > 1 .$$

It will be more convenient to arrange these functions as single sequences g_n and h_n, consisting of succesive blocks of lenght $K^{(m)}$. We denote the parameters used in the construction of g_n and h_n by the corresponding subindex. Hence $|g_n(t)| = \alpha_n$ for all $t \in H_n - E_n$, and $|h_n(t)| \ge (7/8)|g_n(t)| = (7/8)\alpha_n$, again for all $t \in H_n - E_n$. From (5.1) and (5.2) we see easily that the following are true.

(5.3) The series $\sum_n h_n(t)$ is uniformly convergent on $t \in \mathsf{T}$.

(5.4) For all $q < 2$ and for all t that belongs to infinitely many of the sets $\mathsf{T} - B^{(m)}$, we have $\sum_{n=1}^{\infty} |h_n(t)|^q = \infty$. Since $\sum_{m=1}^{\infty} \mu(B^{(m)}) < \infty$, we see that $\sum_{n=1}^{\infty} |h_n(t)|^q = \infty$ for a.e. $t \in \mathsf{T}$.

Recall that each h_n is an analytic polynomial, which will be written as

$$h_n(t) = \sum_{w=x_n}^{y_n} q_w^{(n)} e^{2\pi i w t} ,$$

where x_n and y_n are positive integers, $x_n \le y_n$. We will let $Q_n = \sum_{w=x_n}^{y_n} |q_w^{(n)}|$. We will also write

$$h_n(t, r) = \sum_{w=x_n}^{y_n} q_w^{(n)} r^w e^{2\pi i w t} ,$$

where $r > 0$. Hence, $h_n(t) = h_n(t, 1)$ with this notation. Further approximations will be controlled by a sequence $\varepsilon_k > 0$ such that $\sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \varepsilon_j < 1$, and such that $\sum_{j=k}^{\infty} \varepsilon_j < (1/8)\alpha_{k+1}$ for all $k \ge 1$.

The functions g_n and h_n will be obtained following the basic construction within each block of length $K^{(m)}$, with an additional requirement. The sequence of numbers r_n will be constructed simultaneously. We start with $g_1 = \alpha_1 \psi_1$, where $\psi_1(t) = e^{2\pi i b_1 t}$, with any positive integer b_1, for example with $b_1 = 1$. At this step we take $h_1 = g_1$. Hence $x_1 = y_1 = b_1$ and $Q_1 = \alpha_1$. We then choose r_1 such that $0 < r_1 < 1$ and such that $(1 - r_1^{y_1})Q_1 < \varepsilon_1$. Assume that the functions $g_1, \dots, g_k, h_1, \dots, h_k$, and the numbers $0 < r_1 < \cdots < r_k < 1$ are already chosen.

If both of the integers k and $k+1$ are within the same mth block, then g_{k+1} and h_{k+1} are constructed as in the basic construction, with an additional condition on the choice of the integer b_{k+1}. First note that Q_{k+1} is determined before the choice of b_{k+1}. Now choose b_{k+1} sufficiently large so that x_{k+1} satisfies

$$r_k^{x_{k+1}} Q_{k+1} < \varepsilon_{k+1} .$$

The purpose of this choice is to have

$$|h_{k+1}(t, r_k)| < \varepsilon_{k+1}$$

for all $t \in \mathsf{T}$. Finally we choose r_{k+1}, $r_k < r_{k+1} < 1$ such that

$$(1 - r_{k+1}^{y_{k+1}}) \sum_{j=1}^{k+1} Q_j < \varepsilon_{k+1} .$$

This implies that, for all $t \in \mathsf{T}$,

$$\sum_{j=1}^{k+1} |h_j(t, r_{k+1}) - h_j(t, 1)| < \varepsilon_{k+1} .$$

The situation is simpler if g_k is the last function in the mth block. In this case g_{k+1} will be the first function in the $(m+1)$st block. It will be of the form $g_{k+1} = \alpha_{k+1}\psi_{k+1}$. After choosing the positive integer b_{k+1} appearing in ψ_{k+1}, we are going to let $g_{k+1} = h_{k+1}$, with $x_{k+1} = y_{k+1} = b_{k+1}$ and $Q_{k+1} = \alpha_{m+1}$. This integer b_{k+1} is chosen sufficiently large to satisfy $r_k^{x_{k+1}} Q_{k+1} < \varepsilon_{k+1}$. Finally we choose r_{k+1}, $r_k < r_{k+1} < 1$ such that

$$(1 - r_{k+1}^{y_{k+1}}) \sum_{j=1}^{k+1} Q_j < \varepsilon_{k+1} .$$

Again, we see that

$$|h_{k+1}(t, r_k)| < \varepsilon_{k+1}$$

and

$$\sum_{j=1}^{k+1} |h_j(t, r_{k+1}) - h_j(t, 1)| < \varepsilon_{k+1}$$

for all $t \in \mathsf{T}$.

Finally note that, if $n > k$ then $r_k \le r_{n-1}$. This implies that $|h_n(t, r_k)| < \varepsilon_n$ whenever $n > k$.

We now let $F(re^{2\pi i t}) = \sum_{n=1}^{\infty} h_n(t, r)$, where, as defined earlier,

$$h_n(t, r) = \sum_{w=x_n}^{y_n} q_w^{(n)} r^w e^{2\pi i w t} .$$

Then F is analytic in the open unit disc and continuous in the closed unit disc, as follows from (5.3). We see that

$$|F(r_k e^{2\pi i t}) - \sum_{n=1}^{k} h_n(t, r_k)| \leq \sum_{n=k+1}^{\infty} |h_n(t, r_k)| < \sum_{n=k+1}^{\infty} \varepsilon_n .$$

Also, since

$$\sum_{n=1}^{k} |h_n(t, r_k) - h(t, 1)| < \varepsilon_k ,$$

we see that

$$|F(r_k e^{2\pi i t}) - \sum_{n=1}^{k} h_n(t, 1)| = |F(r_k e^{2\pi i t}) - \sum_{n=1}^{k} h_n(t)| < \sum_{n=k}^{\infty} \varepsilon_n .$$

Hence

$$|[F(r_{k+1} e^{2\pi i t}) - F(r_k e^{2\pi i t})] - h_{k+1}(t)| \leq 2 \sum_{n=k}^{\infty} \varepsilon_n < (1/4)\alpha_{k+1} .$$

If $t \in H_{k+1} - E_{k+1}$ then

$$\alpha_{k+1} = |g_{k+1}(t)| < (8/7)|h_{k+1}(t)| .$$

Hence, for $t \in H_{k+1} - E_{k+1}$,

$$\begin{aligned} |F(r_{k+1} e^{2\pi i t}) - F(r_k e^{2\pi i t})| &> |h_{k+1}(t)| - 2 \sum_{n=k}^{\infty} \varepsilon_n \\ &> |h_{k+1}(t)| - (1/4)\alpha_{k+1} > (5/7)|h_{k+1}(t)| . \end{aligned}$$

Hence (5.4) shows that

$$\sum_{k=1}^{\infty} |F(r_{k+1} e^{2\pi i t}) - F(r_k e^{2\pi i t})|^q = \infty$$

for all $q < 2$ and for all a.e. $t \in \mathsf{T}$. This completes the proof of the theorem. □

If $q = 2$ then the prior construction cannot be made. In fact we have the following positive result in the case $q = 2$.

THEOREM 5.3. *Let $(r_k) \subset [0, 1)$ be an increasing sequence. Let $f \in L^2[0, 1)$ where $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\theta}$. Let $F(r, \theta) = \sum_{n=-\infty}^{\infty} a_n r^n e^{2\pi i n\theta}$. Define the square function $Sf(\theta) = (\sum_{k=1}^{\infty} |F(r_{k+1}, \theta) - F(r_k, \theta)|^2)^{\frac{1}{2}}$. Then we have $\|Sf\|_2 \leq \|f\|_2$.*

Proof. Using Parseval's equality we write

$$\begin{aligned}
\|Sf\|_2^2 &= \int_0^1 \sum_{k=1}^{\infty} |F(r_{k+1},\theta) - F(r_k,\theta)|^2 \, d\theta \\
&= \sum_{k=1}^{\infty} \int_0^1 \left| \sum_{n=-\infty}^{\infty} a_n r_{k+1}^n e^{2\pi i n\theta} - \sum_{n=-\infty}^{\infty} a_n r_k^n e^{2\pi i n\theta} \right|^2 d\theta \\
&= \sum_{k=1}^{\infty} \int_0^1 \left| \sum_{n=-\infty}^{\infty} a_n (r_{k+1}^n - r_k^n) e^{2\pi i n\theta} \right|^2 d\theta \\
&= \sum_{k=1}^{\infty} \sum_{n=-\infty}^{\infty} |a_n|^2 (r_{k+1}^n - r_k^n)^2 \\
&\le \sum_{n=-\infty}^{\infty} |a_n|^2 \left[\sum_{k=1}^{\infty} r_{k+1}^n - r_k^n \right] \\
&\le \sum_{n=-\infty}^{\infty} |a_n|^2 \\
&= \|f\|_2^2.
\end{aligned}$$

□

References

1. M. Akcoglu, A. Bellow, R. Jones, V. Losert, K. Reinhold, and M. Wierdl, *The strong sweeping out property for Riemann sums, convolution powers, and related matters*, To appear in Ergodic Theory and Dynamical Systems.
2. J. Bourgain, *On the radial variation of bounded analytic functions on the disc*, Duke Math. J. **69** (1983), 671–682.
3. ________, *Pointwise ergodic theorems for arithmetic sets*, Publ. Math. Institut Hautes Etudes Scientifiques **69** (1989), 5–45.
4. D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. of Prob. **1** (1973), 19–42.
5. ________, *Martingale transforms*, Ann. Math. Statist. **37** (1966), 1494–1504.
6. ________, Personal communication.
7. D. L. Burkholder, R. Gundy and M. Silverstein, *A maximal function characterization of the class H^p*, Tran. Amer. Math. Soc. **157** (1971), 137–153.
8. A. P. Calderón, *Ergodic theory and translation invariant operators*, Proc. Nat. Acad. Sci. U.S.A. **59** (1968), 349–353.
9. V. F. Gaposhkin, *A theorem on the convergence almost everywhere of a sequence of measurable functions, and its applications to sequences of stochastic integrals*, Math. USSR Sbornik **33** (1977), 1–17.
10. ________, *Individual ergodic theorem for normal operators on L^2*, Functional Anal. Appl. **15** (1981), 14–18.
11. R. L. Jones, *Inequalities for the ergodic maximal function*, Studia Math. **40** (1977), 111–129.
12. R. L. Jones, I. Ostrovskii and J. Rosenblatt, *Square functions in ergodic theory*, to appear in Ergodic Theory and Dynamical Systems, preprint 52 pages.
13. R. L. Jones, R. Kaufman, J. Rosenblatt and M. Wierdl, *Oscillation in ergodic theory*, preprint.
14. J. Rosenblatt and M. Wierdl, *Pointwise ergodic theorems via harmonic analysis*, Proc. Conference on Ergodic Theory, Alexandra Egypt, 1993.

15. W. Rudin, *The radial variation of analytic functions*, Duke Math. J. **22** (1955), 235–242.
16. E. M. Stein, *Singular integrals and differentiablity properties of functions*, Princeton University Press, Princeton, N.J., 1970.
17. ———, *The development of square functions in the work of A. Zygmund*, Bull. Amer. Math. Soc. **7** (1982), 359–376.

Mustafa A. Akcoglu, Department of Mathematics, University of Toronto, Toronto, Ontario M5S 1A1, Canada
akcoglu@math.toronto.edu

Roger L. Jones, Department of Mathematics, DePaul University, 2219 N. Kenmore, Chicago IL 60614 USA
rjones@condor.depaul.edu

Peter O. Schwartz, Department of Mathematics, University of Toronto, Toronto, Ontario M5S 1A1, Canada
schwartz@math.toronto.edu