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CONTINUITY AND ANALYTICITY OF FAMILIES OF SELF-ADJOINT DIRAC OPERATORS ON A MANIFOLD WITH BOUNDARY

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Section 1

Given a continuous or analytic family D_t of self-adjoint elliptic operators on a manifold X, it is often useful to know whether the *spectrum* of D_t also varies continuously or analytically. If X is a closed manifold, the answer to this question is well known to be yes (assuming, in the analytic case, that the parameter space is an interval in the reals). The key point here is that because the domain of the operator is independent of the parameter t, one may apply standard theorems on deformations of self-adjoint operators to conclude that the spectrum varies in as nice a way as the operator. An excellent reference for these theorems is Kato's book, *Perturbation Theory For Linear Operators* [K].

If X is a compact manifold with boundary the situation is not as simple because one must choose boundary conditions in order for the operator to be self-adjoint; if these boundary conditions vary with the value of the parameter then the domain of the operator is changing. It is reasonable to expect a theorem that states essentially that given a continuous (resp. analytic) family of formally self-adjoint operators on a manifold X with boundary, and a continuous (resp. analytic) path of self-adjoint boundary conditions, one may conclude the the spectrum varies continuously (resp. analytically). In the context of Dirac operators and Atiyah-Patodi-Singer (APS) boundary conditions, this is precisely what our main theorem says.

THEOREM. *Let T be a topological space and $D_t = D_0 + A_t$ a family of formally self-adjoint Dirac operators on an odd-dimensional manifold with boundary in cylindrical form near the boundary. Suppose that the dimension of the kernel of the tangential operator $\hat{D}_t$ is independent of t. Choose a family $L(t)$ of Lagrangians in the kernel of $\hat{D}_t$. Then:*

1. *If the map $t \mapsto A_t$ is a continuous map into the space of smooth bundle endomorphisms and the family $L(t)$ is continuous then the self-adjoint family obtained by imposing APS boundary conditions on D_t is continuous in the graph topology; in particular the spectrum of D_t with $P_+(t) + L(t)$ boundary conditions depends continuously on t.*

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2. *If T is an interval and the map $t \mapsto A_t$ is analytic into the space of smooth bundle endomorphisms and the family $L(t)$ is analytic then the self-adjoint family obtained by imposing APS boundary conditions is an analytic family of self-adjoint operators; it follows that the eigenvectors and eigenvalues of D_t with $P_+(t) + L(t)$ boundary conditions can be chosen to depend analytically on t.*

The assumption that the tangential operator has constant dimensional kernel is not essential for this result, and we indicate in the last section how to remove it.

We now say a few words about the importance of this theorem; we begin with part (1). Several papers have recently been written (for example [Y], [N], [CLM], [MW], and [B]) concerning the spectral flow of a path of operators on a manifold with boundary; the goal in these papers is to find a formula expressing the spectral flow on a closed manifold as the sum of spectral flows on two submanifolds with boundary, together with a Maslov index term giving the relationship between the boundary conditions. However, in order to make sense of the spectral flow on a manifold with boundary one needs a theorem stating that in this case the spectrum varies continuously (such as part (1) of our main result). For this reason, and because we also need it for the proof of part (2), we have included part (1) in our main theorem and provided a detailed proof.

A paper of Bismut and Cheeger contains a theorem closely related to part (1) above. To be precise, Theorem 3.2 of [BC] states and proves the special case of (1) in which the tangential operator is assumed to be invertible for all $t \in T$.

The paper of Melrose and Piazza [MP] uses the idea of a "Cl(1)-spectral section" to construct continuous families of boundary conditions for the case in which the kernel of the tangential operator is not always zero and, in fact, not even constant dimensional. Though they use the fact that the resulting family of self-adjoint elliptic operators is continuous, their paper does not appear to contain an explicit proof of this fact.

Our need for part (2) above (the "analytic" result) arose from our own research, and was our main motivation for writing this paper. In [KK3], we consider the path of signature operators D_t arising from an analytic path of flat $SU(2)$-connections on an odd-dimensional manifold with boundary. We show that the first derivatives of those eigenvalues of D_t which pass through 0 at $t = 0$ can be computed using the cup product structure on $H^{even}(X; ad_{su(2)})$. In order to apply our techniques, however, we need to know that the eigenvalues and eigenvectors of D_t vary analytically. Continuous (or even smooth) 1-parameter families of operators are inadaquate for our needs; it is not hard to construct a smooth family of self-adjoint matrices whose eigenvectors cannot be chosen to vary continuously.

A special case of this theorem was proven for the odd signature operator on a 3-manifold with torus boundary in [KK1]. These results provide partial extensions to the bounded case of the main theorems of [KK2] and [FL], which show how to express the spectral flow of the odd signature operator on a closed manifold in terms of homotopy invariants. The theorem also allows us to remove a technical restriction (that a path be "fine") which was used in [KK1]. For that reason we restricted our attention in [KK1] to 3-manifolds with torus boundary. That restriction is now unnecessary.

We remark that the requirement that the parameter space be an interval is also important, since it is not hard to produce a 2-parameter analytic family of self-adjoint matrices whose eigenvectors cannot be chosen to vary analytically.

The paper is organized as follows. In Section 2, we state the definition and basic properties of the Dirac operators we are studying. In Section 3, we show that if the path of "tangential operators" $\hat{D}_t$ varies continuously (resp., analytically) then the corresponding path of positive spectral projections P_t also varies continuously (resp., analytically). In Section 4, we prove the main theorem.

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Section 2

Fix a locally compact parameter space T. In those cases in which we are dealing with analytic families of operators, T will be taken to be an interval. Otherwise T can be arbitrary. A good example to keep in mind is to take T to be the moduli space of flat connections on some bundle.

We will consider generalized Dirac operators in cylindrical form on a manifold with boundary. More precisely we make the following assumptions and notation.

1. X is a smooth odd-dimensional Riemannian manifold with boundary Y. The metric is isometric to a product in a collar of Y; fix an isometry of $I \times Y$ with a collar of Y taking $\{0\} \times Y$ to ∂X ($I = [-1, 0]$).
2. $E \to X$ is a smooth, complex vector bundle equipped with a positive definite Hermitian inner product $\langle\ ,\ \rangle$ on the fibers. This inner product defines an L^2 inner product $(\phi, \tau)_X = \int_X \langle \phi, \tau \rangle dvol_X$ on the space of sections of E.
3. $D_t = B + C_t$ is a T-parameterized family of essentially self-adjoint operators of Dirac type acting on sections of E. What this means is that for each $t \in T$, D_t is a first order, elliptic self-adjoint operator such that the principal symbol of D_t^2 is given by the metric, i.e.,
$$\sigma_{D_t^2}(\xi) = -|\xi|^2 \,\mathrm{Id}\,.$$
Furthermore, B is a fixed operator, C_t is a continuous map C_t: $T \to C^\infty(\hom(E, E))$, and each D_t is essentially self-adjoint in the sense that $(D_t\phi, \tau)_X = (\phi, D_t\tau)_X$ whenever ϕ and τ are smooth sections which vanish near Y.
4. We are given a bundle $\hat{E} \to Y$ and a bundle isomorphism ϕ: $\pi^*\hat{E} \to E_{|I\times Y}$ where π: $I \times Y \to Y$ is the projection to the second factor. The bundle $\hat{E}$ has

a fiberwise Hermitian inner product which defines an L^2 inner product $(\ ,\ )_Y$ on the sections of $\hat{E}$. The fiberwise inner product is consistent with the inner product on E (i.e. ϕ is a bundle isometry). Notice that the principal symbol σ_{D_t} of D_t is independent of t. In particular, if du denotes the unit inward normal covector on the collar, let

$$\sigma\colon \hat{E} \to \hat{E}$$

denote $\sigma_{D_t}(du)$. Then $\sigma^2 = -|du|^2 = -\mathrm{Id}$ and so σ is a bundle automorphism.

5. In the collar $I \times Y$, D_t takes the form

$$D_{t\ |I\times Y} = (\pi^*\sigma)\left(\pi^*\hat{D}_t + \frac{\partial}{\partial u}\right)$$

for any $t \in T$, where $\hat{D}_t = \hat{B} + \hat{C}_t$ is a T-parameter family of 1st order, self-adjoint, elliptic operators acting on sections of $\hat{E}$. Again $\hat{B}$ is a fixed 1st order, self-adjoint, elliptic operator and $\hat{C}_t$ is defined by a continuous map $T \to C^\infty(\mathrm{hom}(\hat{E}, \hat{E}))$.

As is standard we will suppress the π^* from our notation and write $D_t = \sigma(\hat{D}_t + \frac{\partial}{\partial u})$ on the collar. The operator $\hat{D}_t$ is called the *tangential operator* for the operator D_t.

We will call a family of operators D_t with parameter space $T = (-\epsilon, \epsilon)$ *analytic* if it satisfies the conditions 1-5 as above and, in addition, the following two conditions:

6. $D_t = B + \sum_{k=1}^\infty C_k t^k$ and the series converges in $C^\infty(\mathrm{hom}(E, E))$ (i.e., it converges in C^r for all $r > 0$).
7. The tangential operators on the boundary Y are of the form $\hat{D}_t = \hat{B} + \sum_{k=1}^\infty \hat{C}_k t^k$ and the series converges in $C^\infty(\mathrm{hom}(\hat{E}, \hat{E}))$.

Define a pairing on sections of $\hat{E}$ by

$$\{\alpha, \beta\} = (\sigma(\alpha), \beta)_Y$$

where $(\ ,\ )_Y$ denotes the L^2 pairing on sections of $\hat{E}$.

We list some of the properties of D_t, $\hat{D}_t$ and $\{\ ,\ \}$; these are well known.

2.1 LEMMA. 1. $\sigma^* = -\sigma$. *In particular* $\{\alpha, \beta\} = -\overline{\{\beta, \alpha\}}$.
2. $\sigma\hat{D}_t = -\hat{D}_t\sigma$. *In particular, σ interchanges the μ and $-\mu$ eigenspaces of $\hat{D}_t$ and preserves the kernel of $\hat{D}_t$.*
3. *The pairing $\{\ ,\ \}$ measures the failure of D_t to be self-adjoint in the sense that*

$$(D_t\phi, \tau)_X - (\phi, D_t\tau)_X = \{\phi_{|Y}, \tau_{|Y}\} \tag{2.1}$$

for smooth sections ϕ, τ of E.

4. *The kernel $\mathcal{H}_t$ of $\hat{D}_t$ is even-dimensional. The map σ preserves $\mathcal{H}_t$ and the eigenvalues of $\sigma\colon \mathcal{H}_t \to \mathcal{H}_t$ are $\pm i$ with the same multiplicity.*

Proof. The first three assertions are routine. The fourth assertion depends on the fact that $\hat{D}_t$ "bounds". We outline a proof. Consider D_t with Atiyah-Patodi-Singer boundary conditions $P_+ + \mathcal{H}_t$. Thus we consider D_t acting on the space of sections of E whose restriction to the boundary lie in the span of the non-negative eigensections of $\hat{D}_t$. The eta-invariant of $\hat{D}_t$ vanishes since the (non-zero) spectrum of $\hat{D}_t$ is symmetric (by 2.). Since X is odd dimensional the local forms which arise in the Index Theorem vanish. Thus the Atiyah-Patodi Singer theorem for D_t reduces to

$$\operatorname{Index} D_t(P_+ + \mathcal{H}_t) = 1/2 \dim \mathcal{H}_t\,.$$

Since the adjoint of $D_t(P_+ + \mathcal{H}_t)$ is $D_t(P_+)$ (see [APS-I]), one easily sees that the composite

$$\ker D_t(P_+ + \mathcal{H}_t) \xrightarrow{restrict} P_+ + \mathcal{H}_t \xrightarrow{project} \mathcal{H}_t$$

has image a half-dimensional subspace. By 3., the form $\{\ ,\ \}$ must vanish on this subspace. This forces the eigenvalues of $\sigma\colon \mathcal{H}_t \to \mathcal{H}_t$ to have the same multiplicity. □

Suppose $(H, \langle\ \rangle)$ is a $2n$-dimensional Hermitian vector space, and $\sigma\colon H \to H$ is a linear map satisfying $\sigma^2 = -\operatorname{Id}$, $\sigma = -\sigma^*$ and the $\pm i$ eigenspaces of σ each have dimension n. Define a skew-hermitian form by

$$\{\alpha, \beta\} = \langle \sigma(\alpha), \beta \rangle.$$

Then $(H, \{\ ,\ \})$ is a (complex) *symplectic* vector space. A subspace $\mathcal{L} \subset H$ is called *Lagrangian* if the form $\{\ ,\ \}$ vanishes identically on $\mathcal{L}$ and $\mathcal{L}$ is maximal with respect to this property. Let $H^{\pm}$ denote the $\pm i$ eigenspaces of σ, so $H = H^+ \oplus H^-$. It is not too hard to see that every Lagrangian is of the form $\mathcal{L} = \{h + \gamma(h) \mid h \in H^+\}$, where $\gamma\colon H^+ \to H^-$ is an isometry. Thus the space of Lagrangians in H is homeomorphic to $U(n)$.

The previous lemma shows that $\mathcal{H}_t = \ker \hat{D}_t$ is a symplectic vector space. Given a Lagrangian subspace $\mathcal{L}_t$ of $\mathcal{H}_t$, one can restrict the domain of D_t to those sections whose boundary values lies in $P_+(t) + \mathcal{L}_t$. This gives an elliptic self-adjoint operator (e.g., see [BW] and Section 4) which we will denote by $\mathcal{D}_t(\mathcal{L}_t)$.

We turn to the notion of continuity and analyticity for real, discrete spectra. Let $\tilde{\mathcal{S}}$ denote the set of maps $f\colon \mathbf{Z} \to \mathbf{R}$ satisfying:

1. f is finite-to-one.
2. The image $f(\mathbf{Z})$ is a discrete subset of $\mathbf{R}$.

Topologize $\tilde{\mathcal{S}}$ as a subspace of the countable product of $\mathbf{R}$. Equivalently, give $\tilde{\mathcal{S}}$ the metric

$$d(f, g) = \sup_n \frac{d(f(n), g(n))}{|n| + 1}$$

where $d(f(n), g(n)) = inf\{|f(n) - g(n)|, 1\}$

Let $\mathrm{Sym}(\mathbf{Z})$ denote the symmetric group on the set $\mathbf{Z}$. Then $\tilde{\mathcal{S}}$ has a $\mathrm{Sym}(\mathbf{Z})$ action given by $(\beta \cdot f)(m) = f(\beta^{-1}(m))$. Let $\mathcal{S}$ denote the orbit space. Thus points of $\mathcal{S}$ are discrete subsets of $\mathbf{R}$ with finite multiplicity.

If D is a self-adjoint operator whose resolvent $(D - \lambda)^{-1}$ is compact, then the spectrum of D is discrete and each eigenvalue is real with finite multiplicity. Therefore D determines an element Σ_D of $\mathcal{S}$. We say that the spectrum of a family D_t, $t \in T$ of self-adjoint operators with compact resolvents *varies continuously* if there is an open cover $\{U_a\}$ of T and continuous lifts $\tilde{\gamma}_a$ of $\gamma(t) = \Sigma_{D_t}, t \in U_a$ (i.e., so that the functions $t \mapsto \tilde{\gamma}(t)(n)$ are continuous from U_a to $\mathbf{R}$ for each U_a). If T is an open interval in $\mathbf{R}$ we will say that *the spectrum of D_t varies analytically* if the lifts $t \mapsto \tilde{\gamma}(t)(n)$ can be chosen to vary real-analytically in t. (With these notions it is easy to define the define the *spectral flow* of a path of self-adjoint operators with continuously varying spectrum ([APS-III], [T]).

Section 3

In this section we prove a key result about the tangential operators $\hat{D}_t$. These form a family of self-adjoint operators on the *closed* manifold $Y = \partial X$, parameterized by T. We will construct a family of projections to $P_+(t) + \mathcal{L}(t)$ which vary continuously (or analytically) in the operator norm topology.

In this section and in Section 4 we make the assumption that *the kernel of $\hat{D}_t$ is constant dimensional.* This is for ease of exposition, and the reader who would like to understand the general case should read Section 5 which indicates how to make the necessary modifications to obtain the general case.

Let $L^2(\hat{E})$ denote the completion of the space of smooth sections of $\hat{E}$ with respect to the L^2 inner product. Let $L^2_s(\hat{E})$ be the Sobolev space of sections with s derivatives in L^2, interpreted in the usual way when s is not an integer. We remind the readers that there is a compact inclusion $L^2_s(\hat{E}) \subset L^2_r(\hat{E})$ if $s > r$ (Rellich's Theorem). Since we are considering closed manifolds at the moment the domain of $\hat{D}_t$ is taken to be the image $L^2_1(\hat{E}) \to L^2(\hat{E})$. In particular, this domain is independent of $t \in T$.

One remark about notation: the Sobolev L^2_s norm will be denoted by $\|\ \|_s$, and $\|\ \|_{s,t}$ will denote the operator norm for an operator L: $L^2_s \to L^2_t$. We will drop the subscripts when using the L^2 norm.

Let K_t: $L^2(\hat{E}) \to L^2(\hat{E})$ denote the L^2-orthogonal projection onto the kernel of $\hat{D}_t$. It is is well known that K_t is a pseudo-differential operator of order $-\infty$ (e.g., see [BW]).

The map $t \mapsto K_t$ defines a continuous map $T \to \mathrm{Bd}(L^2_s(\hat{E}))$ for any $s \geq 0$, as one can check using the resolvent formula:

$$K_t = \frac{1}{2\pi i}\int_\Gamma (\lambda - \hat{D}_t)^{-1} d\lambda,$$

where Γ is a circle in the complex plane centered at 0 of radius smaller than the smallest non-zero eigenvalue of $\hat{D}_t$ for t is some small neigborhood in T. Continuity then follows from the continuity in norm of $\hat{C}_t$ by applying the formula

$$(\lambda - \hat{D}_t)^{-1} = (\lambda - \hat{D}_{t_0})^{-1}\sum_{k=0}^{\infty}((\hat{C}_t - \hat{C}_{t_0})(\lambda - \hat{D}_{t_0})^{-1})^k.$$

3.1 LEMMA. *The spaces* $\ker(\hat{D}_t)$, $t \in T$, *form a symplectic vector bundle* $\mathcal{H}$ *over* T.

Proof. The subset $\{(t, v) \mid (K_t - \mathrm{Id})v = 0\}$ of $T \times L^2(\hat{E})$ is a vector bundle since K_t is continuous and since the kernel of $K_t - \mathrm{Id}$ is constant dimensional. We have already observed that the fibers are symplectic. Clearly the symplectic structure is continuous from fiber to fiber, since it is the restriction of a symplectic structure on the trivial vector bundle $L^2(\hat{E}) \times T$. □

In the case where D_t is an analytic path of operators, the tangential operators $\hat{D}_t$ form an analytic path of elliptic self-adjoint operators over a closed manifold, and so perturbation theory [K, p. 386] shows that one can find analytic paths of vectors $\psi_i(t)$, $i = 1, \ldots, k$, which form a basis for the kernel of $\hat{D}_t$ for each t (recall we are assuming the kernel of $\hat{D}_t$ is constant dimensional). Thus the bundle of Lemma 3.1 (in this case a bundle over the interval $T = (-\epsilon, \epsilon)$) inherits an analytic structure, defined by declaring a section to be analytic if it can be written in the form $\sum_i f_i(t)\psi_i(t)$ for analytic functions f_i. An *analytic Lagrangian subbundle* is a Lagrangian subbundle locally spanned by analytically varying sections. More generally an arbitrary subbundle of $\mathcal{H}$ is called analytic if it is locally spanned by analytic sections.

Suppose a subbundle $\mathcal{L} \to T$ of the kernel bundle $\mathcal{H} \to T$ is given. In the analytic context, we assume the subbundle is analytic. (We are most interested in the case when $\mathcal{L}$ is a Lagrangian subbundle, but the cases when $\mathcal{L}$ is equal to $\mathcal{H}$, or is the zero subbundle are also interesting.) For $t \in T$, let L_t: $L^2(\hat{E}) \to L^2(\hat{E})$ be the projection to the fiber of $\mathcal{L}$ at t. Since L_t factors through K_t, it is easy to see that $t \mapsto L_t$ is a continuous (resp. analytic) map of T into $\mathrm{Bd}(L^2_s(\hat{E}))$ for all s.

We use L_t to "split the kernel" of $\hat{D}_t$. Let $L_t^\perp$ be the orthogonal projection to the orthogonal complement of $\mathcal{L}_t$ in $\mathcal{H}_t$. Then define

$$R_t = \hat{D}_t + L_t - L_t^\perp.$$

Thus R_t is a continuous (resp. analytic) family of self-adjoint pseudodifferential operators whose spectrum coincides with that of $\hat{D}_t$ except that $\mathcal{L}_t$ lies in the $+1$ eigenspace of R_t and the orthogonal complement of $\mathcal{L}_t$ in $\mathcal{H}_t$ lies in the -1 eigenspace of R_t. Moreover, R_t is invertible.

We remark that R_t is introduced only for technical reasons, namely, to make the proof Theorem 3.2 more transparent. All the ideas of the arguments we give are present in the special case when the kernel of $\hat{D}_t$ is zero for each t. For this case $R_t = \hat{D}_t$.

Let $P_t\colon L^2(\hat{E}) \to L^2(\hat{E})$ be the L^2 projection onto the span of the positive eigenvectors of R_t. Thus P_t is the projection to the sum of the positive eigenspace of $\hat{D}_t$ and $\mathcal{L}_t$. We will show continuity of the P_t, as well as analyticity if D_t (and hence $\hat{D}_t$) is an analytic path of operators. Fix $t_0 \in T$, (where T means $(-\epsilon, \epsilon)$ in the analytic case).

Let $\Gamma_n \subset \mathbf{C}$ be the vertical segment in the complex plane

$$\Gamma_n = \{ir \mid -n \leq r \leq n\}.$$

Define the operators

$$Q_n(t) = \frac{1}{2\pi} \int_{\Gamma_n} (r - R_t)^{-1} - (r - R_{t_0})^{-1} dr.$$

We wish to thank U. Bunke for suggesting a simplified proof of the following theorem. Compare to [B, Lemmas 2.1 and 2.2].

3.2. THEOREM. *As $n \to \infty$, the $Q_n(t)$ converge to $P_{t_0} - P_t$ in the operator norm topology on* $\mathrm{Bd}(L^2_s(\hat{E}))$ *for all s. The projections P_t are continuous in t as functions $T \to$* $\mathrm{Bd}(L^2_s(\hat{E}))$. *Moreover, the projections P_t are analytic in t if $\hat{D}_t$ is analytic, that is, the map $(-\epsilon, \epsilon) \to$* $\mathrm{Bd}(L^2_s(\hat{E}))$ *taking t to P_t is analytic.*

Proof. To begin with, let $A_t = R_t - R_{t_0}$. If D_t is an analytic path, then $A_t = \sum_{i>0} A_i t^i$.

Let $r \in i\mathbf{R}$ denote a purely imaginary number as well as the corresponding multiplication operator $r\colon L^2_s(\hat{E}) \to L^2_s(\hat{E})$. Observe that if $r - R_t$ is invertible for t near t_0,

$$(r - R_t)^{-1} - (r - R_{t_0})^{-1} = [(I - (r - R_{t_0})^{-1} A_t)^{-1} - I](r - R_{t_0})^{-1}.$$

The Neumann series

$$(I - (r - R_{t_0})^{-1} A_t)^{-1} = \sum_{k=0}^{\infty} [(r - R_{t_0})^{-1} A_t]^k \tag{3.1}$$

converges in $\mathrm{Bd}(L^2_s(\hat{E}))$ if $\|(r - R_{t_0})^{-1}\|_{s,s} \cdot \|A_t\|_{s,s} < 1$.

The spectral theorem implies that there is an orthonormal basis $\{\psi_i\}_{i=-\infty}^{\infty}$ of $L^2(\hat{E})$ such that ψ_i is a μ_i eigenvector of R_{t_0}. Since $\hat{D}_{t_0}$ is elliptic and R_{t_0} agrees with $\hat{D}_{t_0}$ away from the finite dimensional space $\mathcal{H}_{t_0}$ the vectors

$$\psi_{i,s} = \psi_i/(1+|\mu_i|^2)^{\frac{s}{2}}$$

form an orthonormal basis for an admissible norm for $L^2_s(\hat{E})$, and,

$$(r-R_{t_0})^{-1}\psi_{i,s} = \frac{1}{r-\mu_i}\psi_{i,s}.$$

Let δ equal half the smallest positive eigenvalue of R_{t_0}. Notice that $|r-\mu_i| > \operatorname{Max}\{|r|,\delta\}$ since r is purely imaginary and μ_i is real. It follows that

$$\|(r-R_{t_0})^{-1}\|_{s,s} \leq \min\left\{\frac{1}{|r|}, \frac{1}{\delta}\right\}.$$

Given $s \geq 0$, there is some neighborhood of t_0 so that for t in this neighborhood, $\|A_t\|_{s,s} < \delta/2$. Thus the Neumann series (3.1) converges for any t in this neighborhood and any $r \in i\mathbf{R}$.

Hence, for any t in this neighborhood and any $r \in i\mathbf{R}$,

$$(1+\|(r-R_{t_0})^{-1}\|_{s,s}\ \|A_t\|_{s,s})^{-1} \leq 2$$

and so

$$\begin{aligned}
&\|(r-R_t)^{-1}-(r-R_{t_0})^{-1}\|_{s,s}\\
&\leq \|(r-R_{t_0})^{-1}\|_{s,s}\sum_{k=1}^{\infty}\|(r-R_{t_0})^{-1}\|_{s,s}^k\|A_t\|_{s,s}^k\\
&= \|(r-R_{t_0})^{-1}\|_{s,s}^2\|A_t\|_{s,s}\sum_{k=0}^{\infty}\|(r-R_{t_0})^{-1}\|_{s,s}^k\|A_t\|_{s,s}^k\\
&= \|(r-R_{t_0})^{-1}\|_{s,s}^2\|A_t\|_{s,s}(1-\|(r-R_{t_0})^{-1}\|_{s,s}\|A_t\|_{s,s})^{-1}\\
&\leq \delta\|(r-R_{t_0})^{-1}\|_{s,s}^2 \leq \min\left\{\frac{\delta}{|r|^2}, \frac{1}{\delta}\right\}
\end{aligned}$$

Since $\int_\delta^\infty r^{-2}dr$ converges, the limit

$$\lim_{n\to\infty}\int_{\Gamma_n}(r-R_t)^{-1}-(r-R_{t_0})^{-1}dr$$

converges in norm, and so

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{\Gamma_n} (r - R_t)^{-1} - (r - R_{t_0})^{-1} dr$$

is a well-defined operator; call it $Q(t)$ temporarily.

The family of operators A_t varies continuously in norm with $t \in T$, and so the integrand

$$(r - R_t)^{-1} - (r - R_{t_0})^{-1} = [(I - (r - R_{t_0})^{-1} A_t)^{-1} - I](r - R_{t_0})^{-1}$$

varies continuously in t. Therefore, $Q(t)$ varies continuously in t. In the analytic case, one concludes that $Q(t)$ varies analytically using the Neumann series (which implies that the map taking a bounded operator to its inverse is an analytic map).

It remains to show that

$$Q(t) = P_{t_0} - P_t.$$

For this one computes that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{\Gamma_n} (r - R_t)^{-1} dr\, \psi_i = \lim_{n\to\infty} \frac{1}{2\pi} \int_{\Gamma_n} \frac{dr}{r - \mu_i}\, \psi_i = \begin{cases} -\frac{1}{2}\psi_i & \text{if } \mu_i > 0 \text{ and} \\ \frac{1}{2}\psi_i & \text{if } \mu_i < 0 \end{cases}$$

Thus

$$Q(t) = \frac{1}{2}(I - P_t) - \frac{1}{2} P_t - \frac{1}{2}(I - P_{t_0}) + \frac{1}{2} P_{t_0} = P_{t_0} - P_t.$$

This finishes the proof of Theorem 3.2. □

Section 4

Assume now that the subbundle $\mathcal{L} \subset \mathcal{H}$ is a Lagrangian subbundle. Let P_t be the family of projections constructed in the previous section. Thus P_t is the projection to the sum of $\mathcal{L}_t$ and the span of the positive eigenvectors of $\hat{D}_t$. The projection $I - P_t \in \mathrm{Bd}(L^2(\hat{E}))$ is continuous (resp. analytic) in t by Theorem 3.2. The kernel of $I - P_t$ will be our space of self-adjoint boundary conditions. We denote this subspace of $L^2(\hat{E})$ by $P_+(t) + \mathcal{L}_t$.

For each $t \in T$, denote by $\mathcal{D}_t(\mathcal{L})$ be the closed (unbounded) operator on $L^2(E)$ equal to D_t with domain

$$\ker((I - P_t) \circ i^*)\colon L^2_1(E) \to L^2_{\frac{1}{2}}(\hat{E}),$$

where $i^*\colon L^2_1(E) \to L^2_{\frac{1}{2}}(\hat{E})$ is the bounded linear map induced by the restriction to

Y. Thus the domain of $\mathcal{D}_t(\mathcal{L})$ is the set of once-differentiable sections of E whose restriction to the boundary lies in the span of the positive eigenvectors of $\hat{D}_t$ and $\mathcal{L}_t$.

The domain of $\mathcal{D}_t(\mathcal{L})$ depends on t. (Notice however that $\mathcal{D}_t(\mathcal{L})$ is a restriction of D_t, and D_t has domain the image of $L^2_1(E)$ in $L^2(E)$. In particular the family $\mathcal{D}_t(\mathcal{L})$ extends to a non self-adjoint family D_t with constant domains.) To deal with the varying boundary conditions, we will construct a parameterization of the boundary values by a fixed Banach space.

Given $t_0 \in T$ define $V_t \in \mathrm{Bd}(L^2(\hat{E}))$ by

$$V_t = \mathrm{Id} + (P_{t_0} - P_t)(\mathrm{Id} - 2P_{t_0}). \tag{4.2}$$

Then V_t is continuous (resp. analytic) in t into $L^2_s(\hat{E})$ for any s, and

$$V_t(I - P_{t_0}) = (I - P_t)V_t.$$

Moreover, V_t is invertible if $\|(P_{t_0} - P_t)(\mathrm{Id} - 2P_{t_0})\|_{L^2} < 1$, and this happens if $\|P_{t_0} - P_t\| < \frac{1}{3}$. It was shown that P_t is continuous in t with respect to the norm topology. Therefore, if t is close enough to t_0, V_t is invertible and so $I - P_{t_0} = V_t^{-1}(I - P_t)V_t$. (We thank K. Wojciechowski for pointing out formula (4.2).)

We now come the the main theorem of this article. It states that the family of operators with Atiyah-Patodi-Singer boundary conditions varies continuously (resp. analytically). Note that the continuity result is similar to a result in [BC] which considers the case when the tangential operators have trivial kernel.

4.1 THEOREM. *Each $\mathcal{D}_t(\mathcal{L})$ is self-adjoint with pure point spectrum. If D_t varies continuously in $t \in T$, then the map from T to the space of self-adjoint, closed operators in the graph topology given by $t \mapsto \mathcal{D}_t(\mathcal{L})$ is continuous, and the spectrum varies continuously in t.*

If D_t varies analytically in $t \in (-\epsilon, \epsilon)$, then the spectrum and eigenvectors of $\mathcal{D}_t(\mathcal{L})$ vary analytically in t. More precisely, there exist real-analytic functions $\lambda(t)$ and smooth sections $\phi_{\lambda,k}$ so that

$$\phi_\lambda(t) = \sum_{k=0}^{\infty} \phi_{\lambda,k}\, t^k,$$

the sum converges in $L^2(E)$,

$$\mathcal{D}_t(\mathcal{L})\phi_\lambda(t) = \lambda(t)\phi_\lambda(t)$$

for all t small enough, and for each t the set $\{\phi_\lambda(t)\}$ forms a complete orthonormal basis of $L^2(E)$.

Proof. That $P_+(t) + \mathcal{L}_t$ is a self-adjoint boundary condition for D_t can be found in [Y], [BW], or [MW]. The essential observation is that if $\phi, \tau \in P_+(t) + \mathcal{L}_t$, then

$$(D_t\phi, \tau)_X - (\phi, D_t\tau)_X = \{\phi_{|Y}, \tau_{|Y}\} = \{proj_{\mathcal{H}_t}\phi_{|Y}, proj_{\mathcal{H}_t}\tau_{|Y}\} = 0$$

since $\mathcal{L}_t$ is a Lagrangian in $\mathcal{H}_t$.

We first deal with the continuous case. Fix $t_0 \in T$. We show continuity at t_0. We begin by constructing operators which parameterize the domains. The restriction $i^*\colon L^2_1(X; E) \to L^2_{\frac{1}{2}}(Y; \hat{E})$ is bounded and onto; e.g., see [P]. Denote by $\mathbf{B}$ the closed subspace $\mathbf{B} = \ker(i^* \circ (I - P_{t_0}))\colon L^2_1(E) \to L^2_{\frac{1}{2}}(\hat{E}) = P_+(t_0) + \mathcal{L}_{t_0}$.

Let $S\colon L^2_{\frac{1}{2}}(\hat{E}) \to L^2_1(E)$ be a bounded right inverse to i^*. Define $J_t\colon \mathbf{B} \to L^2_1(E)$ by

$$J_t(g) = g + S \circ (V_t - \mathrm{Id})(i^*(g)),$$

where V_t is the operator defined by Equation (4.1) . Then

$$i^*(J_t(g)) = i^*(g) + i^*S(V_t - \mathrm{Id})(i^*(g)) = V_t(g).$$

Therefore, $J_t(g) \in \ker\colon i^* \circ (I - P_t)\colon L^2_1(E) \to L^2_{\frac{1}{2}}(\hat{E})$. Composing J_t with the inclusion $L^2_1(E) \subset L^2(E)$ yields a bounded operator $\Psi_t\colon \mathbf{B} \to L^2(E)$ whose image is the domain of $\mathcal{D}_t(\mathcal{L})$. The map $T \to \mathrm{Bd}(\mathbf{B}, L^2(E))$ given by $t \mapsto \Psi_t$ is continuous in t since $t \mapsto V_t$ is continuous.

The composites $\mathcal{D}_t(\mathcal{L}) \circ \Psi_t\colon \mathbf{B} \to L^2(E)$ are bounded. Indeed, $\mathcal{D}_t(\mathcal{L}) \circ \Psi_t = D_t \circ J_t$, and $D_t\colon L^2_1(E) \to L^2(E)$ is bounded since D_t is a first order elliptic operator and X is compact.

Finally, the map $T \to \mathrm{Bd}(\mathbf{B}, L^2(E))$ given by $t \mapsto \mathcal{D}_t(\mathcal{L}) \circ \Psi_t$ is continuous in t since if $\xi \in \mathbf{B}$ then

$$\begin{aligned}\|(D_t \circ J_t - D_{t_0} \circ J_{t_0})\xi\|_{L^2} &\leq \|(D_t \circ J_t - D_t \circ J_{t_0})\xi\|_{L^2} \\ &\quad + \|(D_t \circ J_{t_0} - D_{t_0} \circ J_{t_0})\xi\|_{L^2} \\ &\leq (\|D_t\|_{1,0}\, \|J_t - J_{t_0}\| + \|D_t - D_{t_0}\|_{1,0}\, \|J_{t_0}\|)\|\xi\|_{L^2_1}\end{aligned}$$

The right side goes to zero as t approaches t_0 since J_t is a continuous family and since $D_t\colon L^2_1(E) \to L^2(E)$ is a continuous family of bounded operators.

The fact that $\mathcal{D}_t(\mathcal{L})$ is a continuous family in the graph topology is now a consequence of Theorem IV.2.29 of [K]. The operators $\mathcal{D}_t(\mathcal{L})$ have compact resolvents $R_t(\lambda) = (\lambda - \mathcal{D}_t(\mathcal{L}))^{-1}$ by Rellich's theorem since the resolvents factor through the inclusion $L^2_1(E) \to L^2(E)$. Hence $\mathcal{D}_t(\mathcal{L})$ has pure point spectrum; each eigenvalue has finite multiplicity and the set of eigenvalues forms a discrete subset of $\mathbf{R}$.

Let $t_0 \in T$. Let $\epsilon > 0$ be some number between 0 and the smallest positive eigenvalue of $\mathcal{D}_{t_0}(\mathcal{L})$. Then $\mathcal{D}_{t_0}(\mathcal{L}) - \epsilon$ is invertible. So Theorem IV.2.21 of [K] shows that $\mathcal{D}_t(\mathcal{L}) - \epsilon$ is invertible whenever

$$\delta(\mathcal{D}_{t_0}(\mathcal{L}) - \epsilon, \mathcal{D}_t(\mathcal{L}) - \epsilon) \leq (1 + \|(\mathcal{D}_{t_0}(\mathcal{L}) - \epsilon)^{-1}\|^2)^{-1/2},$$

where $\delta(A, B)$ denotes the gap metric between closed operators, defining the graph topology. Since the $\mathcal{D}_t(\mathcal{L}) - \epsilon$ form a continuous family in the graph topology (with respect to t), there exists a neighborhood U of t_0 in T so that $\mathcal{D}_t(\mathcal{L}) - \epsilon$ is invertible for all t in U.

Define a function

$$f\colon U \to \mathrm{Maps}(\mathbf{Z}, \mathbf{R})$$

as follows: for each t in U, let $f_1(t)$ be the smallest positive eigenvalue of $\mathcal{D}_t(\mathcal{L})$. Then let $f_2(t)$ be the next smallest, etc. always counting with multiplicity, so that for example if the smallest positive eigenvalue of $\mathcal{D}_t(\mathcal{L})$ has multiplicity 2, then $f_1(t) = f_2(t) \neq f_3(t)$. Repeat the definition for the non-positive eigenvalues, so $f_0(t)$ is the largest non-positive eigenvalue of $\mathcal{D}_t(\mathcal{L})$, $f_{-1}(t)$ is the next largest non-positive eigenvalue of $\mathcal{D}_t(\mathcal{L})$, etc.. Thus f parameterizes the spectrum of $\mathcal{D}_t$, and so defines a map $f\colon U \to \tilde{\mathcal{S}}$ lifting the map $U \to \mathcal{S}$ defined by $t \mapsto \Sigma_{\mathcal{D}_t(\mathcal{L})}$.

We must show that each coordinate function $f_n(t)$ is continuous. Theorem IV.2.20 of [K] shows that the family of resolvents $(\mathcal{D}_t(\mathcal{L}) - \lambda)^{-1}$ is continuous in the norm topology on bounded operators. Assume by induction that $f_k(t)$ is continuous in t on U for all $0 < k < n$. Fix $s \in U$. Thus $f_n(s)$ is an eigenvalue of $\mathcal{D}_s(\mathcal{L})$. This eigenvalue may have multiplicity; assume $f_{n+i}(s) = f_n(s)$ for $i = -a, \cdots - 1, 0, 1, \ldots, b$.

Let Γ be a small circle in the complex plane encircling $f_n(s)$, but containing no other eigenvalues. The continuity of the resolvents $(\mathcal{D}_t(\mathcal{L}) - \lambda)^{-1}$ implies that there exists a neighborhood $W \subset U$ of s so that λ does not lie in the spectrum of $\mathcal{D}_s(\mathcal{L})$ for $s \in W$ and $\lambda \in \Gamma$. Then the L^2-projection onto the span of the eigenvectors of $\mathcal{D}_s(\mathcal{L})$ spanned by the eigenvectors inside Γ is given by the formula:

$$\Phi_t = \frac{i}{2\pi} \int_\Gamma (\mathcal{D}_t(\mathcal{L}) - \lambda)^{-1} d\lambda$$

for $t \in W$. Notice that Φ_t is continuous in norm since Γ is compact and since $(\mathcal{D}_t(\mathcal{L}) - \lambda)^{-1}$ is continuous in norm (with respect to both t and λ).

Define

$$M_t = \mathrm{Id} + (\Phi_t - \Phi_s)(2\Phi_s - \mathrm{Id}).$$

Then on a possibly smaller neighborhood $V \subset W$ of s, M_t is invertible for $t \in V$ since Φ_t is continuous in t. Now $M_t \Phi_s M_t^{-1} = \Phi_t$. Let H denote the (finite dimensional) range of Φ_s. Then

$$F_t = (M_t \Phi_s^{-1} M_t^{-1}) D_t (M_t \Phi_s M_t^{-1})\colon H \to H$$

is a continuous family of self-adjoint operators on the finite dimensional space H. Thus the eigenvalues of F_t when ordered in a non-decreasing manner, vary continuously. But these eigenvalues are just $f_{n-a}(t), \ldots, f_{n+b}(t)$. In particular, $f_n(t)$ varies continuously over V. Hence f_n is continuous at s. Since $s \in U$ was arbitrary, $f_n(t)$

is continuous on all of U. This shows f is continuous and so the spectrum varies continuously.

We turn to the analytic case. To begin with, since the projections P_t and the path of Lagrangians $\mathcal{L}_t$ vary analytically, we can extend the function $t \mapsto \mathcal{D}_t(\mathcal{L})$ to an open set $U \subset \mathbf{C}$.

We recall what it means for a family of closed operators to be holomorphic (in the sense of [K]). First, a family B_t of *bounded* operators between two Hilbert spaces H and K defined for t in an open set U in $\mathbf{C}$ is called *bounded holomorphic* if the map $U \to \mathrm{Bd}(H, K)$ is differentiable. This is equivalent to the map $t \mapsto \langle B_t h, k\rangle$ being a differentiable function for any $h \in H$ and $k \in K$. A family $D_t \in Cl(H, K)$ of closed operators is said to be holomorphic if there exists a Banach space $\mathbf{B}$ and a family of *bounded* operators $\Psi_t\colon \mathbf{B} \to H$ such that the image of Ψ_t is the domain of D_t and such that Ψ_t and $D_t \circ \Psi_t$ are bounded holomorphic.

We show that the family $\mathcal{D}_t(\mathcal{L})$, $\{t \in \mathbf{C} \,|\, |t| < \epsilon\}$ is holomorphic. The family $\Psi_t\colon \mathbf{B} \to L^2_1(E)$ defined above is a bounded holomorphic family since $V_t\colon L^2_{\frac{1}{2}}(\hat{E}) \to L^2_{\frac{1}{2}}(\hat{E})$ is a holomorphic family. Furthermore, $\mathcal{D}_t(\mathcal{L}) \circ \Psi_t = D_t \circ J_t$ is bounded holomorphic since both J_t and D_t are bounded holomorphic.

Hence $\mathcal{D}_t(\mathcal{L})$ forms a holomorphic family of operators, which are self adjoint for real t. The results of [K; Ch. VII], especially p. 386, shows that the eigenvalues and eigenvectors of $\mathcal{D}_t(\mathcal{L})$ can be chosen to vary analytically.

Smoothness of the $\phi_{\lambda,k}$ follows from elliptic regularity: $D_0\phi_{\lambda,0} = \lambda(0)\phi_{\lambda,0}$ and so $\phi_{\lambda,0}$ is smooth. Taking coefficients of t in the equation $\mathcal{D}_t(\mathcal{L})\phi_\lambda(t) = \lambda(t)\phi_\lambda(t)$ shows the higher derivatives are smooth also.

This completes the proof of Theorem 4.1. $\square$

It follows from this theorem that given any Lagrangian subbundle $\mathcal{L} \subset \mathcal{H}$ and path $\gamma(t)$ in T the spectral flow $SF(\mathcal{D}(\mathcal{L}), \gamma(t))$ is well defined. Thus the spectral flow of a path of Dirac operators on a manifold with boundary with Atiyah-Patodi-Singer boundary conditions is defined. An analytically varying path has analytically varying eigenvectors and eigenvalues. This is useful when computing the spectral flow, since the local behavior of an analytically varying eigenvalue $\lambda(t)$ with $\lambda(t_0) = 0$ is determined by the order to which λ vanishes at t_0 and the sign of its first non-vanishing derivative at t_0. This principle is used in the study of the odd signature operator in [FL], [KK1], [KK2], and for general Dirac operators on a manifold with boundary in [KK3].

Section 5

We conclude with a brief discussion of how to extend these results when the kernel of the tangential operator jumps up.

First, given a general T-parameterized family of operators as above, one can stratify the set T into subsets

$$T_n = \{t \in T \mid \dim\ker \hat{D}_t = n\}.$$

Then over each stratum one can pick a Lagrangian subbundle and construct continuous self-adjoint families as above. (for example, in the applications of this theorem we consider signature operators parameterized by flat connections, and the T_n are defined in terms of the dimensions of certain cohomology groups.) Then as long as one stays within the strata, The assumption made at the beginning of Section 3 applies.

However, this assumption (that the kernel of $\hat{D}_t$ is constant dimensional) can be weakened in Theorem 4.1 if one takes more care in defining the boundary conditions by enlarging the symplectic vector bundle in the following manner.

Given $t_0 \in T$, Choose a continuous positive function $\epsilon(t)$ with the property that $\pm\epsilon(t)$ misses the spectrum of the tangential operator $\hat{D}_t$. Let $\mathcal{H}_t$ be the the sum of the eigenspaces of $\hat{D}_t$ corresponding to eigenvalues whose absolute value is less than $\epsilon(t)$. The function ϵ can be chosen so that $\mathcal{H}_t$ (the "thickened" harmonic space) forms a symplectic vector bundle in a neighborhood of t. (For example, in the analytic case, choose $\epsilon(t)$ so that $\mathcal{H}_t$ is the span of those $\phi_\lambda(t)$ which satisfy $\lambda(0) = 0$.) Notice that

$$L^2(\hat{E}) = P_{-\epsilon(t)} \oplus \mathcal{H}_t \oplus P_{\epsilon(t)}$$

where $P_{\epsilon(t)}$ (resp. $P_{-\epsilon(t)}$) denotes the span of eigenvectors of $\hat{D}_t$ corresponding to eigenvalues greater than $\epsilon(t)$ (resp. less than $-\epsilon(t)$).

Choose a continuous (resp. analytic) family $\mathcal{L}_t$ of Lagrangians in $\mathcal{H}_t$. Then define $\mathcal{D}_t(\mathcal{L})$ to be the operator D_t restricted to those sections whose boundary values lie in $\mathcal{L}_t + P_{\epsilon(t)}$.

Then the statement of Theorem 4.1 holds for this more general family. The proof is essentially the same. One needs only make use of the gaps in the spectrum of $\hat{D}_t$ corresponding to the function $\epsilon(t)$. Since Theorem 4.1 is a local result, the local compactness of the parameter space makes it possible to replace $\epsilon(t)$ by a small positive constant.

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