ILLINOIS JOURNAL OF MATHEMATICS
Volume 43, Number 2, Summer 1999

POLARIZED VARIETIES WHOSE POINTS ARE JOINED BY RATIONAL CURVES OF SMALL DEGREES

YASUYUKI KACHI AND EIICHI SATO

Dedicated to Professor Shoshichi Kobayashi on his 66th Birthday

ABSTRACT. Let X be a projective variety with $\mathbb{Q}$-factorial singularities, over an algebraically closed field k of characteristic 0, L an ample Cartier divisor on X, and x a non-singular point of X. We prove that if for two general points $y, z \in X$ there exists a rational curve C passing through x, y, z, such that $(L \,.\, C) = 2$, then $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}(1))$ or $(Q^n, \mathcal{O}(1))$, a hyperquadric.

0. Introduction

Let X be a non-singular projective algebraic variety over an algebraically closed field k. If the anti-canonical divisor $-K_X := \bigwedge^{\dim X} T_X$ of X is ample, then X is called a *Fano variety*. It is known that the *Fano index* of X, that is, the largest integer $r > 0$ that divides $-K_X$ in the Picard group Pic X of X, is at most $\dim X + 1$. According to the Kobayashi-Ochiai Theorem [KoO] (see also Theorem 0.7 below), the projective space $\mathbb{P}^n$ and the quadric hypersurface (i.e., the hyperquadric) $Q^n \subset \mathbb{P}^{n+1}$ are the only Fano varieties having Fano indices $\dim X + 1$ and $\dim X$, respectively, if the characteristic of k is 0. Due to this result, $\mathbb{P}^n$ and Q^n are distinguished in the geography of Fano varieties, and have attracted special attention. In particular, several subsequent characterizations were obtained ([Mo1], [Fuj1–5], [W], [ChS2] and [ChM]; cf. [MoS], [CaP], [ChS1], [Zh]; also from a differential geometric point of view, [Si], [SiY]), stated in terms of properties of T_X or $-K_X$ (see 0.6 and (0.7.2) below, also [Kol] Chap. V. 3).

The purpose of this paper is to characterize $\mathbb{P}^n$ and Q^n in a somewhat different flavor, that is, to characterize them by means of bounding degrees of rational curves connecting points (Theorems 0.2, 0.6), after the initial case dealt with in [ABW] (cf. [Zh2], also Theorems 0.3, 0.5 below). This formalism is traced back to Campana and Kollár–Miyaoka–Mori's works [Cam], [KoMiMo1,2,3], where they introduced the notion of *rational connectedness*. In particular, they showed that the class of Fano varieties satisfies this property, strengthening the Miyaoka–Mori's theorem on

Received January 27, 1998.
1991 Mathematics Subject Classification. Primary 14E30, 14E05, 14J35.

uniruledness [MiMo]. In this paper we propose an effective variant of their definition for a polarized variety (0.1, 0.4).

0.0. Let k be an algebraically closed field, X a normal projective algebraic variety of dimension n defined over k, and L an ample Cartier divisor on X.

0.1 (Effective rational connection; cf. [KoMiMo3, §4], [Kol, Chap. IV 3.10.1].) Let $y_1, \dots, y_s \in X$ be s distinct points. Consider the following condition:

$(\mathrm{RC_{gen}}; y_1, \dots, y_s)^r_d$. For r general points $x_1, \dots, x_r \in X$, there exists an irreducible rational curve $C = C_{x_1,\dots,x_r}$ on X such that

$$C_{x_1,\dots,x_r} \ni x_1, \dots, x_r, y_1, \dots, y_s, \quad \text{and} \quad (L \,.\, C) = d.$$

(Compare this with $(\mathrm{RC}; y_1, \dots, y_s)^r_d$ in 0.4 below.) In case $s = 0$, we simply denote this condition by $(\mathrm{RC_{gen}})^r_d$.

We give criteria for the pair (X, L) to be $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ or $(Q^n, \mathcal{O}_{Q^n}(1))$, in terms of the conditions $(\mathrm{RC_{gen}}; y_1, \dots, y_s)^r_d$ for suitable (s, d, r)'s (Theorem 0.2, cf. 0.3). We establish our characterizations in the category not only of non-singular projective varieties, but also of projective varieties having at worst $\mathbb{Q}$-*factorial singularities*, where every purely codimension 1 subvariety D has an integer multiple mD $(m \neq 0)$ which is a Cartier divisor; $mD \in \operatorname{Pic} X$. This category is considered to be substantial from the view point of *minimal models*; it is closed under the birational transforms arising from the minimal model program (e.g., see [R1], [KaMaMa]). This enlargement was made possible, first for the *log-terminal singularity case* by virtue of the improvements of Kobayashi–Ochiai Theorem ([Fuj5], §1, Corollary 1.3, [Sh0], [Al2], cf. [Fuj1], [Al1]) combined with the Kawamata–Viehweg Vanishing Theorem [Kaw1], [V] (cf. [KaMaMa]), and for the general case by virtue of the result of Fujita [Fuj4], as indicated by Andreatta–Ballico–Wiśniewski [ABW] and Mella [Me2]. These are based on the formula describing the tangent spaces of the Hilbert schemes $\operatorname{Hom}(C, X)$ for singular varieties [Kol, Chap. II]. As for the condition "log-terminal singularities", see also (0.9.2) below.

In this paper we prove the following;

THEOREM 0.2 (a weaker version of Theorem 0.6 = Theorem 5.1). *Let* (X, L) *be as in* 0.0. *Assume that* $\operatorname{char} k = 0$, *and that k has an uncountable cardinality. Moreover assume that X has only* $\mathbb{Q}$-*factorial singularities. Assume that* (X, L) *satisfies* $(\mathrm{RC_{gen}}; x)^2_2$, *for some non-singular point x of X. Then* $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ *or* $(Q^n, \mathcal{O}_{Q^n}(1))$, *where* Q^n *is a (possibly singular) hyperquadric in* $\mathbb{P}^{n+1}$

We note that Theorem 0.2 (and Theorem 0.6 below) was proved initially under the extra assumption "X has log-terminal singularities", while Mella pointed out that this was unnecessary [Me2]. The above Theorem 0.2 is a revised statement after taking this into account (see (0.9.2) below).

As to the question of isolating $\mathbb{P}^n$ from Q^n in the above characterization, one has the following result, which is essentially proved by Andreatta–Ballico–Wiśniewski [ABW], p. 194 (cf. Zhang [Zh2]).

THEOREM 0.3 (Andreatta–Ballico–Wiśniewski [ABW], cf. Theorem 0.5 (2) = Corollary 4.2) *Let* (X, L) *be as in* 0.0. *Assume that* $\operatorname{char} k = 0$. *Let* $x \in X$ *be a non-singular point of* X. *Assume the following condition*;

$(\mathrm{RC}_{\mathrm{flat}}; x)_1^1$. *There exists a flat family* $\{C_t\}_{t\in T}$ *of irreducible rational curves of* X *such that for every* $y \in X$, *there exists* $t \in T$ *such that* $C_t \ni x, y$, *and* $(L\,.\,C_t) = 1$.

Then $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

0.4. In the definition of $(\mathrm{RC}_{\mathrm{gen}}; y_1, \dots, y_s)_d^r$ (0.1), we may replace "r general points $x_1, \dots, x_r$ in X" by "r points $x_1, \dots, x_r$ placed in a *sufficiently dense* subset of X", to obtain an analogous but virtually weaker condition. Namely, let $S^r(X)$ be the r-th symmetric product of X, and consider the following assumption;

$(\mathrm{RC}; y_1, \dots, y_s)_d^r$. There exists a subset $\Lambda \subset S^r(X)$ which is not contained in a countable union of proper Zariski closed subsets of $S^r(X)$ (or a *uc-dense subset* Λ of $S^r(X)$, in Terminology 2.4), such that for every $\{x_1, \dots, x_r\} \in \Lambda$, there exists an irreducible rational curve $C_{x_1,\dots,x_r}$ on X such that

$$C_{x_1,\dots,x_r} \supset \{x_1, \dots, x_r, y_1, \dots, y_s\}, \quad \text{and} \quad (L\,.\,C_{x_1,\dots,x_r}) = d$$

(see 2.11). In this paper, we deduce the same conclusions as in Theorems 0.2 and 0.3 under this weaker assumption $(\mathrm{RC}; y_1, \dots, y_s)_d^r$, with $(s, d, r) = (1, 1, 1)$ and $(1, 2, 2)$, respectively:

THEOREM 0.5 (= Proposition 4.1 + Corollary 4.2). *Let* (X, L) *be as in* 0.0. *Assume that* $\operatorname{char} k = 0$, *and that* k *has an uncountable cardinality. Moreover assume that* (X, L) *satisfies* $(\mathrm{RC}; x)_1^1$ (*in* 0.4), *for some* x. *Then*:

(1) $\rho(X) = 1$.
(2) (*a minor modification of* [ABW]) *Furthermore if we assume that* x *is a non-singular point of* X; $x \in \operatorname{Reg} X$, *then* $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

The following is our main result:

THEOREM 0.6 (= Theorem 5.1). *Let* (X, L) *be as in* 0.0. *Assume that* $\operatorname{char} k = 0$, *and that* k *has an uncountable cardinality. Moreover assume that* X *has* $\mathbb{Q}$*-factorial singularities. If* (X, L) *satisfies* $(\mathrm{RC}; x)_2^2$ (*in* 0.4) *for some* $x \in \operatorname{Reg} X$, *then* $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, *or* $(Q^n, \mathcal{O}_{Q^n}(1))$.

We note that Theorem 0.5, seemingly a minor strengthening of the result of [ABW], is indeed used in an essential way to prove Theorems 0.2 and 0.6 (see the proof of Proposition 5.4). Due to this technicality, it is more reasonable to consider $(\mathrm{RC}; y_1, \dots, y_s)_d^r$, rather than $(\mathrm{RC}_{\mathrm{gen}}; y_1, \dots, y_s)_d^r$. This generalization may also be related to the question over non-algebraically closed fields ([Kol] Chap. IV. 6; cf. [Isk2]).

Our proof relies upon the special case of the following theorem, which is proved by Kobayashi-Ochiai for the non-singular case, and by Fujita (an alternative algebraic proof), Shokurov and Alexeev (enlargement for the *log-terminal singularity* case).

THEOREM 0.7 (Kobayashi–Ochiai [KoO], Fujita [Fuj1,4], cf. Shokurov [Sh0], Alexeev [Al2]; cf. [Kac2]) *Let* (X, L) *be as in* 0.0. *Let* $n := \dim X$. *Assume that* $\operatorname{char} k = 0$, *and that* X *has only* $\mathbb{Q}$*-Gorenstein log-terminal singularities.*

(1) *If* $-K_X \equiv rL$ *in* $\operatorname{Pic} X$ *for some* $r \geq n+1$, *then* $X \simeq \mathbb{P}^n$.
(2) *If* $-K_X \equiv rL$ *in* $\operatorname{Pic} X$ *for some* $r \geq n$, *then* $X \simeq \mathbb{P}^n$ *or* Q^n.

0.8 There is a tight relation between Theorem 0.7 and the Theorems 0.2–6, despite their rather distinct appearances. Here we clarify the relation, and the partial logical dependence existing between those, and also explain the feature of our characterization in comparison with the prior characterizations collected above.

(0.8.1) First, each of the assumptions of Theorem 0.7 directly implies that X is Fano, while in Theorems 0.2–0.6 it is not assumed that X is Fano.

(0.8.2) The assumptions of Theorems 0.2–0.6 do not concern the anti-canonical divisor $-K_X$. The condition that L is ample is purely numerical, according to the Kleiman's criterion [Kl]. So our assumption is stated purely in the language of the intersection pairing

$$\operatorname{Pic}(X) \times \mathrm{CH}_1(X) \longrightarrow \mathbb{Z},$$

and the incidence between the Chow scheme of algebraic 1-cycles $\mathrm{Chow}(X)_1$ and the symmetric power $S^r(X)$ of X (i.e., the Chow scheme of algebraic 0-cycles of X). The only non-extrinsic character involved in the assumptions of Theorems 0.2–0.6 is that the curves C are assumed to be rational. (This was pointed out to us by J. Kollár; a generalization along this line is obtained in [Kac2]. Also a relevant example is observed by Iskovskikh–Prokhorov–Shokurov for Del Pezzo surfaces.)

(0.8.3) Assume that X is non-singular. Assume that $-K_X \equiv rL$ for some $r \geq n+1$, as in Theorem 0.7 (1). Then X is Fano (0.8.1), and by the arguments of [Mo1] (cf. [Cam], [KoMiMo3]), for general $x, y \in X$, there exists a rational curve C on X passing through x, y, with $(-K_X \,.\, C) = n+1$. Consequently, we have $(\mathrm{RC}_{\mathrm{gen}})_1^2$, and in particular $(\mathrm{RC}_{\mathrm{gen}}; x)_1^1$. Similarly, if $-K_X \equiv rL$ for some $r \geq n$, then we have $(\mathrm{RC}_{\mathrm{gen}})_2^3$, and in particular $(\mathrm{RC}_{\mathrm{gen}}; x)_2^2$. These show that Theorems 0.2–0.6 seem to generalize Theorem 0.7, at least when X is non-singular. However, the proofs of 0.2–0.6 rely on using a weak form of Theorem 0.7 (plus [Fuj4] when X is not assumed to have log-terminal singularities; [ABW], [Me2]) at the final stage (5.5.5).

(0.8.4) Briefly, the proof of Theorem 0.6 is outlined as follows.

First, starting from X with the assumption $(\mathrm{RC}; x)_1^1$, we verify $(\mathrm{RC}_{\mathrm{flat}}; x)_1^1$ (see Theorem 0.3), by using Theorem 2.6. Using this, combined with the result of §1 (particularly Theorem 1.5 (1)), we prove $\rho(X) = 1$ (Proposition 4.1), irrespective of $x \in \operatorname{Reg} X$ or not (cf. [ABW], [Zh2]).

Keeping this in mind, we then investigate X with $(\mathrm{RC}; x)_2^2$, $x \in \operatorname{Reg} X$. We prove that X satisfies either $(\mathrm{RC}; x)_1^1$ (then $X \simeq \mathbb{P}^n$ by [ABW] or Corollary 4.2), or otherwise there exists a '*uc-dense*' subset $\Lambda \subset X$ (see Terminology 2.4) such that for each $y \in \Lambda$, there exist two irreducible divisors S_x^y and S_y of X, having flat families of rational curves which make S_x^y and S_y satisfy $(\mathrm{RC_{flat}}; x)_1^1$ and $(\mathrm{RC_{flat}}; y)_1^1$, respectively. Moreover such two families $\{l_{x,t}^y\}_{t\in T_y}$ and $\{l_{y,t}\}_{t\in T_y}$ have a common parameter scheme T_y, such that $l_{x,t}^y \cap l_{y,t} \neq \emptyset$ for every $t \in T_y$ (Lemma 5.3). This is done by looking at the conic bundle structure arising from the condition $(\mathrm{RC}; x)_2^2$ (cf. [Isk1], [Kac1], [P1,2]), and making use of Mori's Bend-and-Break method (Lemma 5.2). Now by Theorem 2.10 applied to the collection of divisors S_y $(y \in \Lambda)$, we obtain a flat family $\{S_r\}_{r\in R}$ of codimension 1 subvarieties of X, endowed with a closed subscheme $Z := \coprod_{r\in R} G_r \subset \coprod_{r\in R} \operatorname{Hilb}(S_r)$, flat over R, which extends T_y's. This Z parametrizes a flat family of curves $\{l_z\}_{z\in Z}$ extending the original $l_{y,t}$'s (see (5.4.6)), and we arrive at a dichotomy:

(i) Every two points of X are joined by a finite chain of l_z's from the family $\{l_z\}_{z\in Z}$,

or

(ii) otherwise

(see 1.1–4). In case (ii), Corollary 1.13 applies to produce a surjection $X \to T$ onto a curve T, whose fibers are transversal to S_x^y's, which is absurd by what is formerly proved; namely that $\rho(S_x^y) = 1$. Therefore we have case (i), and Theorem 1.5 (1) yields $\rho(X) = 1$ (Proposition 5.4). Note that until this point Theorem 0.7 is not used. From this, together with the results of §3, it is proved that $(-K_X \,.\, C) \geq \dim X$. It automatically follows that the assumption of Theorem 0.7 is satisfied (in case X has log-terminal singularities; see [Fuj4], [Me2] for the general case), and thus we conclude $X \simeq \mathbb{P}^n$ or Q^n.

0.9 (0.9.1) In Theorem 0.6, the assumption $(\mathrm{RC}; x)_2^2$ is not inadequately strong, namely, neither $(\mathrm{RC}; x)_2^1$ nor $(\mathrm{RC})_2^2$ is enough to deduce $X \simeq \mathbb{P}^n$ or Q^n. In fact, there are a great many examples which satisfy $(\mathrm{RC})_2^2$ (and hence $(\mathrm{RC}; x)_2^1$) but not $(\mathrm{RC}; x)_2^2$. Indeed, even in the case that L is very ample, a large number of series of homogeneous varieties listed in [Kaj], [KOY] and [O] fall into $(\mathrm{RC})_2^2$. Also, most of *Del Pezzo varieties* of Fujita of dimension ≥ 3 seem to fall into $(\mathrm{RC})_2^2$, by virtue of Fujita's *ladder* argument [Fuj1]. (For example, see [Me1], [Am]; note that an enlargement of this notion to *algebraically equivalent* systems of algebraic cycles is found to play, in a certain circumstance, a role alternative to the Kodaira vanishing theorem to provide the necessary lower bound of the dimension of the global sections of L (e.g., (0.9.2)), which is valid in any characteristic; see [Kac2]). So our Theorem 0.6 serves to isolate Q^n in the class $(\mathrm{RC})_2^2$. This may also have a connection with the problem of Lazarsfeld–Van de Ven [LV]. For example, consider the following condition;

(RQ^m) . For two general points $x, y \in X$, there exists an m-dimensional subvariety $V = V_{x,y} \subset X$ such that $V \ni x, y$, and $(V, L|_V)$ satisfies $(\mathrm{RC}; x, y)_2^1$, with either

(a) $\dim X \geq 11,\ m > \frac{1}{2}\dim X$,

or

(b) $\dim X \geq 17, m \geq 9$.

In particular, the primary interest would be the case when L is very ample; $X \underset{|L|}{\hookrightarrow} \mathbb{P}^N$, or X has a fixed closed immersion into $\mathbb{P}^N$ with a fixed N, and the following holds:

($\mathrm{RQ}^m_{\mathrm{satur}}$). For a general point z of the *secant variety* $\mathrm{Sec}\, X \subset \mathbb{P}^N$, the union H_z of all the secant lines passing through z is an $(m+1)$-dimensional linear subspace of $\mathbb{P}^N$, and $H_z \cap X$ is a non-singular hyperquadric Q^m in $H_z \simeq \mathbb{P}^{m+1}$.

(See [FR], [O], which prescribe varieties with ($\mathrm{RQ}^m_{\mathrm{satur}}$), $m \geq 1$, in terms of the *Gauss maps*.) For example, *Severi varieties* of Zak [Za] satisfy the condition ($\mathrm{RQ}^m_{\mathrm{satur}}$), with $m = \frac{1}{2}\dim X$, $\dim X = 2, 4, 8, 16$ (see [LV]). Also the Grassmannian variety $X = \mathrm{Grass}(1, \mathbb{P}^r)$ parametrizing lines in $\mathbb{P}^r$ (with the Plücker embedding) satisfies ($\mathrm{RQ}^m_{\mathrm{satur}}$) with $m = 4$, irrespective of its dimension $\dim X = 2(r-1)$. (Note that $X \simeq Q^4$ when $r = 3$.) Moreover, the 10-dimensional *orthogonal Grassmannian* $X \subset \mathbb{P}^{15}$, which is a homogeneous space of $SO(10)$ (see [H], [Muk]), satisfies ($\mathrm{RQ}^m_{\mathrm{satur}}$) with $(\dim X, m) = (10, 6)$. The above conditions (a) and (b) are posed as counterhypotheses of these familiar examples in projective geometry.

In the meantime, if X satisfies ($\mathrm{RQ}^m_{\mathrm{satur}}$) plus the condition (a) above, then $\mathrm{Sec}\, X = \mathbb{P}^N$, by the theorem of Zak [Za], combined with the result of [FR], [O]. If we pose (b) instead of (a), the problem of [LV] is equivalent to asking whether $\mathrm{Sec}\, X = \mathbb{P}^N$, which is verified when X is a *homogeneous space* by Kaji [Kaj]. It would be of interest to find (or disprove the existence of) such $X \subset \mathbb{P}^N$, besides $Q^n \subset \mathbb{P}^{n+1}$, even under the assumption $\mathrm{Sec}\, X = \mathbb{P}^N$. As to this, a primary case is settled in [KaS].

(0.9.2) As was mentioned above, our Theorems 0.2 and 0.6 were proved originally with the assumption "X has only $\mathbb{Q}$-factorial log-terminal singularities", whereas it turned out that the log-terminality assumption was redundant, as pointed out by Mella [Me2].

In the 2-dimensional case, the log-terminality is stronger than $\mathbb{Q}$-factoriality. Indeed, a log-terminal singularity is nothing but a quotient singularity (see [Kaw2]), while a $\mathbb{Q}$-factorial singularity is the same as a rational singularity. Let X be a normal projective variety of dimension 2. If X is a *log Del Pezzo surface*, i.e., if $-K_X$ is 'numerically ample' in the sense of Mumford [Mum], then it is easily seen that X admits only rational singularities if and only if X is birational to $\mathbb{P}^2$ (cf. [Ab], [AN], [Che], [HW], [KeM], [R2], [Sh3]). So in dimension 2, the assumption "log-terminal" is *a priori* redundant. In case $\dim X \geq 3$, Fujita [Fuj1,5] characterizes (possibly singular) hyperquadric Q^n as a projective variety X having an ample Cartier divisor L, with

$$\dim H^0(X, L) \geq \dim X + 2, \qquad \text{and } (L)^{\dim X} = 2$$

[Fuj1, Theorem 2.2], without any condition on singularities of X. As a consequence, when (X, L) satisfies the cohomology vanishings

$$H^i(X, -tL) = 0 \quad (i \geq 1,\ 0 \leq t \leq \dim X, (i, t) \neq (\dim X, \dim X)),$$

e.g., when X satisfies the assumption in Theorem 0.7 (2) and has only log-terminal singularities, the Riemann–Roch theorem verifies the above pair of conditions on L, and hence $X \simeq Q^n$. This argument works by virtue of the vanishing theorems [Kaw1], [V], which are known to be true only for varieties with log-terminal singularities. For the general case, a further argument is required, since there is no standard way to find the estimate of $\dim H^0(X, L)$ without maintaining the vanishing theorems (cf. [Ish]). Another theorem of Fujita [Fuj4, §2] remedies this obstacle, and in fact our proof for this case (5.5.6) is based on his result [Fuj4, Theorem 2.3], following [Me2]. (cf. One encounters the same difficulty in the case of characteristic $p > 0$, even when X is assumed non-singular. An alternative approach which covers this case is found in [Kac2].)

0.10. This paper is organized as follows.

First in §1, we deal with a projective variety A, endowed with a fiberspace $B \to C$ which parametrizes closed subschemes of A (1.1). For points $x, y \in A$, we consider the following equivalence relation: $x \sim y$ if x and y are joined by a finite chain of fibers of $B \to C$ (cf. [Cam1,2], [KoMiMo1]). Typically, the equivalence class of $x \in A$ is not a Zariski closed subset of A (1.1–2). We prove two things. First, we investigate the Picard group of A when every two points are equivalent. In particular, if the fibration $B \to C$ is a Fano fiber bundle (at least outside a boundary of codimension ≥ 2), then we obtain a bound of the Picard number of A, $\rho(A) \leq \rho(F)$, where F is a general fiber of $B \to C$ (Theorem 1.5 (1)). Second, for the general case, we give one sufficient condition toward the existence of the algebraic quotient with respect to this equivalence relation, modulo a purely inseparable cover of A (Theorem 1.5 (2); cf. [Kol], Chap. IV. 4).

In §2, we formulate several extendability results for collections of closed subschemes or algebraic cycles of an algebraic variety M. These are made in the language of Hilbert schemes and Chow schemes (cf. [Kol]). First, we apply the *Dirichelet principle* to prove Theorem 2.6: For a collection of closed subschemes of M, indexed by a subset Λ of the Chow scheme $\operatorname{Chow} M$, satisfying the incidence relation ((IN) in 2.5), there exists an extending flat family $\{Y_h\}_{h\in H}$ of closed subschemes of M which preserves the incidence over the *uc-closure* (2.4) of $\Lambda \subset \operatorname{Chow} M$. Moreover, by running the Noetherian induction, we may realize the parameter scheme $H \subset \operatorname{Hilb} M$

of this flat family in such a way that the subset of H over which Y_h coincides with the originally given closed subscheme is *uc-dense* (in 2.4) in H (by Key Lemma 2.3). We also prove Proposition 2.7: For a surjective flat morphism $N \to H$, together with a uc-dense subset $J \subset H$, over which an effective algebraic cycle B_h sitting in the fiber is given, there exists a section $H \to \mathrm{Chow}(N/H) \to H$ which extends those B_h's, after a base change of H. The proof is straightforward through the graph construction. As a main consequence of these results, we prove Theorem 2.10: For a collection of closed subschemes $\{\mathcal{E}_\lambda\}_{\lambda\in\Lambda}$ indexed by $\Lambda \subset \mathrm{Chow}\, M$, with each $\mathcal{E}_\lambda$ having a covering flat family of closed subschemes $\{E_{\lambda,t}\}_{t\in T_\lambda}$ whose members satisfy the obvious incidence relation, there exists an extending flat family $\{S_r\}_{r\in R}$ of closed subschemes of M, each of whose members carries a flat family of closed subschemes, preserving the incidence. This Theorem 2.10 has many useful consequences. For instance, this gives rise to the existence of the *virtual flag schemes* for varieties with $(\mathrm{RC})_1^2$, which enables us to work over algebraic equivalence, makes one free from remaining in the fixed linear systems, valid in any characteristic (see [Kac2]). These results are used essentially in §§4–5. Also in 2.11, we redefine the condition $(\mathrm{RC}; y_1, \dots, y_s)_d^r$, in 0.4.

In §3, we investigate the tangent bundle T_X of a variety X which has plenty of rational curves. In [Mo1], Mori developed a technique of deforming rational curves on a variety, showing that if T_X is sufficiently positive, for example, if for a non-constant morphism $\nu\colon C \to X$ from a curve C, the pull-back ν^*T_X is ample, then X has a covering family of rational curves. As an aside, a close observation of his argument allows us to deduce a consequence which is in an opposite direction. We give formulae (Propositions 3.1, 3.3) that carry information about subvector bundles of ν^*T_X, which in particular yield estimates of $c_1(\nu^*T_X)$ when the given collections of rational curves cover X. This is a restatement of Kollár's result [Kol, Chap. IV, 3.7] in a slightly generalized form. This argument works for singular varieties as well, by virtue of Kollár's generalization [Kol], where ν^*T_X is replaced by $(\nu^*\Omega_X^1)^\vee$. According to Lemma 3.4, this replacement does not interfere with the estimate of $(-K_X \,.\, \nu(C))$.

In §4, we prove that the Picard number $\rho(X)$ of X is equal to 1, if X satisfies $(\mathrm{RC}; x)_1^1$, irrespective of the type of singularities of X, and regardless of whether $x \in \mathrm{Reg}\, X$ or not (Proposition 4.1). As a corollary, we reproduce the result of Andreatta–Ballico–Wiśniewski [ABW] (Corollary 4.2), that $X \simeq \mathbb{P}^n$, when $x \in \mathrm{Reg}\, X$ (under the weakened assumption $(\mathrm{RC}; x)_1^1$).

In §5, we prove the main theorem (Theorem 5.1 = Theorem 0.6), a stronger version of Theorem 0.2.

Acknowledgements. The authors are grateful to Professor J. Kollár for suggesting (0.8.2), Professors S. Keel and V. Shokurov for helping the discussion on §1, and Professor H. Kaji for pointing out the error of the proof of 1.7 in the first draft.

We are indebted to the generosity of Professors M. Andreatta, E. Ballico and J. Wiśniewski, during the revision of this paper, toward our first draft which mistreated their result [ABW].

We are grateful to Professor M. Mella who corresponded with the authors [Me2] and pointed out that the assumption that X has log-terminal singularities in Theorem 0.2, made in our first draft, is redundant.

We also thank Professors V. Alexeev, M. Beltrametti, F. Campana, A. Conte, S. D. Cutkosky, T. Fujita, M. Gross, S. Kimura, S. Kovács, K. Matsuki, T. Peternell, B. Shiffman and Q. Zhang for valuable discussions.

Finally, we thank the referee for the thorough and valuable comments.

1. Division of a variety into scrolls

1.1. (cf. [Cam1,2]) (1.1.0) Let A, B, C be normal projective varieties, $n := \dim A$, and let

$$p\colon B \to A \quad \text{and} \quad q\colon B \to C$$

be surjective morphisms. Assume that q is equi-dimensional, with connected fibers, and is flat over a Zariski open subset $C^0 \subset C$ of codimension 2, and that for each $c \in C$, $p|_{q^{-1}(c)}\colon q^{-1}(c) \to A$ is finite. Let B_c be its image: $B_c := p(q^{-1}(c))$. For a Zariski closed subset $Z \subset A$, let

$$S_1(Z) := p q^{-1} q p^{-1}(Z),$$

and inductively define $\{S_m(Z)\}_{m \geq 0}$ by setting $S_m(Z) := S_1(S_{m-1}(Z))$ (and $S_0(Z) := Z$) (*cf.* Campana [Cam2], Kollár–Miyaoka–Mori [KoMiMo1]). Obviously this forms an increasing sequence of Zariski closed subsets of A:

$$Z = S_0(Z) \subset S_1(Z) \subset S_2(Z) \subset \cdots \subset S_m(Z) \subset \cdots \tag{1.1.1}$$

Let

$$S(Z) := \bigcup_{m \geq 1} S_m(Z). \tag{1.1.2}$$

Note that in general $S(Z)$ may not be Zariski closed in A, although the $S_m(Z)$ are. For a point $x \in A$, we often write $S_m(x)$, $S(x)$, instead of $S_m(\{x\})$, $S(\{x\})$, respectively.

(1.1.3) If Z is connected, then so are $S_m(Z)$ and $S(Z)$.

We introduce a relation '$\sim$' on the set of points of A as follows:

$x \sim y$ if and only if there exists a finite set $\{c_1, \ldots, c_r\}$ of points of C such that $x \in B_{c_1}$, $y \in B_{c_r}$, and for $i = 1, \ldots, r-1$, $B_{c_i} \cap B_{c_{i+1}} \neq \emptyset$.

LEMMA 1.2. *$\sim$ is an equivalence relation on the set of points of A. $S(x)$ is the equivalence class containing $x \in A$.*

1.3. The following two cases are of specific interest for the purpose.

Case I. $S_m(x) = A$ for some $x \in A$ and for some $m \geq 1$.

Case II. $\dim S_m(x) = n - 1$ for a general $x \in A$ and for any $m \geq 1$.

Remark 1.4. (1) Every $S(x)$ is a union of at most countably many Zariski closed subsets. (2) In particular, if the cardinality of the base field k is uncountable, and if we are not in Case I, then $S(x) \neq A$, and A is divided into uncountably many such $S(x)$'s.

The following is the main result of this section:

THEOREM 1.5. *With the notation and assumption above, let* $N^1(A) := \operatorname{Pic} A/\equiv$ *be the group of numerical equivalence classes of Cartier divisors of* A, *and* $N^1(A)_{\mathbb{Q}} := N^1(A) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

(1) *Assume that the quintuple* (A, B, C, p, q) *falls into Case I, and that there exists a Zariski open subset* $C^0 \subset C$ *with* $\operatorname{codim}_C(C - C^0) \geq 2$ *such that* $q|_{q^{-1}(C^0)}$: $q^{-1}(C^0) \to C^0$ *is an étale fiber bundle with the fiber* F. *Let* $\operatorname{Pic} A \xrightarrow{\varphi} \operatorname{Pic} F$ *be the natural homomorphism sending* $L \in \operatorname{Pic} A$ *to* $(p^*L)|_F$. *Then*

$$\operatorname{Ker} \varphi \subset \{L \in \operatorname{Pic} A \mid L \equiv 0\}.$$

In particular, if $\operatorname{Pic} F \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \simeq N^1(F)_{\mathbb{Q}}$, *then the induced homomorphism* $N^1(A)_{\mathbb{Q}} \to N^1(F)_{\mathbb{Q}}$ *is injective, and* $\rho(A) \leq \rho(F)$.

(2) (*Existence of the algebraic quotient, modulo a purely inseparable cover of* A.) *Assume that* (A, B, C, p, q) *falls into Case II. Assume that* A *is* $\mathbb{Q}$*-factorial, and that the base field* k *is uncountable. Then there exists an irreducible projective curve* $H \subset \operatorname{Hilb}(A)$ *such that the induced family* α: $\mathcal{H} \to H$ *and the natural projection* β: $\mathcal{H} \to A$ *satisfy the following*:

(1.5.1) β *is a finite morphism.*

(1.5.2) *There exists a non-empty Zariski open subset* $H^0 \subset H$ *such that on* $\mathcal{H}^0 := \alpha^{-1}(H^0)$, β *is set theoretically an injection, and* $\beta(\mathcal{H}^0)$ *is an open dense subset of* A.

(1.5.3) *For any* $x \in \beta(\mathcal{H}^0)$,

$$S(x) = S_1(x) = \beta\alpha^{-1}\alpha\beta^{-1}(x),$$

which is a prime divisor of A.

For Case I, we begin with the following result.

LEMMA 1.6 (see also [Sa], §1)). *Let* p: $B \to A$ *and* q: $B \to C$ *be as in* 1.1. *Let* Z *be a Zariski closed subset of* A, *and* L *and* M *line bundle on* A *and* C *respectively. Assume that* $q^*M \equiv p^*L$ *on* B, *and that either* $\dim Z = 0$, *or* $(L \,.\, l) = 0$ *for any irreducible curve* $l \subset Z$. *In such a case we let* $L|_Z \equiv 0$. *Then for any* $m \geq 0$, $L|_{S_m(Z)} \equiv 0$, *and* $M|_{q\, p^{-1}(S_m(Z))} \equiv 0$.

Proof. Let $C_Z := q\, p^{-1}(Z)$, $q_Z := q|_{p^{-1}(Z)}\colon\ p^{-1}(Z) \to C_Z$, and consider the diagram

$$\begin{array}{ccccc} C_Z & \xleftarrow{q_Z} & p^{-1}Z & \longrightarrow & Z \\ \cap & & \cap & & \cap \\ C & \xleftarrow[q]{} & B & \xrightarrow[p]{} & A. \end{array} \tag{1.6.1}$$

By the assumption $L|_Z \equiv 0$ and the projection formula, $(p^*L)|_{p^{-1}(Z)} \equiv 0$, that is, $(q^*M)|_{p^{-1}(Z)} \equiv 0$. Since q_Z is surjective, we have $M|_{C_Z} \equiv 0$. Repeat the argument, switching the role of (A, L, Z) and (C, M, C_Z), to obtain $L|_{S_1(Z)} \equiv 0$. Then switch (C, M, C_Z) and $(A, L, S_1(Z))$, to obtain $M|_{C_{S_1(Z)}} \equiv 0$, and so on. $\square$

1.7. *Proof of Theorem* 1.5 (1). Let L be a line bundle on A such that $L \in \operatorname{Ker}\varphi$, namely, $p^*L|_F \sim 0$ for a general fiber F of q. By the assumption, there exists a Zariski open subset $C^0 \subset C$ with $\operatorname{codim}_C(C - C^0) \geq 2$ such that $q|_{q^{-1}(C^0)}\colon\ q^{-1}(C^0) \to C^0$ is an étale fiber bundle with the fiber F. Hence by the base change theorem, there exists a line bundle M on C such that $p^*L \equiv q^*M$ (cf. [Sh2]). By Lemma 1.6, $L|_{S_m(x)} \equiv 0$ for all $m \geq 1$. By the assumption, $S_m(x) = A$ for some m, and hence $L \equiv 0$. Thus we have proved $\operatorname{Ker}\varphi \subset \{L \in \operatorname{Pic} A \mid L \equiv 0\}$, and we obtain the commutative diagram

$$\begin{array}{ccc} \operatorname{Pic} A/\operatorname{Ker}\varphi & \hookrightarrow & \operatorname{Pic} F \\ \downarrow & & \downarrow \\ N^1(A) & \longrightarrow & N^1(F) \end{array}$$

with the vertical arrows both surjective. In particular, if $\operatorname{Pic} F \otimes_{\mathbb{Z}} \mathbb{Q} \simeq N^1(F)_{\mathbb{Q}}$, then the lefthand vertical arrow also becomes an isomorphism after tensoring with $\mathbb{Q}$. Hence from the diagram, we have $N^1(A)_{\mathbb{Q}} \hookrightarrow N^1(F)_{\mathbb{Q}}$. $\square$

Now we turn to Case II.

1.8. Let M be an n-dimensional normal projective variety defined over an uncountable field k. Let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a set of closed subschemes of M, indexed by an uncountable set Λ.

LEMMA 1.9 (cf. Theorem 2.6). *Under the assumption of* 1.8, *there exists an irreducible closed subscheme $H \subset \operatorname{Hilb}(M)$, parametrizing closed subschemes $\{Y_h\}_{h\in H}$ of M, an uncountable subset $\Lambda_0 \subset \Lambda$, and an injection $\iota\colon\ \Lambda_0 \hookrightarrow H$ such that*

$$Y_{\iota(\lambda)} = F_\lambda \quad (\lambda \in \Lambda_0).$$

Proof. Let us fix a very ample line bundle $\mathcal{O}_M(1)$ of M, and let us take each $\lambda \in \Lambda$ into the polynomial

$$e(\lambda) := \chi(F_\lambda, \mathcal{O}_{F_\lambda}(1)^{\otimes t}) \in \mathbb{Q}[t].$$

Since $\{\deg e(\lambda) \mid \lambda \in \Lambda\}$ is bounded, $R := e(\Lambda)$ is a countable set, whereas by the assumption Λ is uncountable. Hence there exists an uncountable subset $\Lambda_1 \subset \Lambda$ such that $e(\Lambda_1)$ consists of a single polynomial, $e(\Lambda_1) = \{P_{\Lambda_1}(t)\}$. Let $\mathrm{Hilb}^{P_{\Lambda_1}}(M)$ be the subscheme of $\mathrm{Hilb}(M)$, the Hilbert scheme of M, consisting of those points which have the Hilbert polynomial P_{Λ_1}. $\mathrm{Hilb}^{P_{\Lambda_1}}(M)$ is a union of several connected components of $\mathrm{Hilb}(M)$ (see [Ig]). Let $\{Y_h\}_{h \in \mathrm{Hilb}^{P_{\Lambda_1}}(M)}$ be the corresponding family of subschemes of M. Then there exists an injective map $\Lambda_1 \overset{\iota}{\hookrightarrow} \mathrm{Hilb}^{P_{\Lambda_1}}(M)$ such that $F_\lambda = D_{\iota(\lambda)}$. Since $\mathrm{Hilb}^{P_{\Lambda_1}}(M)$ is a projective scheme, we may choose an irreducible component H of $\mathrm{Hilb}^{P_{\Lambda_1}}(M)$ such that $\Lambda_0 := \Lambda_1 \cap H$ is uncountable. This H and $\Lambda_0 \hookrightarrow H$ satisfy the required property. □

PROPOSITION 1.10 (THEOREM ON ALGEBRAIZATION, PART I). *In* 1.8, *assume further that for each $\lambda \in \Lambda$, F_λ is purely of dimension r, and there exists a set of $(n-r)$ effective Weil divisors $\{E_\lambda^{(1)}, \dots, E_\lambda^{(n-r)}\}$ such that*

$$\begin{cases} E_\lambda^{(1)} \text{ is an irreducible } \mathbb{Q}\text{-Cartier divisor } (\lambda \in \Lambda), \\ E_\lambda^{(1)} \cap E_\mu^{(1)} = \emptyset \quad (\lambda, \mu \in \Lambda, \ \lambda \neq \mu), \quad \text{and} \\ F_\lambda = \bigcap_{i=1}^{n-r} E_\lambda^{(i)} \quad (\text{scheme theoretically}). \end{cases}$$

Moreover assume that $\bigcup_{\lambda \in \Lambda} F_\lambda$ is not contained in a union of countably many proper Zariski closed subsets of M. Then there exists an irreducible closed subscheme H of the Hilbert scheme $\mathrm{Hilb}(M)$, *with* $\dim H = n - r$, *such that the induced family $\alpha \colon \mathcal{H} \to H$ and the natural projection $\beta \colon \mathcal{H} \to M$ satisfy the following*:

(1) *There exists a non-empty Zariski open subset $U \subset H$ such that $\beta|_{\alpha^{-1}(U)} \colon \alpha^{-1}(U) \to M$ is set theoretically an injection.*

(2) *Denote by Y_h the closed subscheme of M corresponding to $h \in H$. Then there exists an uncountable subset $\Lambda_0 \subset \Lambda$ and an injection $\iota \colon \Lambda_0 \hookrightarrow H$ such that for each $\lambda \in \Lambda_0$,*

$$Y_{\iota(\lambda)} = F_\lambda.$$

Proof. Let $H \subset \mathrm{Hilb}(M)$ be the closed subscheme obtained by Lemma 1.9, and $\{Y_h\}_{h \in H}$ the corresponding family over H. Clearly, it is enough to prove the following:

(1.10.0) There exists a Zariski open subset $U \subset H$ such that for $h, h' \in U$ with $h \neq h'$, $Y_h \cap Y_{h'} = \emptyset$.

F_λ is assumed to be the scheme theoretic intersection of $(n-r)$ effective Weil divisors $\{E_\lambda^{(i)}\}_{i=1,\dots,n-r}$. Apply Lemma 1.9 to the collection of divisors $\{E_\lambda^{(1)}\}_{\lambda \in \Lambda}$, to obtain an irreducible closed subscheme $H^{(1)} \subset \mathrm{Hilb}(M)$ parametrizing closed subschemes $\{D_h^{(1)}\}_{h\in H^{(1)}}$, an uncountable subset $\Lambda^{(1)} \subset \Lambda$, and an injection $\iota^{(1)}\colon \Lambda^{(1)} \hookrightarrow H^{(1)}$ such that

$$D^{(1)}_{\iota^{(1)}(\lambda)} = E^{(1)}_\lambda \quad (\lambda \in \Lambda^{(1)}). \tag{1.10.1}$$

Next, apply Lemma 1.9 to the subcollection $\{E_\lambda^{(2)}\}_{\lambda\in\Lambda^{(1)}}$, to obtain $H^{(2)} \subset \mathrm{Hilb}(M)$ parametrizing $\{D_h^{(2)}\}_{h\in H^{(2)}}$, an uncountable subset $\Lambda^{(2)} \subset \Lambda^{(1)}$, and an injection $\iota^{(2)}\colon \Lambda^{(2)} \hookrightarrow H^{(2)}$ such that

$$D^{(2)}_{\iota^{(2)}(\lambda)} = E^{(2)}_\lambda \quad (\lambda \in \Lambda^{(2)}).$$

Then apply Lemma 1.9 to the subcollection $\{E_\lambda^{(3)}\}_{\lambda\in\Lambda^{(2)}}$, and so on. Repeat this procedure $(n-r)$ times, and we obtain $H^{(1)}, \dots, H^{(n-r)} \subset \mathrm{Hilb}(M)$, a chain of uncountable subsets $\Lambda^{(n-r)} \subset \Lambda^{(n-r-1)} \subset \cdots \subset \Lambda^{(1)} \subset \Lambda$, and injections $\iota^{(i)}\colon \Lambda^{(i)} \hookrightarrow H^{(i)}$ $(i = 1, \dots, n-r)$ such that

$$D^{(i)}_{\iota^{(i)}(\lambda)} = E^{(i)}_\lambda, \quad \text{and hence} \quad \bigcap_{i=1}^{n-r} D^{(i)}_{\iota^{(i)}(\lambda)} = F_\lambda \quad (\lambda \in \Lambda^{(i)}). \tag{1.10.2}$$

Let $\Lambda^{(0)} := \Lambda^{(n-r)}$, and $\iota^{(0)} := (\iota^{(1)}, \dots, \iota^{(n-r)})\colon \Lambda^{(0)} \hookrightarrow H^{(1)} \times \cdots \times H^{(n-r)}$. Take the smallest subset H^c of $H^{(1)} \times \cdots \times H^{(n-r)}$ which contains $\iota^{(0)}(\Lambda^{(0)})$ and is a union of at most countably many Zariski closed subsets (cf. §2, 2.4). Furthermore, take an irreducible component H_1 of H^c of maximal dimension, and let

$$\Lambda_1 := H_1 \cap (\iota^{(0)}(\Lambda^{(0)})) \overset{\iota}{\hookrightarrow} H_1. \tag{1.10.3}$$

Denote $D^{(i)}_{\pi^{(i)}(h)}$ simply by $D_h^{(i)}$, where $\pi^{(i)}\colon H_1 \to H^{(1)}$ is the i-th projection. Also for $\lambda \in \Lambda_1$, denote $D^{(i)}_{\iota(\lambda)}$ by $D^{(i)}_\lambda$. By construction we have the following:

(1.10.4) Λ_1 is uncountable, and Λ_1 is not contained in a countable union of proper Zariski closed subsets of H_1.

$$D^{(i)}_\lambda = E^{(i)}_\lambda \quad (i = 1, \dots, n-r), \quad \text{and hence} \quad \bigcap_{i=1}^{n-r} D^{(i)}_\lambda = F_\lambda \quad (\lambda \in \Lambda_1) \tag{1.10.5}$$

(scheme theoretically). Take a Zariski open subset $V \subset H_1$ such that $\{\bigcap_{i=1}^{n-r} D_h^{(i)}\}_{h\in V}$ forms a flat family of closed subschemes of M (where the intersections are the scheme theoretic ones). By (1.10.4) we have the following:

(1.10.6) $\Lambda_0 := V \cap \Lambda_1$ is an uncountable set, and is not contained in a countable union of proper Zariski closed subsets of V.

Let $v\colon V \to \mathrm{Hilb}(M)$ be the induced morphism, and H the closure of $\mathrm{Im}\, v$. H is an irreducible projective scheme. Let $\{Y_h\}_{h\in H}$ be the family parametrized by H. For $h \in U_1 := v(V)$, we have

$$Y_h = \bigcap_{i=1}^{n-r} D_h^{(i)}. \tag{1.10.7}$$

Moreover, by (1.10.5) and the assumption, $v|_{\Lambda_0}$ is injective, and for $\lambda, \lambda' \in \Lambda_0$ with $\lambda \neq \lambda'$, we have $D_\lambda^{(1)} \cap D_{\lambda'}^{(1)} = \emptyset$. This implies

$$(1.10.8) \qquad (D_h^{(1)} . D_{h'}^{(1)} . B_1 . \cdots . B_{n-2}) = 0$$

for any $\mathbb{Q}$-Cartier divisors $B_1, \ldots, B_{n-2}$, and $h, h' \in H$. Since $D_\lambda^{(1)} = E_\lambda^{(1)}$ (by (1.10.5)) and $E_\lambda^{(1)}$ is assumed to be irreducible for $\lambda \in \Lambda_0$, by (1.10.4), there exists a non-empty Zariski open subset $U \subset U_1$ such that for any $h \in U$, $D_h^{(1)}$ is irreducible. In particular, for $h, h' \in U$ with $h \neq h'$, $D_h^{(1)} \cap D_{h'}^{(1)} = \emptyset$, and hence $Y_h \cap Y_{h'} = \emptyset$ [Ful]. Thus (1.10.0) is proved, and we are done. □

1.11. *Proof of Theorem* 1.5 (2). We apply Proposition 1.10 with $M = A$. First, let $\{x_\lambda\}_{\lambda \in \Lambda}$ be an entire collection of representatives of the equivalence relation '$\sim$' on A in Lemma 1.2:

$$(1.11.1) \qquad S(x_\lambda) \cap S(x_\mu) = \emptyset \text{ for } \lambda \neq \mu, \quad \text{and} \quad \bigcup_{\lambda \in \Lambda} S(x_\lambda) = A.$$

Since the base field k is assumed to be uncountable, and since we are not in Case I, we have $S(x_\lambda) \neq A$ by Remark 1.4 (2). Moreover, by assumption, $\dim S_1(x_\lambda) = n - 1$ for each $\lambda \in \Lambda$. Take an irreducible component F_λ of $S_1(x_\lambda)$ with $\dim F_\lambda = n - 1$.

$$(1.11.2) \qquad F_\lambda \cap F_\mu = \emptyset \text{ for } \lambda \neq \mu.$$

Again by Remark 1.4 (2), the index set Λ is uncountable. Moreover, by the assumption, A is $\mathbb{Q}$-factorial, and so every F_λ is $\mathbb{Q}$-Cartier. Hence the assumption of Proposition 1.10 is satisfied. By Proposition 1.10, there exists an irreducible closed subscheme H of $\mathrm{Hilb}(A)$ of dimension 1, with the associated flat morphism α and the projection β,

$$(1.11.3) \qquad \begin{array}{ccc} \mathcal{H} & \xrightarrow{\beta} & A \\ {\scriptstyle\alpha}\downarrow & & \\ H & & \end{array}$$

such that (1.11.4) and (1.11.5) hold as follows:

(1.11.4) There exists a finite set $\{h_1, \ldots, h_s\}$ of points of H such that every fiber $\alpha^{-1}(h)$ of α over $H^0 := \{h_1, \ldots, h_s\}$ is irreducible, and of dimension $n - 1$.

(1.11.5) Let $\mathcal{H}^0 := \alpha^{-1}(H^0)$, then $\beta|_{\mathcal{H}^0}$ is set theoretically an injection. In particular, $\beta \colon \mathcal{H} \to A$ is a finite morphism.

(1.11.6) Let $A^0 := \beta(\mathcal{H}^0)$. This is a Zariski open subset of A.

Consider $D_h := \beta(\alpha^{-1}(h))$ $(h \in H^0)$. D_h is a prime divisor of A by (1.11.4).

We are going to prove

$$S_1(x) = S(x) = D_h \tag{1.11.7}$$

for any $x \in D_h$.

In fact, by assumption $\dim S_1(x) = n - 1$, it is enough to prove $S(x) \subset D_h$, so let us assume to the contrary that $S(x) \not\subset D_h$. By the definition of $S(x)$ (1.1.2), there exists $B_c \subset S(x)$ (1.1.0) such that $B_c \cap D_h \neq \emptyset$ and $B_c \not\subset D_h$ (1.1.3). Thus we may take an irreducible curve $C \subset B_c$ such that $C \cap D_h \neq \emptyset$ and $C \not\subset D_h$. $(D_h . C) > 0$, and hence $(D_{h'} . C) > 0$ for all $h' \in H$. Thus $B_c \cap D_{h'} \neq \emptyset$. By construction, for $h' \in \Lambda_0$, $D_{h'} \subset S_1(y)$ for some $y \in D_{h'}$, and hence $y \in S(x)$ for $y \in \bigcup_{h' \in \Lambda_1} D_{h'}$; i.e.,

$$S(x) \supset \bigcup_{h' \in \Lambda_1} D_{h'}. \tag{1.11.8}$$

This contradicts Remark 1.4 (1), since $S(x) \neq A$ (by Remark 1.4 (2)), the $D_{h'}$'s are all divisors, and Λ_0 is uncountable. Hence (1.11.7). □

COROLLARY 1.12. *Under the same assumption as in Theorem* 1.5 (2), *for any $x \in A$, $S(x)$ is a Zariski closed subset of A, which is purely of codimension* 1.

COROLLARY 1.13 (Existence of the algebraic quotient, $\operatorname{char} k = 0$). *In Theorem* 1.5 (2), *if we also assume that* $\operatorname{char} k = 0$, *then there exists a smooth projective curve $\overline{H}$ and a surjective morphism $\overline{\alpha}$: $A \to \overline{H}$ such that*

$$S(x) = \overline{\alpha}^{-1}\overline{\alpha}(x) \tag{1.13.1}$$

for any $x \in A$.

Proof. Let α: $\mathcal{H} \to H$, β: $\mathcal{H} \to A$, $\mathcal{H}^0 \subset \mathcal{H}$ and $H^0 \subset H$ be as in 1.11. Since $\operatorname{char} k$ is assumed to be 0, by the Zariski Main Theorem, (1.11.5) implies the following:

(1.13.2) β is a finite morphism, and $\beta|_{\mathcal{H}^0}$ is an open immersion. In particular, β is a finite birational morphism.

Since A is assumed to be normal (1.1.0), again by the Zariski Main Theorem, β is an isomorphism:

$$\beta\colon \mathcal{H} \xrightarrow{\sim} A.$$

Through this identification, α can be viewed as a morphism from A: α: $A \to H$.

(1.13.3) Take the normalization of H; ν: $\overline{H} \to H$. By Theorem 1.5 (2), a general fiber of α is of the form $S(x)$ ($x \in A$), and hence is connected by (1.1.3). Thus:

(1.13.4) The induced morphism $\overline{\alpha}$: $A \to \overline{H}$ has connected fibers.

(1.13.5) By (1.5.3), this $\overline{\alpha}$ satisfies (1.13.1) for a point x in the open set $\alpha^{-1}(H^0) = \overline{\alpha}^{-1}(\nu^{-1}(H^0)) \subset A$.

We still have to verify (1.13.1) for an arbitrary $x \in A$. If $S(x) \not\subset \overline{\alpha}^{-1}\overline{\alpha}(x)$, then (1.13.5) plus an argument similar to (1.11.7–8) shows that $S(x) \supset \alpha^{-1}(H^0)$. However this is impossible by Remark 1.4 (2). Hence

$$S(x) \subset \overline{\alpha}^{-1}\overline{\alpha}(x).$$

By (1.13.4), we conclude that this inclusion is in fact an equality:

$$S(x) = \overline{\alpha}^{-1}\overline{\alpha}(x).$$

Hence we have (1.13.1). □

2. Theorems on algebraizations

In this section, we prove several results on algebraic extendability for a collection of cycles or subschemes of a variety (Theorem 2.6, Proposition 2.7, and Theorem 2.10) (cf. Proposition 1.10). Also at the end of this section (2.11), we give a precise definition of the condition (RC; $y_1, \dots, y_s)^r_d$ for a polarized variety (X, L). In particular, the case $(s, d, r) = (1, 1, 1)$ and $(1, 2, 2)$ are exactly the assumptions of our Main Theorems (Theorem 4.1 and 5.1), respectively. These assumptions are apparently weaker than those stated in the introduction, and carry more flexibility (see §5).

Throughout this section, k is an algebraically closed field of uncountable cardinality and of arbitrary characteristic.

First we start with some set theory in 2.0–3. For a set E, denote $\#E$ the cardinality of E.

Definition 2.0. For a set E, let $\mathfrak{P}(E)$ denote the set of all subsets of E.

(2.0.1) Consider the following set of axioms (Z0–3) for a subset $\mathcal{Z}(E) \subset \mathfrak{P}(E)$:

(Z0) $\emptyset, E \in \mathcal{Z}(E)$, $\{x\} \in \mathcal{Z}(E)$ for $x \in E$.

(Z1) If $F_1, F_2 \in \mathcal{Z}(E)$, then $F_1 \cap F_2 \in \mathcal{Z}(E)$.

(Z2) If $\{F_i\}_{i\in I}$ is a collection of countably many elements of $\mathcal{Z}(E)$, then

$$\bigcup_{i\in I} F_i \in \mathcal{Z}(E).$$

(2.0.2) We say that $F \in \mathcal{Z}(E)$ is *$\mathcal{Z}$-irreducible* if for any collection $\{F_\lambda\}_{\lambda\in\Lambda}$ of distinct elements of $\mathcal{Z}(E)$ such that $F_\lambda \subsetneq F$ and $F = \bigcup_{\lambda\in\Lambda} F_\lambda$, Λ is uncountable.

(Z3) For any $F \in \mathcal{Z}(E) - \{\emptyset\}$, there exists a collection $\{F_r\}_{r\in R}$ of $\mathcal{Z}$-irreducible subsets, with R at most countable, such that

$$F = \bigcup_{r\in R} F_r.$$

A set E, equipped with $\mathcal{Z}(E)$ satisfying the axioms (Z0–3) is called a *$\mathcal{Z}$-set*, and is denoted by $(E, \mathcal{Z}(E))$.

(2.0.3) Let $(E, \mathcal{Z}(E))$ be a $\mathcal{Z}$-set, $F \in \mathcal{Z}(E)$. A subset $D \subset F$ (not necessarily $D \in \mathcal{Z}(E)$) is called *$\mathcal{Z}$-dense in F*, if there is no $G \in \mathcal{Z}(E)$ such that $D \subset G \subsetneq F$.

(2.0.4) We sometimes require certain optional conditions, such as (Z4) or (Z5) below, in addition to (Z0–3):

(Z4) E is $\mathcal{Z}$-irreducible.

(Z5) (Quasi-Noetherian condition) Let $\{F^n\}_{n\in\mathbb{N}}$ be a descending chain of $\mathcal{Z}$-irreducible subsets

$$F^0 \supset F^1 \supset \cdots \supset F^n \supset F^{n+1} \supset \cdots.$$

Then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $F^n = F^{n_0}$.

(2.0.5) Let $(E, \mathcal{Z}(E))$, $(E', \mathcal{Z}(E'))$ be $\mathcal{Z}$-sets. A map $e\colon \mathcal{Z}(E) \to \mathcal{Z}(E')$ is called a *$\mathcal{Z}$-map* if the following two conditions are satisfied:

(M1) For $F, G \in \mathcal{Z}(E)$ with $F \subset G$, $e(F) \subset e(G)$.

(M2) For a collection $\{F_\nu\}_{\nu\in\mathcal{N}}$ of elements of $\mathcal{Z}(E)$, such that $\bigcup_{\nu\in\mathcal{N}} F_\nu \in \mathcal{Z}(E)$, we have $\bigcup_{\nu\in\mathcal{N}} e(F_\nu) \in \mathcal{Z}(E')$, and

$$e\left(\bigcup_{\nu\in\mathcal{N}} F_\nu\right) = \bigcup_{\nu\in\mathcal{N}} e(F_\nu).$$

(2.0.6) If E is countable, then it is easily seen that $\mathcal{Z}(E) = \mathfrak{P}(E)$, and the one-point sets $\{x\}$ $(x \in E)$ are the only $\mathcal{Z}$-irreducible subsets, so the whole thing becomes trivial in this case. So from now on let us make one convention: for a $\mathcal{Z}$-set $(E, \mathcal{Z}(E))$, E is always assumed to be uncountable, unless otherwise specified.

LEMMA 2.1. *Let $(E, \mathcal{Z}(E))$ be a $\mathcal{Z}$-set, and $F \in \mathcal{Z}(E)$ a $\mathcal{Z}$-irreducible subset. Let $\{D_i\}_{i\in I}$ be a collection of subsets of F, with the countable index set I, and $D \subset F$ such that*

$$D \subset \bigcup_{i\in I} D_i.$$

Assume that D is $\mathcal{Z}$-dense in F. Then there exists $i \in I$ such that D_i is $\mathcal{Z}$-dense in F.

In particular, if assume that $D_i \in \mathcal{Z}(E)$ for each $i \in I$, then $D_i = F$ for some $i \in I$.

Setting 2.2. Let $(H, \mathcal{Z}(H))$, $(W, \mathcal{Z}(W))$ be $\mathcal{Z}$-sets, and $L\colon \mathcal{Z}(H) \to \mathcal{Z}(W)$ a $\mathcal{Z}$-map.

(A1) Assume that $\mathcal{Z}(H)$ satisfies the axiom (Z5), and $\mathcal{Z}(W)$ satisfies (Z4).

Let Λ be an uncountable set, and $a: \Lambda \to H$ a map. Define a map $c: \Lambda \to \mathcal{Z}(W)$ by $c(\lambda) := L(\{a(\lambda)\})$. This is well defined by (Z0). Assume the following:

$$\bigcup_{\lambda \in \Lambda} c(\lambda) \text{ is } \mathcal{Z}\text{-dense in } W. \tag{A2}$$

KEY LEMMA 2.3. *Under Setting 2.2, there exists a $\mathcal{Z}$-irreducible subset $H_\Lambda \subset H$ such that*

$$L(H_\Lambda) = W \tag{1}$$

and

$$a(\Lambda) \cap H_\Lambda \textit{ is } \mathcal{Z}\textit{-dense in } H_\Lambda. \tag{2}$$

Proof. By (Z3), applied to $\mathcal{Z}(H)$,

$$H = \bigcup_{r \in R} H_r, \tag{2.3.0}$$

where

$$R \text{ is countable and } H_r \text{ is } \mathcal{Z}\text{-irreducible.} \tag{2.3.1}$$

Let

$$\Lambda_r := a^{-1}(H_r). \tag{2.3.2}$$

Assume that for every $r \in R$, either

$$L(H_r) \neq W \tag{2.3.3}$$

or

(2.3.4) $L(H_r) = W$, and there exists $K_r \in \mathcal{Z}(W) - \{W\}$ such that $\bigcup_{\lambda \in \Lambda_r} c(\lambda) \subset K_r$. In case (2.3.3), put $K_r := L(H_r)$. Note that also in this case, by (M1),

$$\bigcup_{\lambda \in \Lambda_r} c(\lambda) \subset K_r. \tag{2.3.5}$$

Then

$$\begin{aligned} \bigcup_{\lambda \in \Lambda} c(\lambda) &= \bigcup_{r \in R} \bigcup_{\lambda \in \Lambda_r} c(\lambda) && \text{(by (2.3.2))} \\ &\subset \bigcup_{r \in R} K_r && \text{(by (2.3.4), (2.3.5)).} \end{aligned}$$

Since $\bigcup_{\lambda \in \Lambda} c(\lambda)$ is $\mathcal{Z}$-dense by (A2), by Lemma 2.1 we have $W = K_r$ for some $r \in R$. However, this contradicts the choice of $\{K_r\}_{r \in R}$. Hence there exists $r \in R$ such that

$$L(H_r) = W \quad \text{and} \quad \bigcup_{\lambda \in \Lambda_r} c(\lambda) \text{ is } \mathcal{Z}\text{-dense in } W.$$

For such r, rewrite $H^{(1)} := H_r$, $\Lambda^{(1)} := \Lambda_r (= a^{-1}(H^{(1)}))$ so we have

$$L(H^{(1)}) = W, \quad \text{and} \quad \bigcup_{\lambda \in \Lambda^{(1)}} c(\lambda) \text{ is } \mathcal{Z}\text{-dense in } W. \tag{2.3.6}$$

Case A. $a(\Lambda^{(1)})$ is $\mathcal{Z}$-dense in $H^{(1)}$.

Then $H^{(1)} =: H_\Lambda$ satisfies the conclusion of the lemma, and the proof is completed.

Case B. $a(\Lambda^{(1)})$ is not $\mathcal{Z}$-dense in $H^{(1)}$, i.e., there exists an element $H' \in \mathcal{Z}(H)$ such that

$$a(\Lambda^{(1)}) \subset H' \subsetneq H^{(1)}. \tag{2.3.7}$$

By (Z3), decompose H' as $H' = \bigcup_{s \in S} H_s^{(1)}$, where

$$S \text{ is countable and } H_s^{(1)} \text{ is } \mathcal{Z}\text{-irreducible.} \tag{2.3.8}$$

Note that

$$H_s^{(1)} \subsetneq H^{(1)} \tag{2.3.9}$$

by (2.3.7). Set $\Lambda_s^{(1)} := a^{-1}(H_s^{(1)})$; then $\bigcup_{\lambda \in \Lambda_s^{(1)}} c(\lambda) \subset L(H_s^{(1)})$ by (M1), and

$$\bigcup_{\lambda \in \Lambda^{(1)}} c(\lambda) = \bigcup_{s \in S} \left(\bigcup_{\lambda \in \Lambda_s^{(1)}} c(\lambda) \right) \subset \bigcup_{s \in S} L(H_s^{(1)}) = L(H') \quad \text{(by (M2)).}$$

By (2.3.6) the left-hand side $\bigcup_{\lambda \in \Lambda^{(1)}} c(\lambda)$ is $\mathcal{Z}$-dense in W, and hence by Lemma 2.1, there exists $s \in S$ such that

$$L(H_s^{(1)}) = W \quad \text{and} \quad \bigcup_{\lambda \in \Lambda_s^{(1)}} c(\lambda) \text{ is } \mathcal{Z}\text{-dense in } W.$$

For such s, let $H^{(2)} := H_s^{(1)}$, $\Lambda^{(2)} := \Lambda_s^{(1)} (= a^{-1}(H^{(2)}))$ so that

$$L(H^{(2)}) = W \quad \text{and} \quad \bigcup_{\lambda \in \Lambda^{(2)}} c(\lambda) \text{ is } \mathcal{Z}\text{-dense in } W. \tag{2.3.10}$$

Case A. $a(\Lambda^{(2)})$ is $\mathcal{Z}$-dense in $H^{(2)}$.

Then $H^{(2)} =: H_\Lambda$ satisfies the conclusion of the lemma, and the proof is complete.

Case B. $a(\Lambda^{(2)})$ is not $\mathcal{Z}$-dense in $H^{(2)}$.

Then repeat the above argument (2.3.7–10), replacing $(H^{(1)}, \Lambda^{(1)})$ by $(H^{(2)}, \Lambda^{(2)})$, to obtain a $\mathcal{Z}$-irreducible $H^{(3)} \subsetneq H^{(2)}$ such that $L(H^{(3)}) = W$, and $\Lambda^{(3)} := a^{-1}(H^{(3)}) \subset \Lambda^{(2)}$.

Again consider two cases:

Case A. $a(\Lambda^{(3)})$ is $\mathcal{Z}$-dense in $H^{(3)}$.

Case B. Otherwise.

Repeat this procedure as long as Case B occurs. If we assume that Case A never occurs, then we have an infinite strictly decreasing chain of $\mathcal{Z}$-irreducible subsets

$$H \supset H^{(1)} \supsetneq H^{(2)} \supsetneq \cdots \supsetneq H^{(n)} \supsetneq \cdots$$

with $L(H^{(n)}) = W$ for every n. This contradicts the assumption (A1) that $(H, \mathcal{Z}(H))$ satisfies the quasi-Noetherian condition (Z5). Hence at some stage, Case A must occur: $L(H^{(n)}) = W$, and $a(\Lambda^{(n)})$ is $\mathcal{Z}$-dense in $H^{(n)}$. This completes the proof. $\square$

Terminology 2.4. Let M be an algebraic scheme over a field k. Assume that k is uncountable. A subset $D \subset M$ of closed points of M is said to be *uncountably dense*, or *uc-dense*, if D is not contained in a union of countably many proper Zariski closed subsets of M. If M is the one-point scheme Spec$k$, then $D = M$ is understood to be a uc-dense subset.

For an arbitrary subset $A \subset M$, there exists a smallest subset $A^{-} \subset M$, called the *uc-closure* of A, which contains A and is a union of at most countably many Zariski closed subsets of M.

2.5. Let M be a projective algebraic scheme over an uncountable field k of characteristic 0. Let W be an irreducible closed Zariski closed subset of the Chow scheme Chow(M). (For the precise definition and the fundamental properties, see for example [Kol], Chap. I; cf. [Cat].) For $w \in W$, denote the corresponding effective cycle on M by Z_w. Given a uc-dense subset $\Lambda \subset W$, satisfying the following property:

(IN) There exists a collection $\{E_\lambda\}_{\lambda\in\Lambda}$ of Zariski closed subsets of M, indexed by Λ, such that

$$E_\lambda \supset \operatorname{Supp} Z_\lambda \quad (\lambda \in \Lambda).$$

Let

(FL) $$F := \bigcap_{\lambda\in\Lambda} E_\lambda$$

(F might be empty).

THEOREM 2.6 (THEOREM ON ALGEBRAIZATION, Part II). *Under the condition and assumption of* 2.5, *there exists an irreducible closed subscheme* $H_\Lambda \subset \operatorname{Hilb}(M)$ *such* *that the corresponding algebraic family* $\{Y_h\}_{h\in H_\Lambda}$ *of closed subschemes of* M *satisfies the following*:

(1) *For any* $w \in W$, *there exists* $h \in H_\Lambda$ *such that*

$$Y_h \supset (\operatorname{Supp} Z_w) \cup F.$$

(2) *There exists a uc-dense subset* $J \subset H_\Lambda$ *such that for every* $h \in J$, Y_h *is reduced, and there exists* $\lambda \in \Lambda$ *such that*

$$Y_h = E_\lambda.$$

Proof. Let $\mathcal{Z}(W)$ be the set of all unions of countably many Zariski closed subsets of W. Since the base field k is assumed to be uncountable, it is easy to see that $(W, \mathcal{Z}(W))$ satisfies the axioms (Z0–4). Let $H := \operatorname{Hilb}(M)$, and define $\mathcal{Z}(H)$ similarly; then $(H, \mathcal{Z}(H))$ satisfies (Z0–3, 5). For $h \in H$, denote by Y_h the corresponding closed subscheme of M. Define $L\colon H \to \mathcal{Z}(W)$ by

$$L(h) := \{[Z] \in W \mid Y_h \supset \operatorname{Supp} Z\}.$$

It is easily seen that $\bigcup_{h\in F} L(h) \in \mathcal{Z}(W)$ for $F \in \mathcal{Z}(H)$, and hence this uniquely extends to a map $L\colon \mathcal{Z}(H) \to \mathcal{Z}(W)$ satisfying (M1), (M2) (see (2.0.5)). Define a map $a\colon \Lambda \to H$ by sending $\lambda \in \Lambda$ to $[E_\lambda] \in H$, the point corresponding to the reduced closed subscheme E_λ of M. Then $(H, \mathcal{Z}(H))$, $(W, \mathcal{Z}(W))$, Λ, a, L satisfy the conditions of 2.2. Hence by Lemma 2.3 we have an irreducible closed subscheme $H_\Lambda \subset H = \operatorname{Hilb}(M)$ such that $L(H_\Lambda) = W$, and $D := a(\Lambda)$ is $\mathcal{Z}$-dense in H_Λ, as required. $\square$

PROPOSITION 2.7 (THEOREM ON ALGEBRAIZATION, PART III). *Let* N, H *be algebraic schemes over an uncountable field* k, *and* $N \to H$ *a projective surjective flat morphism. Denote by* Y_h *the fiber at* $h \in H$. *Assume that there exists a uc-dense subset* $J \subset H$ *such that for every* $h \in J$, Y_h *is reduced, and there exists an effective cycle* $B = B_h$ *in* N *such that*

$$Y_h \supset \operatorname{Supp} B_h.$$

Then there exist a scheme R, *together with a uc-dense subset* $\Gamma \subset R$, *and proper morphisms* $\varphi\colon R \to H$, $\psi_R\colon R \to \operatorname{Chow}(N_R/R)$, *where* $N_R := N \underset{H}{\times} R$, *such that the following hold*:

(1) $\varphi|_\Gamma\colon \Gamma \to J$ *is set theoretically an injection,* $\varphi(\Gamma)$ *is uc-dense in* H.

(2) *The composite* $R \xrightarrow{\psi_R} \operatorname{Chow}(N_R/R) \xrightarrow{\pi_R} R$ *is the identity* id_R, *where* π_R *is the structure morphism.*

(3) *Denote by* Z_r *the effective cycle corresponding to* $\psi_R(r) \in \operatorname{Chow}(N_R/R)$. *Then for any* $\gamma \in \Gamma$, $Z_\gamma = B_{\varphi(\gamma)}$.

Moreover, if there exists an irreducible Zariski closed subset $V \subset \mathrm{Chow}(N/H)$ *such that every* B_h $(h \in J)$ *belongs to* V, *then we may find* ψ_R *in such a way that* $\mathrm{Im}\,\psi_R \subset V_R$.

Proof. Let $j\colon J \to \mathrm{Chow}(N/H)$ be the map sending h to $[B_h]$, the class of the effective cycle B_h, $\Gamma_0 \subset H \times \mathrm{Chow}(N/H)$ the graph of j, and $(\Gamma_0)^-$ its uc-closure in $H \times \mathrm{Chow}(N/H)$ (see 2.4). Take an irreducible Zariski closed subset R of $(\Gamma_0)^-$ of maximal dimension which dominates H. Let $\varphi\colon R \to H$ and $\psi_H\colon R \to \mathrm{Chow}(N/H)$ be the natural projections, and $\pi_H\colon \mathrm{Chow}(N/H) \to H$ the structure morphism. Then by construction,

$$\varphi = \pi_H \circ \psi_H\colon R \to \mathrm{Chow}(N/H) \to H. \tag{2.7.1}$$

Since we have an R-isomorphism $\mathrm{Chow}(N/H) \underset{H}{\times} R \simeq \mathrm{Chow}(N_R/R)$ (see [Kol], Chap. I, 3.21), by (2.7.1), ψ_H lifts to $\psi_R\colon R \to \mathrm{Chow}(N_R/R)$ so as to satisfy $\pi_R \circ \psi_R = \mathrm{id}_R$. $(\Gamma \subset R, \varphi, \psi_R)$ satisfies the required properties. □

Remark 2.8. It is not true in general that a fiber of $\varphi\colon R \to H$ is finite. Indeed, consider the natural projection $\mathbb{A}^2_k \to \mathbb{A}^1_k =: H$. Then for a set theoretic section $s\colon H \to \mathbb{A}^2_k$ (which might be highly discontinuous), normally it is impossible to find an algebraic section $s_{\mathrm{alg}}\colon H \to \mathbb{A}^2_k$ which agrees with s over an uncountable set of points of H.

Notation 2.9. Let M be as above, and D a Zariski closed subset of $\mathrm{Hilb}(M)$. Let Y_h be the closed subscheme of M corresponding to $h \in D$. Define

$$\mathrm{Locus}\,D := \bigcup_{h\in D} \mathrm{Supp}\,Y_h \quad \text{and} \quad \mathrm{Bs}\,D := \bigcap_{h\in D} Y_h.$$

We have $\quad \mathrm{Locus}\,D \supset \mathrm{Bs}\,D$.

Let M be an algebraic scheme, and $\Lambda \subset M$ a uc-dense subset. In the following we deal with an extendability problem for a collection of cycles $\{\mathcal{E}_\lambda\}_{\lambda\in\Lambda}$ on M of the following kind: Assume that for $\lambda \in \Lambda$, we are given an algebraic family $\{E_{\lambda,t}\}_t$ of closed subschemes which sweeps out $\mathcal{E}_\lambda$ and such that each passes through λ. Then the $\mathcal{E}_\lambda$'s are extended (with few discardings) to an algebraic family $\{S_r\}_{r\in R}$, where each S_r brings an algebraic family $\{L_{r,t}\}_t$ which sweeps out S_r in such a way that for any $z \in M$, there exists $r \in R$ such that every $L_{r,t}$ passes through z. Moreover, all the $L_{r,t}$'s form an algebraic family which extends $\{E_{\lambda,t}\}_{\lambda,t}$. This is stated in a more precise and generalized form as follows.

THEOREM 2.10 (THEOREM ON ALGEBRAIZATION, PART IV). *Let* M *be a projective algebraic scheme over* k, $W \subset \mathrm{Chow}(M)$ *an irreducible Zariski closed subset,* *parametrizing effective cycles* $\{Z_w\}_{w\in W}$. *Assume that there exists a uc-dense subset* $\Lambda \subset W$, *such that for any* $\lambda \in \Lambda$, *there exists a complete family* $\{E_{\lambda,t}\}_{t\in T_\lambda}$ *of closed subschemes with an irreducible base scheme* T_λ *satisfying*

$$\bigcap_{t\in T_\lambda} E_{\lambda,t} \supset \operatorname{Supp} Z_\lambda.$$

Let

$$\bigcup_{t\in T_\lambda} E_{\lambda,t} =: \mathcal{E}_\lambda.$$

Then by suitably replacing Λ *by a subset of* Λ *which is uc-dense in* W (*which we denote by the same symbol* Λ), *there exist a flat morphism* $\mathcal{S} \to R$ *parametrizing closed subschemes* $\{S_r\}_{r\in R}$ *of* M *over a projective algebraic scheme* R *over* k, *a uc-dense subset* $\Gamma \subset R$, *and an* R-*morphism* $\psi\colon R \longrightarrow \operatorname{Chow}(\operatorname{Hilb}(\mathcal{S}/R)/R)$, *with the corresponding effective cycles* $\{G_r\}_{r\in R}$, *such that the following hold*:

(1) *For any* $\gamma \in \Gamma$, S_γ *and* G_γ *are reduced, and there exists unique* $\lambda \in \Lambda$ *such that* $S_\gamma = \mathcal{E}_\lambda$, *and* $\operatorname{Supp} G_\gamma = T_\lambda$ (*in* $\operatorname{Hilb}(S_\gamma)$).

(2) *For any* $r \in R$, *Locus*$(\operatorname{Supp} G_r) = S_r$ (*see Notation* 2.9).

(3) *For any* $w \in W$, *there exists* $r \in R$ *such that* $Bs(\operatorname{Supp} G_r) \supset \operatorname{Supp} Z_w$ (*see Notation* 2.9).

Proof. By Theorem 2.6 (Part II), there exists a flat family $\mathcal{S} \to H$ of closed subschemes $\{S_h\}_{h\in H}$ of M such that:

(2.10.1) For any $w \in W$, there exists $h \in H$ such that $S_h \supset \operatorname{Supp} Z_w$.

(2.10.2) There exists a uc-dense subset $J \subset H$ such that for any $h \in J$, S_h is reduced, and there exists $\lambda = \lambda(h) \in \Lambda$ such that $S_h = \mathcal{E}_{\lambda(h)}$.

Consider $\mathcal{H} := \operatorname{Hilb}(\mathcal{S}/H) \to H$. By (2.10.2) above, for $h \in J$, we have $S_h = \mathcal{E}_{\lambda(h)} = \bigcup_{t\in T_\lambda} E_{\lambda,t}$, so regard $E_{\lambda,t}$ as a point of $\mathcal{H}$.

(2.10.3) As t sweeps out T_λ, we obtain a subset T_h ($h \in J$) of $\mathcal{H}$, which is an irreducible reduced cycle sitting on the fiber of $\mathcal{H} \to H$ over h. By the assumption,

$$\text{(2.10.4)} \qquad \operatorname{Locus}(\operatorname{Supp} T_h) = S_h \ \text{ and } \ \operatorname{Bs}(\operatorname{Supp} T_h) \supset \operatorname{Supp} Z_{\lambda(h)} \quad (\text{for } h \in J).$$

Take an irreducible component $\mathcal{H}_1$ of $\mathcal{H}$ such that $\{h \in J \mid \operatorname{Supp} T_h \subset \mathcal{H}_1\} \subset J$ is uc-dense in H.

(2.10.5) Consider a surjective morphism $\varphi\colon R \twoheadrightarrow H$ from a projective algebraic scheme R, endowed with a uc-dense subset $\Gamma \subset R$ such that $\varphi(\Gamma) =: J_0 (\subset J)$ is uc-dense in H. Denote the resulting base-changes by $\mathcal{H}_1 \underset{H}{\times} R =: (\mathcal{H}_1)_R \to R$ and $\mathcal{S} \underset{H}{\times} R =: \mathcal{S}_R \to R$. Let $S_r := S_{\varphi(r)}$ ($r \in R$), $Z_\gamma := Z_{\lambda(\varphi(\gamma))}$, and $T_\gamma := T_{\varphi(\gamma)}$ ($\gamma \in \Gamma$).

Note that we have a natural R-isomorphism

$$\text{(2.10.6)} \qquad \operatorname{Hilb}(\mathcal{S}/H) \underset{H}{\times} R \simeq \operatorname{Hilb}(\mathcal{S}_R/R)$$

([Kol], Chap. I (1.4.1.5)), and this induces a closed immersion; $(\mathcal{H}_1)_R \hookrightarrow \mathrm{Hilb}(\mathcal{S}_R/R)$. By Proposition 2.7 (Part III) applied to $(N \to H \supset J, \{B_h\}_{h\in J}) = (\mathcal{H}_1 \to H \supset J, \{T_h\}_{h\in J})$ for a suitable such $(\Gamma \subset R \xrightarrow{\varphi} H)$, we have (2.10.7) and (2.10.8) as follows:

(2.10.7) There exists an R-morphism ψ_R: $R \to \mathrm{Chow}((\mathcal{H}_1)_R/R)$, namely, a morphism ψ_R such that the composite

$$R \xrightarrow{\psi_R} \mathrm{Chow}((\mathcal{H}_1)_R/R) \xrightarrow{\pi_R} R \quad (\pi_R \text{ is the structure morphism })$$

is the identity id_R.

(2.10.8) Denote by G_r the effective cycle corresponding to $\psi_R(r) \in \mathrm{Chow}((\mathcal{H}_1)_R/R)$. Then for any $\gamma \in \Gamma$, $G_\gamma = T_\gamma$ (see the notation in (2.10.5)).

By construction,

(2.10.9) $$\mathrm{Locus}(\mathrm{Supp}\, G_r) \subset S_r \quad (r \in R)$$

and

(2.10.10) G_γ is irreducible, reduced, and $\mathrm{Locus}(\mathrm{Supp}\, G_\gamma) = S_\gamma \quad (\gamma \in \Gamma)$

(by (2.10.3), (2.10.8)). Take a Zariski open subset $R^0 \subset R$ such that $\{\mathrm{Supp}\, G_r\}_{r\in R^0}$ is a flat family of irreducible reduced closed subschemes of $(\mathcal{H}_1)_R$. $R^0 \neq \emptyset$ by (2.10.10) (see [Kol], Chap. I, (3.10.3)). Consider $\mathcal{S} \underset{R}{\times} R^0 =: \mathcal{S}^0 \to R^0$. This is flat, and by (2.10.9–10), there exists a non-empty Zariski open subset R^{00} of R^0 such that

(2.10.11) $$\mathrm{Locus}(\mathrm{Supp}\, G_r) = S_r \quad (r \in R^{00}).$$

Rewrite R^{00} as R^0 for simplicity, and let

(2.10.12) $$Y_r := \mathrm{Bs}(\mathrm{Supp}\, G_r) \quad \text{for } r \in R^0.$$

By the assumption and (2.10.8), we have

(2.10.13) $$Y_\gamma = \mathrm{Bs}(\mathrm{Supp}\, T_\gamma) \supset \mathrm{Supp}\, Z_\gamma \quad \text{for } \gamma \in \Gamma^0 := \Gamma \cap R^0$$

(see the notation in (2.10.5)). Moreover, there exists a Zariski open subset of R^0, which is denoted again by R^0, such that $\{Y_r\}_{r\in R^0}$ forms a flat family of irreducible reduced closed subschemes of M.

(2.10.14) Consider the associated morphism $R^0 \xrightarrow{\mu^0} \mathrm{Hilb}(\mathcal{S}_R/R)$. By construction, the composite

$$R^0 \xrightarrow{\mu^0} \mathrm{Hilb}(\mathcal{S}_R/R) \xrightarrow{h_R} R$$

of μ^0 and the structure morphism h_R coincides with the open immersion $R^0 \hookrightarrow R$. Hence there exists an extension $R \xrightarrow{\mu} \mathrm{Hilb}(\mathcal{S}_R/R)$ such that $h_R \circ \mu = \mathrm{id}_R$.

Denote by Y_r the closed subscheme of S_r corresponding to $\mu(r)$ $(r \in R)$, and let $\mathcal{Y} := \bigcup_{r \in R} Y_r \subset \mathcal{S}_R$. By Proposition 2.7 (Part III), applied to $(N \to H \supset J, \{B_h\}_{h \in J}) = (\mathcal{Y} \to R \supset \Gamma^0, \{Z_\gamma\}_{\gamma \in \Gamma^0})$ we have the following.

(2.10.15) There exist a morphism φ': $R' \to R$ from a projective algebraic scheme R', a uc-dense subset $\Gamma' \subset R'$ whose image is in Γ^0 and is uc-dense in R, and an R'-morphism $\psi_{R'}$: $R' \to \mathrm{Chow}(\mathcal{Y}_{R'}/R')$, such that the cycle corresponding to $\psi_{R'}(\gamma) \in \mathrm{Chow}((\mathcal{H}_1)_{R'}/R')$ coincides with $Z_{\varphi'(\gamma)}$ for $\gamma \in \Gamma'$.

This $\mathcal{S}_{R'} \to R' \supset \Gamma'$ gives a family with the required properties. $\square$

2.11. Let X be a projective algebraic scheme over an algebraically closed field k, and L an ample Cartier divisor on X. Let $S^r(X)$ denote the *r-th symmetric product* $(r \geq 1)$ whose closed points are in one to one correspondence with r-tuples $(x_1, \dots, x_r)$ of closed points of X, modulo permutation $\mathfrak{S}_r$. This is naturally identified with the Chow scheme of 0-cycles of X of degree r (see [Kol], Chap. I 3.22). Denote by pr^r: $X \times \cdots \times X \to S^r(X)$ the natural projection. For a set $D \subset X$ of closed points of X, let $S^r(D) := \{\mathrm{pr}(x_1, \dots, x_r) \in S^r(X) \mid x_1, \dots, x_r \in D\} \subset S^r(X)$.

Let $\Delta^r(X) \subset S^r(X)$ be the set of closed points $(x_1, \dots, x_r) \bmod \mathfrak{S}_r$ with $x_i = x_j$ for some $i \neq j$. Let $S^r_0(X) := S^r(X) - \Delta^r(X)$, an element of which is identified with a set of distinct r points $\{x_1, \dots, x_r\}$ of X. Let $\{y_1, \dots, y_s\} \in S^s_0(X)$. This gives rise to a natural injective morphism $s^r_{\{y_1, \dots, y_s\}}$: $S^r(X) \hookrightarrow S^{r+s}(X)$. Let $S^r_{\{y_1, \dots, y_s\}}(X) := (s^r_{\{y_1, \dots, y_s\}})^{-1}(S^{r+s}_0(X))$. Given a uc-dense subset $\Lambda \subset S^r_{\{y_1, \dots, y_s\}}(X)$ (in the sense of 2.4), satisfying the following condition;

$(\mathrm{RC}; y_1, \dots, y_s; \Lambda)^r_d$. There exists an irreducible rational curve $C_{x_1, \dots, x_r}$ on X for every $\{x_1, \dots, x_r\} \in \Lambda$ such that

$$C_{x_1, \dots, x_r} \supset \{x_1, \dots, x_r, y_1, \dots, y_s\}, \quad \text{and} \quad (L \,.\, C_{x_1, \dots, x_r}) = d.$$

We sometimes do not specify the set Λ, and denote this condition simply $(\mathrm{RC}; y_1, \dots, y_s)^r_d$. In particular, if $s = 0$, we use the notation $(\mathrm{RC})^r_d$.

Remark 2.12. If (X, L) satisfies $(\mathrm{RC}; y_1, \dots, y_s; \Lambda)^r_d$, and $\{x_1, \dots, x_r\} \in \Lambda$, then (X, L) satisfies also $(\mathrm{RC}; y_1, \dots, y_r, x_1, \dots, x_t; \Lambda')^{r-t}_d$, with $\Lambda' := (s^{r-t}_{\{x_1, \dots, x_t\}})^{-1}(\Lambda)$.

COROLLARY 2.13. *Assume that (X, L) satisfies $(\mathrm{RC}; y_1, \dots, y_s)^r_d$ in 2.11. Assume that the base field k is uncountable. Then there exists an irreducible closed subscheme $H \subset \mathrm{Hilb}\, X$ such that the corresponding algebraic family $\{C_h\}_{h \in H}$ satisfies the following*:

(1) *For any $\{x_1, \dots, x_r\} \in S^r_{\{y_1, \dots, y_s\}}(X)$, there exists $h \in H$ such that*

$$C_h \supset \{x_1, \dots, x_r, y_1, \dots, y_s\}.$$

(2) *For a general $h \in H$, C_h is an irreducible rational curve with $(L \,.\, C_h) = d$.*

3. (Co-)Tangent bundle of X pulled back to $\mathbb{P}^1$ (char $k = 0$)

In [Mo1] Mori developed what he called the 'Bend-and-Break' method (see also Lemma 5.2), and observed that if the tangent bundle T_X of a smooth variety X is sufficiently positive, then X has a large family of rational curves, while a little close attention to his argument leads us to a consequence in a somewhat opposite direction. Namely, if there is a family of rational curves on X, then T_X, restricted to each of those curves, has a positive (or semi-positive) subbundle, whose rank is equal to the dimension of the locus of those curves, if $\operatorname{char} k = 0$ (Propositions 3.1, 3.3 below). This is regarded as a modification of Kollar's [Kol], Chap. IV Theorem 3.7. We prove this for singular varieties as well. Indeed the proof for the non-singular case works with only a few minor change. This is made possible by virtue of Kollár's generalization [Kol] of Mori's argument.

Here we are unable to remove the assumption $\operatorname{char} k = 0$, because of the neccesity of deducing the surjectivity of the derivative of φ_a (3.1.6). Indeed this forces us to restrict the validity of our Main Theorems (Theorem 4.1, 5.1) to the case of $\operatorname{char} k = 0$, although all the other parts of our proof are valid in an arbitrary characteristic. (Of course, we may alternatively assume the separability of the evaluation map Ψ (3.0), instead of assuming $\operatorname{char} k = 0$; (cf. [Kac2] for a treatment applicable to the arbitrary characteristic case.)

We follow Mori and Kollár's notation and terminology ([Mo1], [Kol], Chap. II).

3.0. (3.0.0) Let X be a normal projective variety defined over an algebraically closed field k. Fix a closed point $x \in \operatorname{Reg} X$. Fix a coordinate of $\mathbb{P}^1_k = \mathbb{P}^1$, so that the point $0 \in \mathbb{P}^1$ is specified. Let S be a connected component of $\operatorname{Hom}(\mathbb{P}^1, X; \iota)_{\mathrm{red}}$, where ι: $\{0\} \to \{x\}$ (see [Mo1] §2). For $s \in S$, the corresponding morphism $\mathbb{P}^1 \to X$ is denoted by ν_s:

(3.0.1) $$s = [\nu_s].$$

Let $\Phi : S \times \mathbb{P}^1 \to X$ be the evaluation map, and let

(3.0.2) $$t := \dim \operatorname{Im} \Phi.$$

Then there exists an irreducible component S^0 of $\operatorname{Reg} S$ such that

(3.0.3) $$\dim \operatorname{Im}(\Phi|_{S^0 \times \mathbb{P}^1}) = t.$$

In fact, take an irreducible component S_1 of S with

$$\dim \operatorname{Im}(\Phi|_{S_1 \times \mathbb{P}^1}) = t.$$

(Such S_1 exists, since the base field k is assumed to be uncountable (3.0.0), and S has at most countably many irreducible components.) Then $S^0 := (\operatorname{Reg} S) \cap S_1$ satisfies (3.0.3).

PROPOSITION 3.1 (char $k = 0$). *With the notation above, assume that the base field k is of characteristic 0. Then for a general $s \in S^0$,*

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(t-1)} \subset (\nu_s^*\Omega_X^1)^\vee.$$

Proof. Let

$$\mathcal{A} := \mathrm{Aut}(\mathbb{P}^1, \{0\}) = \{\sigma\colon \mathbb{P}^1 \to \mathbb{P}^1 \mid \sigma\colon \text{ isomorphism}, \sigma(0) = 0\} = \{\sigma_\alpha\}_{\alpha\in\mathcal{A}}.$$

This is a 2-dimensional scheme. Define

$$f_{s,\alpha} := \nu_s \circ \sigma_\alpha\colon \mathbb{P}^1 \to X$$

($s \in S$). This has the following properties:

$$\mathrm{Im}\, f_{s,\alpha} = \mathrm{Im}\, \nu_s \tag{3.1.1}$$

(for all $s \in S, \alpha \in \mathcal{A}$),

$$[f_{s,\alpha}] \in S \tag{3.1.2}$$

(for all $s \in S, \alpha \in \mathcal{A}$), and

$$[f_{s,\alpha}] \in S^0 \tag{3.1.3}$$

(for all $s \in S^0$, and for a general $\alpha \in \mathcal{A}$).

Define

$$\mathcal{A}^s := \{\alpha \in \mathcal{A} \mid [f_{s,\alpha}] \in S^0\} \subset \mathcal{A}.$$

This is dense in $\mathcal{A}$, hence:

(3.1.4) $\{f_{s,\alpha}(a) \mid \alpha \in \mathcal{A}^s\}$ is dense in $\mathrm{Im}\,\nu_s$ for any $a \in \mathbb{P}^1 - \{0\}$;

(3.1.5) $\mathrm{Im}(\Phi|_{S^0\times\{a\}}) = \bigcup_{s\in S^0}\{f_{s,\alpha}(a) \mid \alpha \in \mathcal{A}^s\}$ is dense in $\bigcup_{s\in S^0}\mathrm{Im}\,\nu_s = \mathrm{Im}(\Phi|_{S^0\times\mathbb{P}^1})$.

Since $\mathrm{Im}(\Phi|_{S^0\times\{a\}})$ is locally closed in $\mathrm{Im}(\Phi|_{S^0\times\mathbb{P}^1})$, by (3.0.3) we conclude that

$$\dim \mathrm{Im}(\Phi|_{S^0\times\{a\}}) = \dim \mathrm{Im}(\Phi|_{S^0\times\mathbb{P}^1}) = t.$$

Let $\mathrm{Im}(\Phi|_{S^0\times\{a\}}) =: \mathrm{Tr}(a)$, and $\varphi_a := \Phi|_{S^0\times\{a\}}\colon S^0 \to \mathrm{Tr}(a)$ which maps s to $\nu_s(a)$. Take $s \in S^0$ so that $\varphi_a(s) \in \mathrm{Reg}\,\mathrm{Tr}(a)$.

(3.1.6) Since char $k = 0$, the derivative of φ_a is surjective, as a homomorphism between the Zariski tangent spaces $T_{S^0,s}$ and $T_{\mathrm{Tr}(a),\varphi_a(s)}$:

$$(d\varphi_a)_s\colon T_{S^0,s} \twoheadrightarrow T_{\mathrm{Tr}(a),\varphi_a(s)}.$$

Here $T_{\mathrm{Tr}(a),\varphi_a(s)}$ is a t-dimensional k-vector space, while we have a canonical isomorphism

(3.1.7) $T_{S^0,s} \simeq \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\nu_s^*\Omega_X^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq H^0(\mathbb{P}^1, (\nu_s^*\Omega_X^1)^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$

([Mo1], Proposition 3, [Kol], Chap. I, 2.16 Theorem). Hence:

(3.1.8) There exists a t-dimensional subvector space $V \subset H^0(\mathbb{P}^1, (\nu_s^*\Omega_X^1)^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$ which is mapped by $(d\varphi_a)_s$ isomorphically onto $T_{\mathrm{Tr}(a), \varphi_a(s)}$. This V generates a rank t locally free subsheaf $\mathcal{F}_V$ of $(\nu_s^*\Omega_X^1)^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$. Since $\mathcal{F}_V$ is globally generated, we have

$$\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus t} \subset (\nu_s^*\Omega_X^1)^\vee. \tag{3.1.9}$$

On the other hand, we have a natural homomorphism $\nu_s^*\Omega_X^1 \to \Omega_{\mathbb{P}^1}^1$ which is generically surjective. By taking the dual, we have the generically injective homomorphism

$$\mathcal{O}_{\mathbb{P}^1}(2) \simeq T_{\mathbb{P}^1} \longrightarrow (\nu_s^*\Omega_X^1)^\vee. \tag{3.1.10}$$

Since both $T_{\mathbb{P}^1}$ and $(\nu_s^*\Omega_X^1)^\vee$ are locally free $\mathcal{O}_{\mathbb{P}^1}$-modules, this homomorphism is injective. (3.1.9) and (3.1.10) prove the proposition. □

3.2. By exactly the same argument, we can show Proposition 3.3 below.

Let S' be a connected component of $\mathrm{Hom}(\mathbb{P}^1, X)_{\mathrm{red}}$, Φ: $S' \times \mathbb{P}^1 \to X$ the evaluation map, and

$$t' := \dim \operatorname{Im} \Phi.$$

For $s' \in S'$, let $\nu_{s'}$: $\mathbb{P}^1 \to X$ be the corresponding homomorphism:

$$s' = [\nu_{s'}]. \tag{3.2.0}$$

Let $(S')^0$ be an irreducible component of $\operatorname{Reg} S'$ such that

$$\dim \operatorname{Im}(\Phi|_{S'^0 \times \mathbb{P}^1}) = t'. \tag{3.2.1}$$

(3.2.2) Assume that for a general $s' \in (S')^0$, $\nu_{s'}(\mathbb{P}^1) \not\subset \operatorname{Sing} X$.

PROPOSITION 3.3 ($\operatorname{char} k = 0$). *Under the notation of* 3.2, *assume that the base field k is of characteristic* 0. *Then for a general* $s' \in (S')^0$,

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(t'-1)} \subset (\nu_{s'}^*\Omega_X^1)^\vee.$$

Proof. Proceed as in the proof of Proposition 3.1, replacing $\mathrm{Hom}(\mathbb{P}^1, X; \iota)$ by $\mathrm{Hom}(\mathbb{P}^1, X)$. Then analogous with (3.17) we have

$$T_{(S')^0, s'} \simeq H^0(\mathbb{P}^1, (\nu_{s'}^*\Omega_X^1)^\vee) \tag{3.3.1}$$

([Mo1], [Kol]), and there exists a globally generated locally free subsheaf $\mathcal{F}' = \mathcal{F}_{V'} \subset (\nu_{s'}^*\Omega_X^1)^\vee$, of rank t'. The rest is the same as before. □

Let $C \subset X$ be an irreducible curve on a variety X, and ν: $\widetilde{C} \to C$ the normalization of C. The following formula compares $c_1((\nu^*\Omega_X^1)^\vee)$ and $(-K_X \,.\, C)$, which coincide when X is non-singular (cf. [Mo3], [Sh1]).

LEMMA 3.4. *Let $C \subset X$ be an irreducible curve on X, and $\nu\colon \widetilde{C} \to X$ the composite of the normalization morphism of C and the closed immersion $C \hookrightarrow X$. Assume that $C \not\subset \operatorname{Sing} X$, and assume that X is $\mathbb{Q}$-Gorenstein; i.e., X is Cohen–Macaulay and $(\omega_X^{\otimes m})^{\vee\vee}$ is an invertible sheaf on X for some integer $m > 0$. Then*

$$c_1\left((\nu^*\Omega_X^1)^\vee\right) \le (-K_X \,.\, C).$$

Proof. Let $U := \operatorname{Reg} X$, $\nu_0 := \nu|_{\nu^{-1}(U)}$, $i\colon U \hookrightarrow X$ and $j\colon \nu^{-1}(U) \hookrightarrow \widetilde{C}$ be the open immersions

$$\begin{tikzcd} \nu^{-1}(U) \arrow[r, hook, "j"] \arrow[d, "\nu_0"'] & \widetilde{C} \arrow[d, "\nu"] \\ U \arrow[r, hook, "i"'] & X \end{tikzcd}$$

Then we have a sequence of $\mathcal{O}_{\widetilde{C}}$-homomorphisms:

(3.4.1)

$$\begin{aligned} \bigwedge^n((\nu^*\Omega_X^1)^\vee) \xrightarrow{\ \alpha\ } j_*\bigwedge^n((\nu_0^*\Omega_U^1)^\vee) &\simeq j_*\nu_0^* \bigwedge^n(\Omega_U^{1\vee}) \\ &\simeq \nu^* i_* \bigwedge^n(\Omega_U^{1\vee}) \\ &\simeq \nu^*\omega_X^\vee \xrightarrow{\ \beta\ } (\nu^*\omega_X^\vee)^{\vee\vee} \simeq (\nu^*\omega_X)^\vee. \end{aligned}$$

Since both $\bigwedge^n((\nu^*\Omega_X^1)^\vee)$ and $(\nu^*\omega_X)^\vee$ are invertible $\mathcal{O}_{\widetilde{C}}$-modules and $(\beta\circ\alpha)|_{\nu^{-1}(U)}$ is an isomorphism, $\beta\circ\alpha$ is injective. Hence

$$c_1\left((\nu^*\Omega_X^1)^\vee\right) \le \deg(\nu^*\omega_X)^\vee. \tag{3.4.2}$$

Next, let $m > 0$ be an integer such that $(\omega_X^{\otimes m})^{\vee\vee} = i_*(\omega_U^{\otimes m})$ is invertible. Then we have a sequence of homomorphisms of $\mathcal{O}_{\widetilde{C}}$-modules,

$$(\nu^*\omega_X^\vee)^{\otimes m} = (j_*\nu_0^*\omega_U^\vee)^{\otimes m} \longrightarrow j_*\nu_0^*(\omega_U^{\vee\otimes m}) \simeq \nu^* i_*(\omega_U^{\vee\otimes m}) \simeq \nu^*(\omega_X^{\vee\otimes m}),$$

which is an isomorphism on $\nu^{-1}(U)$ (cf. [Mo3], §1). By taking the double-dual $(\,\cdot\,)^{\vee\vee}$, we have an injective homomorphism of $\mathcal{O}_{\widetilde{C}}$-modules:

$$(\nu^*\omega_X)^{\vee\otimes m} \hookrightarrow \nu^*(\omega_X^{\vee\otimes m}) \tag{3.4.3}$$

Hence

$$m\cdot\deg(\nu^*\omega_X)^\vee = \deg\left((\nu^*\omega_X)^{\vee\otimes m}\right) \le \deg\left(\nu^*(\omega_X^{\vee\otimes m})\right) = (-mK_X \,.\, C). \tag{3.4.4}$$

(3.4.2) and (3.4.4) prove the lemma. □

4. Varieties with $(\mathrm{RC}; x)^1_1$ (after Andreatta–Ballico–Wiśniewski)

In this section we prove Proposition 4.1.

4.0. Let X be an n-dimensional projective algebraic scheme, defined over an algebraically closed uncountable field k, and L an ample Cartier divisor on X.

PROPOSITION 4.1 (cf. Andreatta–Ballico–Wiśniewski [ABW]). *Let (X, L) be as in 4.0. Assume that (X, L) satisfies $(\mathrm{RC}; x)^1_1$ in 2.11 for some $x \in X$. Let $\nu: \widetilde{X} \to X$ be the normalization of X. Then $\rho(\widetilde{X}) = 1$.*

Proof. If X satisfies $(\mathrm{RC}; x)^1_1$, then $\widetilde{X}$ satisfies $(\mathrm{RC}; \widetilde{x})^1_1$ for some $\widetilde{x} \in \nu^{-1}(x)$. Hence we may assume that X is normal.

By Corollary 2.13, there exists a flat family $\{C_h\}_{h\in H}$ of 1-dimensional closed subschemes of X such that:

(4.1.1) For any $y \in X - \{x\}$, there exists $h \in H$ such that $C_h \supset \{x, y\}$.

(4.1.2) For a general $h \in H$, C_h is reduced and is an irreducible rational curve with $(L \,.\, C_h) = 1$.

Let $\mathcal{H} \to H$ be the flat morphism defining the family $\{C_h\}_{h\in H}$, $\widetilde{\mathcal{H}}$ and $\widetilde{H}$ the normalization of $\mathcal{H}$, H, respectively, and $q: \widetilde{\mathcal{H}} \to \widetilde{H}$ the morphism induced from $\mathcal{H} \to H$. Let $p: \widetilde{H} \to X$ be the composition of the normalization morphism $\nu: \widetilde{\mathcal{H}} \to \mathcal{H}$ and the natural projection $\mathcal{H} \to X$. We have the following diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{H}} & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \\ \widetilde{H} & & \end{array} \tag{4.1.3}$$

For $h \in \widetilde{H}$, denote the fiber of q at h by B_h, so $p(B_h) = C_h$. Since $\operatorname{char} k$ is assumed to be 0 we have the following:

(4.1.4) There exists a Zariski open subset $\widetilde{H}^0 \subset \operatorname{Reg} \widetilde{H}$ such that $q^{-1}(\widetilde{H}^0)$ is smooth, and $q|_{q^{-1}(\widetilde{H}^0)}$ is a $\mathbb{P}^1$-bundle. Let

$$S := \{\xi \in \widetilde{\mathcal{H}} \mid \mathcal{O}_{\widetilde{\mathcal{H}},\xi} \text{ is not Cohen–Macaulay}\}.$$

Since $\widetilde{\mathcal{H}}$ is normal, this is of codimension at least 3 in $\widetilde{\mathcal{H}}$, and hence

$$\operatorname{codim}_{\widetilde{H}} q(S) \geq 2.$$

Also, since $\widetilde{H}$ is normal,

$$\operatorname{codim}_{\widetilde{H}} \left(q(S) \cup \operatorname{Sing}\widetilde{H}\right) \geq 2 \tag{4.1.5}$$

(cf. [Kol], Chap. II, (2.8.5)). Let $\widetilde{H}^{00} := \widetilde{H} - (q(S) \cup \operatorname{Sing}\widetilde{H})$, and consider the restriction $q|_{q^{-1}(\widetilde{H}^{00})}: q^{-1}(\widetilde{H}^{00}) \to \widetilde{H}^{00}$.

(4.1.6) $q^{-1}(\widetilde{H}^{00})$ is Cohen–Macaulay, and $\widetilde{H}^{00}$ is smooth. In particular, $q|_{q^{-1}(\widetilde{H}^{00})}$ is flat.

By (4.1.2), B_h $(h \in \widetilde{H})$ is a generically reduced irreducible curve. Hence by (4.1.6), if $h \in \widetilde{H}^{00}$, then B_h does not have an embedded point; in particular, $\dim H^0(\mathcal{O}_{B_h}) = 1$. By (4.1.4), a general fiber of q is isomorphic to the reduced $\mathbb{P}^1$, and thus B_h has to satisfy $\chi(\mathcal{O}_{B_h}) = 1$. Hence $p_a(B_h) := \dim H^1(\mathcal{O}_{B_h}) = 0$. Namely, B_h is isomorphic to the reduced $\mathbb{P}^1$ (cf. [Mo2], pp. 154, 158). Therefore:

(4.1.7) q is a $\mathbb{P}^1$-bundle over $\widetilde{H}^{00}$, with $\operatorname{codim}_{\widetilde{H}}(\widetilde{H} - \widetilde{H}^{00}) \geq 2$.

By (4.1.1) and (4.1.7), for $A = X$, $B = \widetilde{\mathcal{H}}$, and $C = \widetilde{H}$, we have Case I of 1.3 (indeed $S_1(x) = A$ in the present case), and the assumption of Theorem 1.5 (1) is satisfied. Hence by Theorem 1.5 (1), we have

$$\rho(X) = 1. \qquad \square$$

The following reproduces the result of Andreatta–Ballico–Wiśniewski [ABW] (also Theorem 0.3 of this article) under the slightly generalized assumption $(\mathrm{RC}; x)_1^1$;

COROLLARY 4.2 (Andreatta-Ballico-Wiśniewski [ABW], p. 194). *In Proposition 4.1, assume furthermore that* $x \in \operatorname{Reg} X$. *Assume that* $\operatorname{char} k = 0$. *Then*

$$(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)).$$

Proof. By (4.1.1), p is surjective, and hence $t = n$ in (3.0.2). Thus by Proposition 3.1 and Lemma 3.4,

(4.2.1) $(-K_X \,.\, C_h) \geq c_1\left((\nu_s^*\Omega_X^1)^\vee\right) \geq c_1(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n-1)}) = n + 1.$

If X has at worst log-terminal singularities, Proposition 4.1, (4.2.1), and $(L \,.\, C_h) = 1$ imply that $-K_X \equiv rL$ for some $r \geq n + 1$. By Theorem 0.7, we conclude that $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(4.2.2) (taken from Andreatta–Ballico–Wiśniewski [ABW]). If X admits singularities worse than log-terminal, apply Fujita's result ([Fuj4] Theorem 2.2) to obtain the same conclusion. $\square$

5. Proof of the Main Theorem

In this section we prove our main result (Theorem 5.1).

THEOREM 5.1 (Main Theorem). *In* 4.0, *assume that X has only normal* $\mathbb{Q}$*-factorial singularities, and that* (X, L) *satisfies the condition* $(\mathrm{RC}; x)_2^2$ *in* 2.11 *for some* $x \in \operatorname{Reg} X$. *Assume that* $\operatorname{char} k = 0$. *Then*

$$(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \text{ or } (Q^n, \mathcal{O}_{Q^n}(1)),$$

where Q^n *is a quadric hypersurface in* $\mathbb{P}^{n+1}$.

In the course of the proof of Theorem 5.1, we need the 'Bend-and-Break Lemma' of Mori ([Mo1], p. 599), in a slightly modified form, as follows (cf. [Kol], Chap. II.5):

LEMMA 5.2 (BEND-AND-BREAK LEMMA, A VARIANT). *Let S be a projective surface with only Du Val singularities (i.e., rational double points), T a non-singular projective curve, and $f\colon S \to T$ a surjective morphism. Assume that $-K_S$ is f-ample, and that there exist two irreducible curves s and s' on S such that $s \cap s' = \emptyset$, $f(s) = f(s') = T$, $(s^2) < 0$, and $(s'^2) < 0$.*

Then there exists a fiber $F = F_1 \cup F_2$ of f, with $F_1 \simeq F_2 \simeq \mathbb{P}^1$, such that

$$s \cap F_1 \neq \emptyset, \quad \textit{and} \quad s' \cap F_2 \neq \emptyset.$$

First we prove:

LEMMA 5.3. *Let (X, L) be as in 4.0. Assume that (X, L) satisfies (RC; $x)_2^2$ in 2.11 for some $x \in X$. Then for any $y \in X - \{x\}$, one of the following holds:*

(1) *There exists an irreducible rational curve $l_{x,y} \subset X$ such that*

$$l_{x,y} \ni x, y, \quad \textit{and} \quad (L \,.\, l_{x,y}) = 1.$$

(2) *There exist flat families $\{l^y_{x,t}\}_{t\in T_y}$, $\{l_{y,t}\}_{t\in T_y}$ of irreducible rational curves on X, parametrized by a common irreducible projective scheme T_y, such that*

$$l^y_{x,t} \cap l_{y,t} \neq \emptyset, \qquad l^y_{x,t} \ni x, \qquad l_{y,t} \ni y, \qquad (L \,.\, l^y_{x,t}) = (L \,.\, l_{y,t}) = 1,$$

and

$$\dim \bigcup_{t\in T_y} l^y_{x,t} = \dim \bigcup_{t\in T_y} l_{y,t} = n - 1.$$

Proof. As in Proposition 4.2, we may assume that X is normal. By Corollary 2.13, there exists a flat family $\{C_h\}_{h\in H}$ of 1-dimensional closed subschemes of X such that:

(5.3.1) For any $y, z \in X - \{x\}$ $(y \neq z)$, there exists $h \in H$ such that $C_h \supset \{x, y, z\}$.

(5.3.2) For a general $h \in H$, C_h is an irreducible rational curve with $(L \,.\, C_h) = 2$.

Let $\mathcal{H} \to H$ be the flat morphism defining the family $\{C_h\}_{h\in H}$. For $y \in X - \{x\}$, let

$$H_y := q(p^{-1}(y)) = \{h \in H \mid C_h \ni y\} \subset H, \tag{5.3.3}$$

and let $q^{-1}(H_y) =: \mathcal{H}_y \to H_y$ be the restriction of the family q over H_y. As in the proof of Theorem 4.1, we may assume that both $\mathcal{H}_y$ and H_y are normal (otherwise we may just replace them by their normalizations). Let

$$\begin{tikzcd} \mathcal{H}_y \arrow[r, "p"] \arrow[d, "q"'] & X \\ H_y \end{tikzcd} \tag{5.3.4}$$

be the induced diagram, as (4.1.3). For $h \in H$, denote the fiber of q at h by B_h, so that $p(B_h) = C_h$. Let

(5.3.5) $\Delta = \Delta_y := \{h \in H_y \mid C_h = C_{h,1} \cup C_{h,2},\ C_{h,j}$ is an irreducible rational curve, $(L \,.\, C_{h,j}) = 1\ (j = 1, 2)\}$.
This is a Zariski closed subset of H_y. By Lemma 5.2:

(5.3.6) For any irreducible closed subset $T \subset H_y$ of dimension 1, $T \cap \Delta \neq \emptyset$. In particular, Δ has an irreducible component Δ_i of codimension 1.

(5.3.7) Let $H^0 \subset \operatorname{Reg} H_y$ be a Zariski open subset of H_y such that $q^{-1}(H^0)$ is Cohen–Macaulay, and in particular, $q|_{q^{-1}(H^0)}$ is flat. As in (4.1.5–6), $H^0 \neq \emptyset$, and $\operatorname{codim}_{H_y}(H_y - H^0) \geq 2$.
Let

$$(5.3.8)\qquad \begin{cases} \operatorname{Dbl}(\Delta) & := \{h \in \Delta \mid B_h \text{ is not generically reduced}\}, \\ (\Delta_i)^0 & := \operatorname{Reg}(\Delta_i \cap H^0), \\ (\Delta_i)^{00} & := \operatorname{Reg}(\Delta_i \cap H^0) - \operatorname{Dbl}(\Delta), \\ \mathcal{D}_i & := q^{-1}(\Delta_i), \quad \text{and} \quad (\mathcal{D}_i)^{00} := q^{-1}((\Delta_i)^{00}). \end{cases}$$

Note that $(\Delta_i)^0 \neq \emptyset$ by (5.3.7).

Claim 1. In (5.3.6), for an irreducible component Δ_i of Δ with $\operatorname{codim}_{H_y} \Delta_i = 1$, the following two conditions are equivalent:

(a) $p(\mathcal{D}_i)$ has an irreducible component of codimension 1.

(b) For any complete irreducible curve $T \subset H_y$, $T \cap \Delta_i \neq \emptyset$.

In particular, by (5.3.6), condition (a) holds for some i.

Proof. We only have to prove the implication (b) $\Longrightarrow$ (a). Assume that $p(\mathcal{D}_i) = p(q^{-1}(\Delta_i))$ has no irreducible component of codimension 1. Take a complete irreducible curve $Z \subset X$ such that $Z \cap p(\mathcal{D}_i) = \emptyset$, and consider $H_Z := q(p^{-1}(Z)) \subset H$. If Z is taken sufficiently general so that the geometric genus $p_g(Z) \geq 1$, then Z is not contained in any C_h, and hence H_Z is a complete curve in H_y such that $H_Z \cap \Delta_i = \emptyset$. Hence Claim 1 is proved.

(5.3.9) Let $\{\Delta_i\}_{i \in I}$ be the set of all those irreducible components of Δ satisfying condition (a) (or equivalently, condition (b)) of Claim 1. If for some $i \in I$, $(\Delta_i)^{00} = \emptyset$, i.e., a general fiber of q over Δ_i is not generically reduced, then we immediately have assertion (1) of the lemma. Hence we may assume in what follows that for every $i \in I$, $(\Delta_i)^{00} \neq \emptyset$.

(5.3.10) Let $s_x := (p|_{\mathcal{D}_i})^{-1}(x)$, and $s_y := (p|_{\mathcal{D}_i})^{-1}(y)$. By construction, $\dim s_x = \dim s_y = n - 2$.

By (5.3.7) and (5.3.8), $(\Delta_i)^{00}$ is non-singular, and $q^{-1}((\Delta_i)^{00})$ is Cohen–Macaulay. Moreover, q is flat over H^0, and therefore $p_a(B_h) = 0$ for any $h \in (\Delta_i)^{00}$. Hence:

(5.3.11) For any $h \in (\Delta_i)^{00}$, B_h is reduced, $B_h = B_{h,1} \cup B_{h,2}$, $B_{h,1} \simeq B_{h,2} \simeq \mathbb{P}^1$, and $B_{h,1} \cap B_{h,2} = \{w_h\}$ (one point), meeting transversally. Also, there exists a commutative diagram

$$\begin{tikzcd}
(\mathcal{D}_i)^{00\sim} \arrow[r, "\nu_i"'] \arrow[d, "b"'] & (\mathcal{D}_i)^{00} \arrow[d, "q|_{(\mathcal{D}_i)^{00}}"] \\
(\Delta_i)^{00\sim} \arrow[r, "d"'] & (\Delta_i)^{00}
\end{tikzcd}$$

where ν_i is the normalization, d is an étale double cover (either $(\Delta_i)^{00\sim}$ is connected or $(\Delta_i)^{00\sim} \simeq (\Delta_i)^{00} \amalg (\Delta_i)^{00}$) and b is a $\mathbb{P}^1$-bundle such that the the fiber at h of $(d \circ b)$: $(\mathcal{D}_i)^{00\sim} \longrightarrow (\Delta_i)^{00}$ gives the normalization of B_h: $B_{h,1} \amalg B_{h,2} \longrightarrow B_h$. Let $C_{h,1} := p(B_{h,1})$ and $C_{h,2} := p(B_{h,2})$, so that $C_h = C_{h,1} \cup C_{h,2}$.

By (5.3.11), we may take a general complete non-singular curve $T \subset H_y$ such that $T \cap \Delta_i \subset (\Delta_i)^{00}$ and $q^{-1}(T)$ has only Du Val singularities. By Lemma 5.2, applied to $q|_{q^{-1}(T)}$: $q^{-1}(T) \to T$, there exist $i \in I$ and $h \in (\Delta_i)^{00}$ such that

$$(5.3.12) \qquad s_x \cap B_{h,1} \neq \emptyset, \quad \text{and } s_y \cap B_{h,2} \neq \emptyset.$$

In (5.3.12), if either $s_y \cap B_{h,1} \neq \emptyset$ or $s_x \cap B_{h,2} \neq \emptyset$, then accordingly $x, y \in C_{h,1}$ or $x, y \in C_{h,2}$, and hence we have (1) of the assertion. Thus we may assume the following:

$$(5.3.13) \quad s_x \cap B_h \subset B_{h,1} - B_{h,2}, \quad \text{and} \quad s_y \cap B_h \subset B_{h,2} - B_{h,1} \quad \text{for any } h \in (\Delta_i)^{00}.$$

In particular, $(\Delta_i)^{00\sim} \simeq (\Delta_i)^{00} \amalg (\Delta_i)^{00}$ in (5.3.11). Let

$$\mathcal{D}_i^{\mathrm{int}} := \{w_h \mid h \in (\Delta_i)^{00}\}^{-} \subset \mathcal{H}_y.$$

(5.3.14) $\mathcal{D}_i^{\mathrm{int}}$ is irreducible, $\dim \mathcal{D}_i^{\mathrm{int}} = n - 2$, and $q|_{\mathcal{D}_i^{\mathrm{int}}}$: $\mathcal{D}_i^{\mathrm{int}} \to \Delta_i$ is a birational morphism which is an isomorphism over $(\Delta_i)^{00}$.

Claim 2. $p|_{\mathcal{D}_i^{\mathrm{int}}}$: $\mathcal{D}_i^{\mathrm{int}} \to p(\mathcal{D}_i^{\mathrm{int}})$ is generically finite.

Proof. Assume to the contrary that any fiber F of $p|_{\mathcal{D}_i^{\mathrm{int}}}$ is of dimension ≥ 1. Hence $\dim((\Delta_i)^{00} \cap q(F)) \geq 1$ for a general F. Let

$$(5.3.15) \quad \mathfrak{F} := \left\{F \mid \text{fiber of } p|_{\mathcal{D}_i^{\mathrm{int}}} \text{ such that } \dim\left((\Delta_i)^{00} \cap q(F)\right) \geq 1\right\}.$$

(5.3.16) Take $F \in \mathfrak{F}$, and let $z := p(F)$. By (5.3.13), $z \neq x$, and $z \neq y$. Moreover,

$$(5.3.17) \qquad C_{h,1} \ni x, z \quad \text{and} \quad C_{h,2} \ni y, z \quad \left(h \in (\Delta_i)^{00} \cap q(F)\right).$$

Let

$$\mathcal{C}_{F,j} := \left(\bigcup_{h \in (\Delta_i)^{(x)} \cap q(F)} C_{h,j} \right)^{-} \qquad (j = 1, 2). \tag{5.3.18}$$

By (5.3.14), we have

$$p(\mathcal{D}_i) = \left(\bigcup_{F \in \mathfrak{F}} (\mathcal{C}_{F,1} \cup \mathcal{C}_{F,2}) \right)^{-}. \tag{5.3.19}$$

Case (i). If $\dim \mathcal{C}_{F,j} = 1 \quad (j = 1, 2)$ for any $F \in \mathfrak{F}$, then by (5.3.19) we have $\dim p(\mathcal{D}_i) \leq n - 2$, a contradiction to the choice of Δ_i in (5.3.9).

Case (ii). If $\dim \mathcal{C}_{F,1} \geq 2$ or $\dim \mathcal{C}_{F,2} \geq 2$ for some $F \in \mathfrak{F}$, then we have a complete family $\{C_{h,j}\}_{h \in q(F)}$ of irreducible rational curves passing through $\{x, z\}$ (case $j = 1$) or $\{y, z\}$ (case $j = 2$). Note that by (5.3.16), $x \neq z$ and $y \neq z$, so this contradicts Lemma 5.2. Hence Claim 2 is proved.

By (5.3.11), (5.3.13), $\{C_{h,j}\}_{h \in (\Delta_i)^{(x)}}$ gives a family of irreducible rational curves on X $(j = 1, 2)$ such that $C_{h,1} \ni x$, $C_{h,2} \ni y$, and $(L \,.\, C_{h,1}) = (L \,.\, C_{h,2}) = 1$. Moreover by Claim 2, $\dim p(\bigcup_{h \in (\Delta_i)^{(x)}} B_{h,j}) = n - 1$, namely,

$$\dim \left(\bigcup_{h \in (\Delta_i)^{(x)}} C_{h,j} \right) = n - 1 \quad (j = 1, 2). \tag{5.3.20}$$

Thus the assertion (2) of Lemma 5.2 is satisfied, with $T_y = \Delta_i$, $l^y_{x,t} = C_{h,1}$, $l_{y,t} = C_{h,2}$, and the proof of the Lemma 5.2 is completed. $\square$

PROPOSITION 5.4. *Under the same assumption as in Lemma* 5.3, *let* $\widetilde{X} \to X$ *be the normalization of* X. *Then* $\rho(\widetilde{X}) = 1$.

Proof. We may assume that X is normal. By Lemma 5.3, we have a uc-dense subset $\Lambda \subset \operatorname{Reg} X$ such that (X, L) either satisfies $(\mathrm{RC}; x; \Lambda)^{1}_{1}$, or the following:

(5.4.1) For every $y \in \Lambda$, there exist flat families $\{l^y_{x,t}\}_{t \in T_y}$, $\{l_{y,t}\}_{t \in T_y}$ of irreducible rational curves on X such that

$$l^y_{x,t} \cap l_{y,t} \neq \emptyset, \quad l^y_{x,t} \ni x, \quad l_{y,t} \ni y, \quad (L \,.\, l^y_{x,t}) = (L \,.\, l_{y,t}) = 1,$$

and

$$\dim \bigcup_{t \in T_y} l^y_{x,t} = \dim \bigcup_{t \in T_y} l_{y,t} = n - 1.$$

Here we may assume that the parameter scheme T_y is irreducible, reduced. In the former case, we have $\rho(X) = 1$, by Proposition 4.2. So we may assume (5.4.1) in what follows.

By Theorem 2.10, applied to $\{l_{y,t}\}_{t\in T_y}$ $(y \in \Lambda)$, there exist a flat morphism $\mathcal{S} \to R$ parametrizing closed subschemes $\{S_r\}_{r\in R}$ of X over a projective base scheme R, together with a uc-dense subset $\Gamma \subset R$, and an R-morphism $\psi\colon R \to \mathrm{Chow}(\mathrm{Hilb}(\mathcal{S}/R)/R)$, with the corresponding family of effective cycles $\{G_r\}_{r\in R}$, such that properties (5.4.2–4) hold as follows:

(5.4.2) For any $\gamma \in \Gamma$, S_γ and G_γ are reduced, and there exists $y \in \Lambda$ such that $S_\gamma = \bigcup_{t\in T_y} l_{y,t}$, and $\mathrm{Supp}\, G_\gamma = T_y$ in $\mathrm{Hilb}(S_\gamma)$.

(5.4.3) For any $r \in R$, $\mathrm{Locus}(\mathrm{Supp}\, G_r) = S_r$ (see Notation 2.9).

(5.4.4) For any $y \in X$, there exists $r \in R$ such that $\mathrm{Bs}(\mathrm{Supp}\, G_r) \ni y$ (see Notation 2.9).

By (5.4.1–2), we have

$$\dim S_r = n - 1 \quad \text{for any } r \in R. \tag{5.4.5}$$

Since G_γ is an irreducible reduced cycle for $\gamma \in \Gamma$, as in (2.10.8–11), there exists a non-empty Zariski open subset $R^0 \subset R$ such that $\{\mathrm{Supp}\, G_r\}_{r\in R^0}$ forms a flat family of reduced closed subschemes of $\mathrm{Hilb}(\mathcal{S}/R)$ ([Kol], Chap. I (3.10.3)). Let $Z^0 \xrightarrow{g^0} R^0$ be the flat morphism parametrizing $\{\mathrm{Supp}\, G_r\}_{r\in R^0}$, and $Z^0 \xrightarrow{\mu^0} \mathrm{Hilb}(\mathcal{S}/R)$ the projection. Note that since $\psi\colon R \longrightarrow \mathrm{Chow}(\mathrm{Hilb}(\mathcal{S}/R)/R)$ is an R-morphism, it follows that the composite

$$Z^0 \xrightarrow{\mu^0} \mathrm{Hilb}(\mathcal{S}/R) \longrightarrow R$$

(where $\mathrm{Hilb}(\mathcal{S}/R) \longrightarrow R$ is the structure morphism) coincides with g^0. Hence there is an extension $Z \xrightarrow{\mu} \mathrm{Hilb}(\mathcal{S}/R)$ from a projective algebraic scheme Z which contains Z^0 as an open subscheme, such that $\mu|_{Z^0} = \mu^0$. Denote the composite $Z \xrightarrow{\mu} \mathrm{Hilb}(\mathcal{S}/R) \longrightarrow R$ by g. We have $g|_{Z^0} = g^0$. Let $Y \xrightarrow{\alpha} Z$ be the flat family of closed subschemes of $\mathcal{S}$ induced from μ. If Y or Z is not normal, then as in (4.2.3), we take the normalization and replace α by the induced one, so we may assume that both Y and Z are normal, that α is equi-dimensional and flat over a Zariski open subset of Z whose complement is of codimension at least 2 (see (4.2.4)). Thus we have the following diagram:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\alpha"'] & \mathcal{S} \arrow[r] \arrow[d] & X \\ Z \arrow[r, "g"'] & R & \end{tikzcd} \tag{5.4.6}$$

Denote the composite of the top line of (5.4.6) by $Y \xrightarrow{\beta} X$. Apply the results in §1 to the diagram

$$Z \xleftarrow{\alpha} Y \xrightarrow{\beta} X.$$

For $z \in Z$, let B_z be the fiber of α at z.

(5.4.7) By construction, β is a surjective morphism such that for a general $z \in Z$, $B_z \simeq \mathbb{P}^1$, and $\nu_z := \beta|_{B_z}\colon B_z \longrightarrow \beta(B_z) =: l_z$ is a birational morphism. By (5.4.3),

$$\bigcup_{z\in g^{-1}(r)} l_z = S_r \quad \text{for } r \in R.$$

Following the notation of 1.1, let

$$S_1(y) := \beta\alpha^{-1}\alpha\beta^{-1}(y).$$

By (5.4.4) we have the following:

(5.4.8) For a general $y \in X$, there exists $r \in R$ such that

$$S_1(y) \supset \operatorname{Locus}(\operatorname{Supp} G_r) = S_r.$$

Hence $\dim S_1(y) = n - 1$ for a general $y \in X$, and we are either in Case I or Case II of 1.3.

(5.4.9) If we are in Case II of 1.3, then by Corollary 1.13, we have a surjective morphism $f\colon X \to T$ to a non-singular curve T such that for a general fiber F of f and any $y \in F$, $F = S_1(y)$. On the other hand, by (5.4.1) and (5.4.8), for every $y \in \Lambda$, $S_1(x)$ has an irreducible component S_x^y of dimension $n-1$, such that $S_x^y \cap S_1(y) \neq \emptyset$; in particular, the restriction $f|_{S_x^y}\colon S_x^y \to T$ of f to S_x^y is surjective, while S_x^y satisfies $(\mathrm{RC}; x)_1^1$, and hence by Proposition 4.2, $\rho(S_x^{y\sim}) = 1$, which is absurd.

Hence we are in Case I in 1.3. By Theorem 1.5 (1), we have

$$\rho(X) = 1,$$

as required. □

5.5. *Proof of Theorem* 5.1. Let $z \in Z$ be a general point, so that $B_z \simeq \mathbb{P}^1$ (cf. (4.2.7)). Consider $\mathbb{P}^1 \simeq B_z \xrightarrow{\nu_z} C_z \subset X$ (see (5.4.7)). Let

$$(\nu_z^*\Omega_X^1)^\vee = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i), \quad a_1 \geq \cdots \geq a_n. \tag{5.5.1}$$

By (5.4.7), $\beta\colon Y \to X$ is surjective, hence we have $t' = n$ in (3.2.1). Thus by Proposition 3.3,

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus n-1} \subset (\nu_z^*\Omega_X^1)^\vee,$$

i.e.,

$$a_1 \geq 2, \quad a_n \geq 0. \tag{5.5.2}$$

Also, by (5.4.1), $t \geq n-1$ in (3.0.3). Thus

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-2} \subset (\nu_z^* \Omega_X^1)^\vee,$$

i.e.,

$$a_{n-1} \geq 1. \tag{5.5.3}$$

By (5.5.1–3) and Lemma 3.4, we have

$$(-K_X \cdot C_h) \ \geq \ c_1\big((\nu_z^* \Omega_X^1)^\vee\big) \geq \sum_{i=1}^{n} a_i \geq n. \tag{5.5.4}$$

Proposition 5.4, (5.5.4), and $(L \cdot C_h) = 1$ imply that $-K_X \equiv rL$ for some $r \geq n$.

(5.5.5) In case X has only log-terminal singularities, by Theorem 0.7, it follows that $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, or $(Q^n, \mathcal{O}_{Q^n}(1))$.

(5.5.6) (following the suggestion by Mella [Me2]). Finally, if X has singularities worse than log-terminal, then (as indicated by [Me2]) we apply Fujita's result (Theorem 2.3 of [Fuj4]) to obtain the same conclusion. □

References

[Ab] T. Abe, *Classification of exceptional complements*, preprint, 1997.

[Al1] V. A. Alexeev, *Fractional indices of log Del Pezzo surfaces*, Math. USSR Izv. **33** (1989), 613–629.

[Al2] V. A. Alexeev, "Theorems about good divisors of log Fano varieties (Case of index $r > n-2$)" in *Algebraic Geometry Proceedings*, US–USSR Symposium, Chicago, 1989, Lect. Notes in Math., vol. 1479 (S. Bloch, I. Dolgachev and W. Fulton, ed.), Springer–Verlag, New York, 1991, pp. 1–9.

[AN] V. A. Alexeev and V. V. Nikulin, *Classification of Del Pezzo surfaces with log-terminal singularities of index ≤ 2 and involutions on K3 surfaces*, Soviet. Math. Dokl. **39** (1989), 507–518.

[Am] F. Ambro, *Ladders on Fano varieties*, preprint, 1997.

[ABW] M. Andreatta, E. Ballico and J. Wiśniewski, *Two theorems on elementary contractions*, Math. Ann. **297** (1993), 191–198.

[BS] M. Beltrametti and A. Sommese, *The adjunction theory of complex projective manifolds*, De Gruyter Expositions in Math., vol. 16, Walter de Gruyter, New York, 1995.

[Cam1] F. Campana, *Connexité rationnelle des variété de Fano*, Ann. Sci. École Norm. Sup. **25** (1992), 539–545.

[Cam2] ———, *Une version géométrique généralizée du théorème du produit de Nedel*, Bull. Soc. Math. France **119** (1991), 479–493.

[CaP] F. Campana and T. Peternell, *Rational curves and ampleness properties of the tangent bundle of algebraic varieties*, preprint, 1995.

[Cat] F. Catanese, *Chow varieties, Hilbert schemes and moduli spaces of surfaces of general type*, J. Alg. Geom. **1** (1992), 561–596.

[Che] I. Cheltsov, *Numerical Del Pezzo surface with non-rational singularities*, preprint, 1997 (in Russian).

[ChM] K. Cho and Y. Miyaoka, *A characterization of projective spaces in terms the minimal degress of rational curves*, preprint, 1997.

[ChS1] K. Cho and E. Sato, *Smooth projective varieties dominated by smooth quadric hypersurfaces in any characteristic*, Math. Zeitschr. **217** (1994), 553–565.

[ChS2] ———, *Smooth projective varieties with ample vector bundle $\bigwedge^2 T_X$ in any characteristic*, J. Math. Kyoto U. **35** (1995), 1–33.

[Cu] S. D. Cutkosky, *Local factorization and monomialization of morphisms*, preprint, 1997.

[Fuj1] T. Fujita, *On the structure of polarized varieties with Δ-genera zero*, J. Fac. Sci. Univ. Tokyo **22** (1975), 103–115.

[Fuj2] ———, *Theorems of Bertini type for certain types of polarized manifolds*, J. Math. Soc. Japan **34** (1982), 709–718.

[Fuj3] ———, "On polarized manifolds whose adjoint bundles are not semipositive" in *Algebraic geometry*, Sendai 1985 (T. Oda, ed.), Adv. Stud. Pure Math., vol. 10, Kinokuniya & North–Holland, 1987, 167–178.

[Fuj4] ———, *Remarks on quasi-polarized varieties*, Nagoya Math. J. **115** (1989), 105–123.

[Fuj5] ———, *Classification theories of polarized varieties*, London Mathematical Society Lecture Note Series, vol. 155, Cambridge University Press, Cambridge, England, 1990.

[Ful] W. Fulton, *Intersection theory*, Ergeb. Math. Grenzgb, 3 Folge, vol. 2, Springer-Verlag, New York, 1983.

[FR] T. Fujita and J. Roberts, *Varieties with small secant varieties: Extremal case*, Amer. J. Math. **103** (1981), 953–976.

[H] R. Hartshorne, *Varieties of small codimension in projective space*, Bull. Amer. Math. Soc. **80** (1974), 1017–1032.

[HW] F. Hidaka and K.-i. Watanabe, *Normal Gorenstein surfaces with ample anti-canonical divisor*, Tokyo J. Math. **4** (1981), 319–330.

[Ig] J. Igusa, *Arithmetic genera of normal varieties in an algebraic family*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 34–37.

[Ish] S. Ishii, *Quasi-Gorenstein Fano 3-folds with isolated non-rational loci*, Compositio Math. **77** (1991), 335–341.

[Isk1] V. A. Iskovskikh, *Fano 3-folds, I*, Math. USSR. Izv. **11** (1977), 485–527; Fano 3-folds, *II*, Math. USSR. Izv. **12** (1978), 469–506.

[Isk2] V. A. Iskovskikh, *Minimal models of rational surfaces over arbitrary fields*, Math. USSR Izv. **14** (1980), 17–39.

[Kac1] Y. Kachi, *Extremal contractions from 4-dimensional manifolds to 3-folds*, Ann. Scuola Norm. Sup. Pisa (4) XXIV (1997), 63–131.

[Kac2] ———, *Kollár's conjecture on characterization of projective spaces and the Hodge index theorem*, preprint, 1998.

[KaS] Y. Kachi and E. Sato, in preparation.

[Kaj] H. Kaji, *Homogeneous projective varieties with degenerate secants*, preprint, 1995.

[KOY] H. Kaji, M. Ohno, and O. Yasukura, *Adjoint varieties and their secant varieties*, preprint, 1995.

[Kaw1] Y. Kawamata, *A generalization of Kodaira–Ramanujam's vanishing theorem*, Math. Ann. **261** (1982), 43–46.

[Kaw2] ———, *The cone of curves of algebraic varieties*, Ann. of Math. **119** (1984), 603–633.

[KaMaMa] Y. Kawamata, K. Matsuda, and K. Matsuki, "*Introduction to the minimal model problem*" in Algebraic Geometry, Sendai 1985 (T. Oda, ed.), Adv. Stud. Pure Math., vol. 10, Kinokuniya & North–Holland, 1987, pp. 283–360.

[KeM] S. Keel and J. M^cKernan, *Rational curves on quasi-projective surfaces*, preprint, 1996.

[Kl] S. Kleiman, *Toward a numerical theory of ampleness*, Ann. of Math. **84** (1966), 293–344.

[KoO] S. Kobayashi and T. Ochiai, *Characterization of the complex projective space and hyperquadrics*, J. Math. Kyoto U. **13** (1972), 31–47.

[Kol] J. Kollár, *Rational curves on algebraic varieties*, Springer–Verlag, New York, 1995.

[KoMiMo1] J. Kollár, Y. Miyaoka and S. Mori, *Rational curves on Fano varieties*, Lecture Notes in Math. vol. 1515, Springer–Verlag, New York, 1992, 100–105.

[KoMiMo2] ________, *Rationally connected varieties*, J. Algebraic Geom. **1** (1992), 429–448.

[KoMiMo3] ________, *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geom. **36** (1992), 765–779.

[LV] R. Lazarsfeld and A. Van de Ven, *Topics in the geometry of projective space – Recent work of F. L. Zak*, DMV Seminar, vol. 4, Birkhäuser, 1984.

[Me1] M. Mella, *Existence of good divisors on Mukai manifolds*, preprint, 1996.

[Me2] ________, private communication, March, 1998.

[MiMo] Y. Miyaoka and S. Mori, *Numerical criterion for unirulesness*, Ann. of Math. **124** (1986), 65–69.

[Mo1] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. **110** (1979), 593–606.

[Mo2] ________, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. **116** (1982), 133–176.

[Mo3] ________, *Flip theorem and the existence of minimal models for 3-folds*, J. Amer. Math. Soc. **1** (1988), 117–253.

[MoS] S. Mori and H. Sumihiro, *On Hartshorne's conjecture*, J. Math. Kyoto. Univ. **18** (1978), 523–533.

[Muk] S. Mukai, *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3*, Proc. Nat. Acad. Sci. U.S.A. **86** (1989), 3000–3002.

[Mum] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Publ. Math. I.H.E.S. **9** (1961), 5–22.

[O] M. Ohno, *On degenerate secant varieties which have the smallest contact loci with their tangent spaces*, preprint, 1995.

[P1] Y. Prokhorov, *On Q-Fano fiber spaces with two-dimensional base*, preprint, Max-Planck-Inst., 1995.

[P2] ________, "On extremal contractions from threefolds to surfaces, the case of one non-Gorenstein point" in *Birational algebraic geometry*, A Conference on Algebraic Geometry in Memory of Wei-Liang Chow, Johns Hopkins University, Contemp. Math. vol. 207 (Y. Kawamata, V. V. Shokurov, ed.), 1997, pp. 119–141.

[R1] M. Reid, "Minimal models of canonical 3-folds" in *Algebraic varieties and analytic varieties*, Adv. Stud. Pure Math., vol. 1 (S. Iitaka, ed.), Kinokuniya & North–Holland, 1983, pp. 131–180.

[R2] ________, *Nonnormal Del Pezzo surfaces*, Publ. Res. Inst. Math. Sci. **30** (1994), 695–727.

[Sa] E. Sato, *Adjoint bundle of ample vector bundles in any characteristic*, preprint, 1996.

[Sh0] V. V. Shokurov, *The non-vanishing theorem*, Math. USSR.-Izv. **26** (1986), 591–604.

[Sh1] ________, *3-fold log flips*, Math. USSR.-Izv. **40** (1993), 95–202.

[Sh2] ________, *Special flips*, in preparation.

[Sh3] ________, *Complements on surfaces*, preprint, 1997.

[Si] Y.-T. Siu, *Curvature characterization of hyperquadrics*, Duke Math. J. **47** (1980), 641–654.

[SiY] Y.-T. Siu and S.-T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980), 189–204.

[V] E. Viehweg, *Vanishing theorems* J. Reine Angew. Math. **335** (1982), 1–8.

[W] J. Wahl, *A cohomological characterization of* $\mathbb{P}^n$, Invent. Math. **72** (1983), 315–322.

[Za] F. L. Zak, *Tangents and secants of algebraic varieties*, Trans. Math. Monographs, vol. 127, Amer. Math. Soc., Providence, RI, 1993.

[Zh1] Q. Zhang, *A theorem on the adjoint system for vector bundles*, Manuscripta Math. **70** (1991), 189–202.

[Zh2] ________, *A remark on elementary contractions*, Math. Proc. Camb. Philos. Soc. **118** (1995), 183–188.

Yasuyuki Kachi, Department of Mathematics, The Johns Hopkins University, Baltimore, MD 21218
kachi@chow.mat.jhu.edu

Eiichi Sato, Graduate School of Mathematics, Kyushu University, Hakozaki, Fukuoka 812, Japan
esato@math.kyushu-u.ac.jp