ILLINOIS JOURNAL OF MATHEMATICS
Volume 43, Number 2, Summer 1999

INDICES OF CENTRALIZERS FOR HALL-SUBGROUPS OF LINEAR GROUPS

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ABSTRACT. Suppose that P is a Sylow-p-subgroup of a solvable group G. If G is a transitive permutation group of degree n, then the number of P-orbits is at most $2n/(p+1)$. This is used to prove that if G is a faithful irreducible linear group of degree n, then the dimension of the centralizer of P is at most $2n/(p+1)$. The latter result generalizes results of Isaacs and Navarro and is also used to affirmatively answer a question of Monasur and Iranzo regarding indices of centralizers in coprime operator groups.

Suppose that V is a faithful irreducible module for a solvable group G. While there is no universal non-trivial upper bound for the dimension of the centralizer of a non-identity element of G (i.e., one may find V, G, and $1 \neq g \in G$ such that $\dim(\mathbf{C}_V(g))/\dim(V)$ is arbitrarily close to 1), we do show that if $1 \neq P \in \mathrm{Syl}_p(G)$, then $\dim(\mathbf{C}_V(P) \leq 2\dim(V)/(p+1)$. Likewise if G is a transitive solvable permutation group on Ω, there are no non-trivial bounds for the number of orbits of a non-identity element of G, but we do show that if $1 \neq P \in \mathrm{Syl}_p(G)$, then the number of orbits of P on Ω is at most $2|\Omega|/(p+1)$. In fact, this result aids the proof of the result on linear groups. We use this to positively answer a question posed by Profs. F. Perez Monasor and M. J. Iranzo. Their question and this paper begin with a paper of Isaacs and Navarro [IN].

HYPOTHESIS CP. *We assume that A acts on G via automorphisms, that $(|A|, |G|) = 1$. We let $C = \mathbf{C}_G(A)$ and $\pi = \pi(G)$ be the set of prime divisors of $|G|$.*

Assuming Hypothesis CP, Isaacs and Navarro [IN] prove a pretty result that states if an irreducible character of G is induced from an irreducible character of C, then indeed $C = G$ (i.e., A acts trivially on G). To this end, they prove that if V is a faithful irreducible GA-module with G solvable, $A \neq 1$, and the characteristic of V is coprime to $|A|$, then $\dim(\mathbf{C}_V(A)) \leq 2\dim(V)/3$. We improve this result in Theorem 1.2 by removing the restriction on the characteristic and showing even that $\dim(\mathbf{C}_V(A)) \leq 2\dim(V)/(p+1)$ where p is the largest prime divisor of $|A|$. Furthermore, we show that if GA is a transitive permutation group on Ω, the number of A-orbits is at

Received January 26, 1998.
1991 Mathematics Subject Classification. Primary 20C20.

most $2|\Omega|/(p+1)$, and this, in turn, helps prove the result on linear groups. Besides removing the restriction on the characteristic and improving the bound, our techniques seem simpler and more direct than those in [IN]. The aforementioned results about Sylow subgroups of solvable linear groups and permutation groups are Corollary 1.4, an easy Corollary to Theorem 1.2.

Assume Hypothesis CP with G solvable and let $F = \mathbf{F}(G)$, the Fitting subgroup of G. If A centralizes F, then indeed A centralizes G, (i.e., $|F: F \cap C| = 1$ implies that $|G: C| = 1$). The question posed by Perez and Iranzo is whether $|G: C|$ is bounded by a function of $|F: F \cap C|$. It is a consequence of Theorem 1.2 that indeed $|G: C| \leq |F: F \cap C|^{\beta}$ for some β, but the first such approximation is much too large and with a little additional work, we show that $|G: C| \leq |F: F \cap C|^{\alpha+1}$ for a constant α with $2.24 < \alpha < 2.25$ and this is best possible. In a minimal counterexample, F is a faithful irreducible GA/F-module.

1. Orbits and centralizers

We first quote Lemma 2.2 of [IN], which is used several times.

LEMMA 1.1. *Assume that A acts on $X = X_i \otimes \cdots \otimes X_n$ for subgroups (or submodules) X_i of X that are permuted transitively by A. Let $A_i = \mathbf{N}_A(X_i)$ and let $C_i = \mathbf{C}_{X_i}(A_i)$. Then $\mathbf{C}_X(A) \cong C_i$.*

We state Theorem 1.2 in a little more generality than mentioned above for convenience of applications and to avoid frequent repetition of a routine argument. We will apply some Hall-Higman results at the end of the proof and have even put these in a separate Lemma 1.3 that appears afterwards.

THEOREM 1.2. *Assume Hypothesis CP with GA normal in Γ, with G solvable and $A \neq 1$. Let p be the largest prime divisor of $|A|$.*

(a) If Γ transitively and faithfully permutes a set Ω, then the number of A-orbits on Ω is at most $2|\Omega|/(p+1)$.

(b) If V is a faithful irreducible Γ-module, then $\dim(\mathbf{C}_V(A)) \leq 2\dim(V)/(p+1)$.

Proof. Our proof will use part (a) to prove part (b). We argue by induction on $|GA||\Omega|$ for part (a) and by induction on $|GA|$ for part (b). The first reduction is common to both parts. Choose K normal in Γ with $K \subseteq GA$ minimal such that p divides $|K|$. Assume that $K < \Gamma$. If Γ is a transitive permutation group, we may write Ω as a disjoint union $\Omega = \Delta_1 \cup \cdots \cup \Delta_t$ for K-orbits Δ_i. Because K is normal in Γ and Γ is transitive on Ω, it follows that the groups $K/\mathbf{C}_K(\Delta_i)$ are all isomorphic. Because K acts faithfully on Ω, we have that $\cap_i \mathbf{C}_K(\Delta_i) = 1$ and that $K/\mathbf{C}_K(\Delta_i)$ has a normal and proper Hall-π-subgroup. Even p divides $|K/\mathbf{C}_K(\Delta_i)|$ for all i. Let $A_0 = A \cap K \in \mathrm{Hall}_{\pi'}(K)$. Now $1 \neq \mathbf{C}_K(\Delta_i)A_0/\mathbf{C}_K(\Delta_i) \in \mathrm{Hall}_{\pi'}(K/\mathbf{C}_K(\Delta_i))$ and the inductive argument shows that A_0 has at most $2|\Delta_i|/(p+1)$ orbits on Δ_i. Hence

A_0 has at most $\Sigma_i 2|\Delta_i|/(p+1) = 2|\Omega|/(p+1)$ orbits on Ω. Since $A_0 \subseteq A$, A has at most $2|\Omega|/(p+1)$ orbits on Ω, proving (a). For (b), we may similarly write $V = V_1 \oplus \cdots \oplus V_m$ for irreducible K-modules V_i with the groups $K/\mathbf{C}_K(V_i)$ all isomorphic and argue by induction that $\dim(\mathbf{C}_V(A)) \leq \dim(\mathbf{C}_V(A_0)) = \Sigma_i \dim(\mathbf{C}_{V_i}(A_0)) \leq \Sigma_i\, 2\dim(V_i)/(p+1) = 2\dim(V)/(p+1)$. Both parts (a) and (b) follow when $K < \Gamma$. Hence we may assume that p does not divide $|N|$ whenever N is a proper normal subgroup of Γ. In particular, $GA = \Gamma$.

(a) Suppose that GA transitively permutes Ω. First assume that GA is an imprimitive permutation group. Write Ω as a disjoint union $\Omega = \Delta_1 \cup \cdots \cup \Delta_t$ for subsets Δ_i that are permuted transitively and faithfully by GA/M for a normal subgroup M of GA and with $1 < |\Delta_i| < |\Omega|$ and $M < G$. By the last paragraph, $p\,||G/M|$. Since $t < |\Omega|$, we apply the inductive hypothesis to the action of G/M on $\{\Delta_1, \ldots, \Delta_t\}$ to conclude that the number of A-orbits on $\{\Delta_1, \ldots, \Delta_t\}$ is at most $2t/(p+1)$. Then the number of A-orbits on Ω is at most $2t|\Delta_i|/(p+1) = 2|\Omega|/(p+1)$, as desired. Conclusion (a) follows if GA is imprimitive on Ω.

Let $n = |\Omega|$. Of course p divides $n!$, and so $1 \leq 2|\Omega|/(p+1)$. We may thus assume that A is not transitive on Ω and $G > 1$. We let $s = s(A)$ be the number of A-orbits on Ω.

Next, assume that GA is a primitive permutation group and let N be a minimal normal subgroup of GA with $N \subseteq G$. The solvability of G forces N to be an elementary abelian group. Now N is transitive on Ω since otherwise the N-orbits form a non-trivial system of imprimitivity for GA. Let $P \in \mathrm{Syl}_p(A)$ and observe that NP is transitive on Ω. If $NP < GA$, the inductive hypothesis implies that $s \leq s(P) \leq 2|\Omega|/(p+1)$. So we may assume that $GA = NP$, whence $N = G$ and $A = P$. Now NP is a solvable primitive permutation group with unique minimal normal subgroup N. It is well known that $NH = GA$ and $N \cap H = 1$ where H is a point stabilizer, that N is a faithful irreducible H-module and the actions of H on Ω and N are permutation isomorphic. In particular $n = |N|$. The complements to N in GA are all conjugate and indeed P is a point stabilizer in GA. Since N is abelian and a minimal normal subgroup of $GA = NP$, it follows that $\mathbf{C}_N(P) = 1$. Since $\mathbf{C}_N(P) = 1$ and the actions of P on N and Ω are permutation isomorphic, then

$$s \leq ((n-1)/p) + 1 = 2n/(p+1) - [(p-1)(n-(p+1))/p(p+1)] \leq 2n/(p+1)$$

as desired.

(b) Now suppose that V is an irreducible GA-module. If V is not quasiprimitive, choose N normal in GA maximal with respect to V_N not homogeneous and write $V_N = W_1 \oplus \cdots \oplus W_m$ for the $m > 1$ homogeneous components $W_1, \ldots, W_m$ of V_N. The maximality of N shows that G/N transitively and faithfully permutes the W_i. By Lemma 1.1, we have that $\dim(\mathbf{C}_V(A)) \leq t\dim(W_1)$, where t is the number of A-orbits on $\{W_1, \ldots, W_m\}$. By the first paragraph, $p\,||G/N|$. Note that $N > 1$ because V_N is not homogeneous. Applying part (a) to the action of GA/N on $\{W_1, \ldots, W_m\}$, we have $t \leq 2m/(p+1)$. So $\dim(\mathbf{C}_V(A)) \leq 2m\dim(W_1)/(p+1) = 2\dim(V)/(p+1)$. Thus we may assume that V is a primitive GA-module.

Suppose that $\mathbf{O}_{\pi'}(GA) \neq 1$. Because V is a faithful irreducible GA-module, $\mathbf{C}_V(A) \subseteq \mathbf{C}_V(\mathbf{O}_{\pi'}(GA)) = 1$ and $\dim(\mathbf{C}_V(A)) = 0$ and part (b) follows. Hence we may assume that $\mathbf{O}_{\pi'}(GA) = 1$.

Let $F = \mathbf{F}(G)$ so that $\mathbf{C}_G(F) \subseteq F$ by the solvability of G. Because $\mathbf{O}_{\pi'}(GA) = 1$, it follows that $\mathbf{C}_{GA}(F) \subseteq F$ and that A acts faithfully on F. We choose $a \in A$ of order p and show $\dim(\mathbf{C}_V(a)) \leq 2\dim(V)/(p+1)$. Since V is quasiprimitive, $\mathbf{Z}(F)$ is cyclic. If a does not centralize $\mathbf{Z}(F)$, then $Z < a >$ is a Frobenius group for some subgroup Z of $\mathbf{Z}(F)$ and then $\dim(\mathbf{C}_V(a)) = \dim(V)/p$ (see Theorem 15.16 of [Is]). We assume then that a centralizes $\mathbf{Z}(F)$ and choose a Sylow subgroup Q of F (for some prime q) that is not centralized by a. Now every characteristic abelian subgroup of Q is cyclic. We apply Theorem 1.9 of [MW]. If q is odd, then $Q = E\mathbf{Z}(Q)$ for an extra-special q-group E that is characteristic in Q and not centralized by a. If $q = 2$, then the hypotheses imply that GA is solvable and there exists an extra-special group E normal in GA such that $Q = ET$ with T normal in GA and $\mathrm{Aut}(T)$ a 2-group. In all cases E is normal in GA, V_E is homogeneous and a does not centralize E. Applying Lemma 1.3 (below), $\dim(\mathbf{C}_V(a)) \leq 2\dim(V)/(p+1)$ and thus $\dim(\mathbf{C}_V(A)) \leq 2\dim(V)/(p+1)$. □

LEMMA 1.3. *Suppose that $G = QP$ where Q is an extra-special q-group and $|P| = p$ for a prime $p \neq q$. Assume V is a G-module and V_Q is a direct sum of faithful irreducible Q-modules. Then* $\dim(\mathbf{C}_V(P)) \leq 2\dim(V)/(p+1)$.

Proof. Observe that $q \neq \mathrm{char}(V)$. Set $Z = \mathbf{Z}(Q)$ so that Z is the unique minimal normal subgroup of Q and $\mathbf{C}_V(Z) = 1$. We may assume that P centralizes Z since otherwise $\dim(\mathbf{C}_V(P)) \leq \dim(V)/p$ (see Theorem 15.16 of [Is]). Since P centralizes Z and $p \neq q$, the commutator group [Q,P] is an extra-special group containing Z (e.g., see Lemma 12.4 of [MW]). In particular, Z is the unique minimal normal subgroup of [Q,P]. Since $\mathbf{C}_V(Z) = 1$, it follows that $V_{[Q,P]}$ is a direct sum of faithful irreducible [Q,P] - modules. So we may assume that $Q = [Q, P]$ and thus $\mathbf{C}_{Q/Z}(P) = 1$.

Let F be the underlying field and let K be an extension field of F. Note that $V \otimes K$ is a direct sum of faithful irreducible Q-modules. Since $\dim_K(V \otimes K) = \dim_F(V)$ and $\dim_K(\mathbf{C}_{V\otimes K}(P)) = \dim_F(\mathbf{C}_V(P))$, it is no loss to assume that F is algebraically closed. Also, it is no loss to assume that V is indecomposable and thus absolutely indecomposable (even absolutely irreducible if $p \neq \mathrm{char}(F)$). Applying Hall-Higman techniques {specifically [Hu, V,17.13] when $p \neq \mathrm{char}(F)$ and [HB IX,2.6 and VII,5.3] when $p = \mathrm{char}(F)$}; we conclude there exists a non-negative integer m such that $\dim(\mathbf{C}_V(P))$ and $\dim(V)$ satisfy one of the following:

$\dim(\mathbf{C}_V(P)) = m$ and $\dim(V) = mp + 1$;
$\dim(\mathbf{C}_V(P)) = m$ and $\dim(V) = mp - 1$;
$\dim(\mathbf{C}_V(P)) = m + 1$ and $\dim(V) = mp + 1$; or
$\dim(\mathbf{C}_V(P)) = m + 1$ and $\dim(V) = mp + p - 1$.

Since Q is non-abelian, $q \mid \dim(V)$ and thus $m > 0$ in the first 3 cases. Also $mp > 2$ in the second case. In the last case, $m > 0$ or $p > 2$. In all four cases, it is easily verified that $\dim(\mathbf{C}_V(P))/\dim(V) \leq 2/(p+1)$, as desired. □

COROLLARY 1.4. *Assume* $1 \neq P \in \mathrm{Syl}_p(G)$ *for a solvable group* G.

(*a*) *If* G *transitively and faithfully permutes a set* Ω, *then the number of* P-*orbits on* Ω *is at most* $2|\Omega|/(p+1)$.

(*b*) *If* V *is a faithful irreducible* G-*module, then* $\dim(\mathbf{C}_V(P)) \leq 2\dim(V)/(p+1)$.

Proof. Let $L = \mathbf{O}_{p'}(G)$, let $M = \mathbf{O}_{p'p}(G)$ and $P \in \mathrm{Syl}_p(M)$. This corollary now follows by applying Theorem 1.2 to M, P, and G in place of G, A, and Γ (respectively). □

If W is a faithful irreducible H-module and S is a transitive permutation group on n letters, then the wreath product $H wr S$ is an irreducible linear group of degree $n^* \dim(W)$. Corollary 1.4 yields restrictions on which subgroups of $H wr S$ can be irreducible. An argument very similar to that of Corollary 1.4 gives the following.

COROLLARY 1.5. *Suppose that* $1 \neq H$ *is a Hall-*π'*-subgroup of a* π*-solvable group* G. *Let* p *be the smallest prime divisor of* $|H|$.

(*a*) *If* G *transitively and faithfully permutes a set* Ω, *then the number of* H-*orbits on* Ω *is at most* $2|\Omega|/(p+1)$.

(*b*) *If* V *is a faithful irreducible* G-*module, then* $\dim(\mathbf{C}_V(H)) \leq 2\dim(V)/(p+1)$.

In Theorem 1.2, we used part (a) to prove part (b). Next we use part 1.2(b) to prove a result similar to that of 1.2(a).

COROLLARY 1.6. *Assume* G *is a solvable primitive permutation group on* Ω *with point stabilizer* H. *If* $1 \neq P \in \mathrm{Syl}_p(H)$ *for a prime* p, *then the number* s *of* P-*orbits on* Ω *is at most* $2|\Omega|/(p+1)$ *unless* (*i*) $|\Omega| = 4$, $G \cong S_4$, $|p| = 2$ *and* $s = 3$; *or* (*ii*) $|\Omega| = 9$, $|P| = 3$ *and* $s = 5$.

Proof. Since G is solvable primitive permutation group on Ω, G has a unique minimal normal subgroup N that acts transitively and regularly on Ω. Furthermore $G = NH$, $N \cap H = 1$ and N is a faithful irreducible H-module and an elementary abelian r-group for a prime r. If $r \neq p$, then $P \in \mathrm{Syl}_p(G)$ and the result follows from Corollary 1.4. We thus assume that N is an elementary abelian p-group. Because $p \,||\mathrm{Aut}(N)|$, in fact $|N| = p^n$ for an integer $n > 1$. Since $P \subseteq H$, the actions of P on N and on Ω are permutation isomorphic. By Corollary 1.4, $|\mathbf{C}_N(P)| \leq |N|^{2/(p+1)}$. Hence the number of orbits of P on N or Ω is at most

$$|N|^{2/(p+1)} + (|N| - |N|^{2/(p+1)})/p = |\Omega|/p + (p-1)|\Omega|^{2/(p+1)}/p.$$

Now $|\Omega|/p+(p-1)|\Omega|^{2/(p+1)}/p \leq 2|\Omega|/(p+1)$ if and only if $p^n = |N| = |\Omega| \geq (p+1)^{(p+1)/(p-1)}$. This inequality is valid except when $p^n = 2^2, 2^3, 2^4$ or 3^2 and so it suffices to establish the corollary in these four cases. If $p^n = 3^2$, then $H \subseteq GL(2,3)$ and exception (ii) is easily verified. If $p^n = 2^2, 2^3$, or 2^4, then G is a subgroup of the semi-linear group $\Gamma(p^n)$ of order $(p^n-1)n$ (see Corollary 2.13 and Theorem 2.14 of [MW]). Thus $|P|\,|n$ and $\mathbf{C}_N(P) = |N|^{1/|P|}$. In particular, $n \neq 3$ and exception (i) occurs when $n = 2$. Finally, when $p^n = 2^4$, P is cyclic of order 2 or 4 and P has at most 10 orbits, whence the conclusion of this corollary is valid. □

2. Indices of centralizers

We will show that $|G{:}A| \leq |F{:}F \cap C|^{\alpha+1}$ for a constant α (defined after Lemma 2.1) if G is solvable, $F = \mathbf{F}(G)$, and Hypothesis CP applies. Theorem 2.4, the key result in this direction, shows that if V is an irreducible GA-module, then $|G{:}C| \leq |V{:}\mathbf{C}_V(A)|^{\alpha}$. The argument involves applying induction when V is an induced module. Lemma 2.1 uses Lemma 1.1 to help control indices of centralizers in this situation.

2.1 LEMMA. *Assume Hypotheses CP. Suppose that V is a faithful GA-module and $V_G = W_1 \oplus \cdots \oplus W_m$ for submodules W_i that are permuted by A in s orbits. Label the W_i so that $W_1, \ldots, W_s$ lie in the distinct orbits of A. Let $H_i = G/\mathbf{C}_G(W_i)$, let $A_i = \mathbf{N}_A(W_i)$ and $C_i = \mathbf{C}_{Hi}(A_i)$. Assume that* $\dim(W_i) = \dim(W_1)$ *and* $|H_i| = |H|$ *for all i where $H = H_1$. Then $|G{:}C| \leq |H|^m/(\Pi_{i=1 \text{ to } s}|C_i|)$.*

Proof. For $j = 1$ to s, we let Y_j be the sum of all X_i in the A-orbit of X_j, so that each Y_j is GA-invariant and $V = Y_1 \oplus \cdots \oplus Y_s$. Let $D_j = \mathbf{C}_G(Y_j)$. If $s > 1$, we argue by induction on $\dim(V)$ that $|G/D_j{:}\mathbf{C}_{G/Dj}(A)| = |G{:}D_jC| \leq |H|^{t(j)}/|C_j|$ where $t(j) = \dim(Y_j)/\dim(W_j)$. Since $(|G|,|A|) = 1$ and $\cap_{j=1 \text{ to } s} D_j = 1$ and, we have

$$|G{:}C| \leq \Pi_{j=1 \text{ to } s}|G/D_j{:}\mathbf{C}_{G/Dj}(A)| \leq |H|^m/\Pi_{j=1 \text{ to } s}|C_j|,$$

as desired. Thus we may assume that $s = 1$.

Now $V_G = W_1 \oplus \cdots \oplus W_m$ for submodules W_i that are permuted transitively by A/A_0 for some normal subgroup A_0 of A. Since $A_1 = \mathbf{N}_A(W_1)$, there is an injection $\varphi{:}GA \to GA_1/\mathbf{C}_{GA}(W_1) \wr A/A_0$ (e.g., see Lemma 2.8 of [MW]). Now $|G{:}C| = |\varphi(G){:}\varphi(C)| = |\varphi(G){:}\mathbf{C}_G(\varphi(A))|$. The normal Hall-$\pi$-subgroup, say R, of the wreath product $GA_1/\mathbf{C}_{GA}(W_1) \wr A/A_0$ is a direct product of m groups isomorphic to $H = GA_1/\mathbf{C}_{GA}(W_1)$ that are permuted transitively by A/A_0. By Lemma 1.1, $|G{:}C| = |\varphi(G){:}\mathbf{C}_{\varphi(G)}(\varphi(A))| \leq |R{:}\mathbf{C}_R(\varphi(A))| = |H|^m/|C_H(A_1) = |H|_1^m/C_1$. □

Notation. We let $\lambda = 24^{1/3}$ and $\alpha = (\ln(48)+\ln(\lambda))/\ln(9)$ and note that $9^\alpha = 48/\lambda$ and $2.24 < \alpha < 2.25$. Also $\lambda \cong 2.88$

The bounds given in the next lemma are best possible for groups of even order.

2.2 LEMMA. *(i) If $V \neq 0$ is a faithful completely reducible G-module for a solvable group G, then $|G| \leq |V|^{\alpha}/\lambda$. If $|G|$ is odd, then $|G| \leq |V|^{\alpha}/4.5$.*
(ii) If $G \neq 1$ is a normal subgroup of a primitive permutation group Γ on Ω and G is solvable, then $|G| \leq |\Omega|^{\alpha+1}/\lambda$.

Proof. The first statement in (i) is [MW, Theorem 3.5], which also shows that $|G| \leq |V|^2/\lambda$ when $|V|$ is odd. But $|V|^2/\lambda < |V|^{\alpha}/4.5$ for $|V| > 6$. The second statement of part (i) then easily follows via inspection.

For part (ii), choose a minimal normal subgroup M of GA with $M \subseteq G$. Then M is abelian, M regularly and transitively permutes Ω, $MH = GA$ and $M \cap H = 1$ where H is a point stabilizer in GA. Also, $\mathbf{C}_H(M)$ is normal in G and fixes every point of Ω, whence $\mathbf{C}_H(M) = 1$ and M is a faithful $H \cong \Gamma/M$-module. Then M is a completely reducible and faithful G/M-module. By part (i), $|G| = |G/M||M| \leq |M|^{\alpha+1}/\lambda = |\Omega|^{\alpha+1}/\lambda$. □

The next bound, used in Theorem 2.4 when V is an induced module, is an easy consequence of Lemma 2.2 and Theorem 1.2(b).

2.3 PROPOSITION. *Assume Hypotheses CP with G solvable and $A \neq 1$. Suppose that GA is a transitive permutation group on Ω and assume also that $G = 1$ or GA is primitive. Let $m = |\Omega|$ and s be the number of A-orbits on Ω. Then $|G{:}\ \mathbf{C}_G(A)| \leq \lambda^{m-s}/1.5$ or $GA \cong S_3$, $|A| = 2$, $s = 2$, $m = 3 = |G{:}\,\mathbf{C}_G(A)|$.*

Proof. Certainly $m > 1$. We assume that $G \neq 1$ since otherwise $s = 1$ and the inequality is trivial. Hence $m > 2$. The conclusion is evident if $m = 3$. If $m = 4$, the hypotheses imply that $GA = A_4$ with $|A| = 3$, so that $s = 2$ and $|G{:}\,\mathbf{C}_G(A)| = 4 = 2^{m-s} \leq \lambda^{m-s}/1.5$. We thus assume that $m > 4$.

Applying Lemma 2.2(ii), we have that $|G{:}\,C| \leq |G| \leq m^{\alpha+1}/\lambda$. By Theorem 1.2(a), $m - s \geq m/3$. For $m > 29$, an easy computation shows that $1.5m^{\alpha+1} < 24^{(m+3)/9} = \lambda^{(m+3)/3}$. Thus, for $m > 29$, it follows that

$$|G{:}\,C| \leq m^{\alpha+1}/\lambda \leq \lambda^{(m+3)/3}/1.5\lambda = \lambda^{m/3}/1.5 \leq \lambda^{m-s}/1.5,$$

as desired. If A is not a 2-group, then Theorem 1.2(a) even shows that $m - s \leq m/2$. For $m > 7$, an easy computation shows that $1.5m^{\alpha+1} < 24^{(m+2)/6} = \lambda^{(m+2)/2}$. When $m > 7$ and A is not a 2-group, it follows that

$$|G{:}\,C| \leq m^{\alpha+1}/\lambda \leq \lambda^{(m+2)/2}/1.5\lambda \leq \lambda^{m-s}/1.5,$$

as desired. Thus the proposition is valid when $m > 7$ provided A is not a 2-group and is always valid when $m > 29$.

Choose a minimal normal subgroup M of GA with $M \subseteq G$. Then M is abelian, M regularly and transitively permutes Ω, $MH = GA$ and $M \cap H = 1$ for a point stabilizer H in GA. In particular, $m = |\Omega| = |M| = p^n$ a prime p and integer n. As in Lemma 2.2, observe that M is a faithful irreducible H-module. By the coprimeness hypothesis, we can assume that $A \subseteq H$ and so that the permutation actions of A on M and Ω are isomorphic.

Suppose that MA/M is normal in GA/M. Then $\mathbf{C}_M(A) = \mathbf{C}_M(MA/M) = 1$ because M is a faithful irreducible GA/M-module and $A \neq 1$. Hence the number s of A-orbits on M is at most $1 + ((m-1)/2) = (m+1)/2$. Also A centralizes G/M and so $|G{:}C| = |M{:}M \cap C| = |M| = m$. Because $m > 4$, it follows that $|G{:}C| = m < \lambda^{(m-1)/2}/1.5 \leq \lambda^{m-s}/1.5$. The conclusion of the proposition is satisfied if MA/M is normal in G/M. If $m = p$, then GA/M is abelian because M is a faithful irreducible GA/M-module and so MA/M is normal in M.

Summarizing, we have that the proposition is valid provided that $m < 5$, $m > 29$ or m is prime. If, in addition, A is not a 2-group, the proposition is valid for $m > 7$. But $m = p^n$ is a prime power. Since $(p, |A|) = 1$, we may assume that A is a 2-group and $m = 3^2, 5^2$, or 3^3. Because M is a faithful irreducible GA/M-module, $|GA/M| \,\big|\, |M|$ and $\mathbf{O}_p(GA/M) = 1$. When $m = 3^2, 5^2$, or 3^3, it follows that $|G/M|$ divides 1, 3, or 13 (respectively). In all three cases, we now have

$$|G{:}C| \leq |G| = m|G/M| < 24^{m/9}/1.5 = \lambda^{m/3}/1.5 \leq \lambda^{m-s}/1.5,$$

where the last equality follows from Theorem 1.2(a). □

2.4 THEOREM. *Suppose that V is a faithful completely reducible GA-module with $A \neq 1$. Then $|G{:}\mathbf{C}_G(A)| \leq |V{:}\mathbf{C}_V(A)|^\alpha/1.5$.*

Note. Even if $A = 1$, we have $|G{:}\mathbf{C}_G(A)| \leq |V{:}\mathbf{C}_V(A)|^\alpha$. We will use the induction argument this way.

Proof. We will argue by induction on $|V|$. Since $(|G|, |A|) = 1$, we have that $GA/[G,A] = G/[G,A] \oplus A[G,A]/[G,A]$ and $[G,A]$ is the normal Hall-π-subgroup of $A[G,A]$. Also we have $G = [G,A]C$ and so $|G{:}C| = |[G,A]{:}\mathbf{C}_{[G,A]}(A)|$. But V is a faithful completely reducible $A[G,A]$-module and so it is no loss to assume that $G = [G,A]$. In particular, $GA = A[G,A] = \mathbf{O}^\pi(GA)$. Since $(|G|, |A|) = 1$, we have that $NC/N = \mathbf{C}_{G/N}(A)$ whenever N is an A-invariant subgroup of G.

First suppose that $V = X \oplus Y$ for GA-modules X and Y. Set $K = \mathbf{C}_{GA}(X)$ and $L = \mathbf{C}_{GA}(Y)$. If GA acts faithfully on X, the argument follows by induction as $|X{:}\mathbf{C}_X(A)| \leq |V{:}\mathbf{C}_V(A)|$. Thus we may assume neither X nor Y is a faithful GA-module. Since $V = X \oplus Y$ is a faithful GA-module, neither X nor Y is a trivial GA-module. Both L and K are proper non-trivial normal subgroups of GA. We apply the inductive hypothesis to the action of GA on X to conclude that:

$$|G{:}(K \cap G)C| = |GK{:}KC| = |GK/K{:}KC/K| \leq |X{:}\mathbf{C}_X(A)|^\alpha/1.5$$

and also that $|G:(L\cap G)C| \le |Y:\mathbf{C}_Y(A)|^\alpha/1.5$. Since $K\cap L = 1$, the group $K\cap G$ is A-isomorphic to a subgroup of $LG/L \cong G/L\cap G$ and thus

$$|K\cap G: K\cap G\cap C| = |K\cap G:\mathbf{C}_{K\cap G}(A)| \le |G:(L\cap G)C| \le |Y:\mathbf{C}_Y(A)|^\alpha/1.5.$$

Then

$$|G:C| = |G:(K\cap G)C||K\cap G: K\cap G\cap C| \le |X:\mathbf{C}_X(A)|^\alpha|Y:\mathbf{C}_Y(A)|^\alpha/1.5^2.$$

So we may assume that V is an irreducible GA-module.

Suppose that V is not quasi-primitive and choose N maximal in GA such that V_N is not homogeneous. Write $V_N = W_1\oplus\cdots\oplus W_m$ for the $m > 1$ homogeneous components $W_1,\dots,W_m$ of V_N. The maximality of N shows that GA/N transitively and faithfully permutes the W_i.

Label the W_i so that $W_1,\dots,W_s$ lie in the distinct orbits of A and set $A_i = \mathbf{N}_A(W_i)$. Let L be the normal Hall-π-subgroup of N, let $H_i = L/\mathbf{C}_L(W_i)$ so that H_i is A_i-isomorphic to the normal Hall-π-subgroup of $N/\mathbf{C}_N(W_i)$. If $C_i = \mathbf{C}_{Hi}(A_i)$, then Lemma 2.1 applied to LA yields

$$|L:L\cap C| \le |H_1|^m/(\Pi_{i=1\text{ to }s}|C_i|) = |H_1|^{m-s}\Pi_{i=1\text{ to }s}|H_i:C_i|.$$

Lemma 1.1 shows that $\mathbf{C}_V(A) \cong D_1\oplus\cdots\oplus D_s$ where $D_i = \mathbf{C}_{Wi}(A_i)$ and so $|V:\mathbf{C}_V(A)| = |W_1|^{m-s}\Pi_{i=1\text{ to }s}|W_i:D_i|$. By Lemma 2.2, $|H_1| \le |W_1|^\alpha/\lambda$. By induction applied to the action of LA_i (or even NA_i) on W_i, we have that $|H_i:C_i| \le |W_i:D_i|^\alpha$ for each i. Thus

$$\begin{aligned}|L:L\cap C| &\le |H_1|^{m-s}\Pi_{i=1\text{ to }s}|H_i:C_i| \le |W_1|^{\alpha(m-s)}/\lambda^{(m-s)}\Pi_{i=1\text{ to }s}|W_i:D_i|^\alpha\\ &= |V:\mathbf{C}_V(A)|^\alpha/\lambda^{m-s}.\end{aligned}$$

If $|A|$ is even, then $|H_i|$ must be odd and we similarly apply Lemma 2.2 to conclude that $|L:L\cap C| \le |V:\mathbf{C}_V(A)|^\alpha/(4.5)^{m-s}$. Now

$$|G:C| = |G/G\cap N:\mathbf{C}_{G/G\cap N}(A)||G\cap N:C\cap N| = |NG/N:\mathbf{C}_{NG/N}(A)||L:L\cap C|.$$

Thus we have

$$|G:C| \le |NG/N:\mathbf{C}_{NG/N}(A)||V:\mathbf{C}_V(A)|^\alpha/\lambda^{m-s}$$

and also

$$G:C \le |NG/N:\mathbf{C}_{NG/N}(A)||V:\mathbf{C}_V(A)|^\alpha/(4.5)^{m-s}$$

when $|A|$ is even. Assume first that GA/N is not isomorphic to S_3. Then Proposition 2.3 shows that $|NG/N:\mathbf{C}_{NG/N}(A)| \le \lambda^{m-s}/1.5$ and thus $|G:C| \le |V:\mathbf{C}_V(A)|^\alpha/1.5$ when V is imprimitive unless $GA/N \cong S_3$ and $|AN/N| = 2$. On the other hand, if $GA/N \cong S_3$ and $|AN/N| = 2$, then $m = 3$, $s = 2$, $|NG/N: \mathbf{C}_{NG/N}(A)| = 3$ and $|A|$ is even. Hence

$$|G:C| \leq 3|V:\mathbf{C}_V(A)|^\alpha/(4.5)^{m-s} = |V:\mathbf{C}_V(A)|^\alpha/1.5.$$

The conclusion $|G:C| \leq |V:\mathbf{C}_V(A)|^\alpha/1.5$ holds whenever V is an imprimitive GA-module.

If $\mathbf{O}_{\pi'}(GA) \neq 1$, then $\mathbf{C}_V(A) \subseteq \mathbf{C}_V(\mathbf{O}_{\pi'}(GA)) = 1$ and Lemma 2.2 yields $|G:C| \leq |G| \leq |V|^\alpha/1.5 = |V:\mathbf{C}_V(A)|^\alpha/1.5$. So we assume that $\mathbf{O}_{\pi'}(GA) = 1$.

Now we assume that V is a primitive GA-module. We let $F = \mathbf{F}(GA)$ and note $F = \mathbf{F}(G)$ because $\mathbf{O}_{\pi'}(GA) = 1$. First assume that F is abelian. Since V_F is homogeneous, GA may be identified as a subgroup of the semi-linear group $\Gamma(V)$ with $F \subseteq \Gamma_0(V)$; i.e., the elements of V may be labeled by those $GF(q^n)$ in a one-to-one fashion (where $q^n = |V|$) such that $GA \subseteq \Gamma(V) = \{x \to ax^\sigma | 0 \neq a \in GF(q^n), \sigma \in \mathrm{Gal}(GF(q^n)/GF(q))\}$ and $Z \subseteq \Gamma_0(V)$, the cyclic normal subgroup of multiplications with order q^n (see [MW, Corollary 2.3]). So A is isomorphic to a subgroup of $\Gamma(V)/\Gamma_0(V)$, whence A is cyclic. If $1 \neq P$ is a Sylow subgroup of A, then P is cyclic and we may find a characteristic subgroup Y of Z such that YP is a Frobenius group. Because $C_V(Y) = \{0\}$, we have that $\dim(\mathbf{C}_V(P)) = \dim(V)/|P|$ (see [Is, Theorem 15.16]). If $|P| > 2$, then

$$|V:\mathbf{C}_V(A)|^\alpha \geq |V:\mathbf{C}_V(P)|^\alpha \geq |V|^{2\alpha/3} \geq 3|V|/2 \geq 3|Z|/2 \geq 3|G:\mathbf{C}_G(A)|/2$$

and the conclusion of the theorem holds. If $|A| = 2$ and $1 \neq a \in A$, then $a = x\sigma$ for some x in $\Gamma_0(V)$ and field automorphism σ of order 2 and the centralizers in $\Gamma_0(V)$ of a and σ coincide as a group of order $q^{n/2} - 1$ (where $q^n = |V|$). Then

$$|G:C| \leq q^{n/2} + 1 \leq q^{n\alpha/2}/1.5 = |V:\mathbf{C}_V(A)|^\alpha/1.5,$$

as desired. Hence we may assume that F is non-abelian.

Because V is a primitive GA-module, every abelian normal subgroup Y of GA is cyclic and also $Y \subseteq G$ because $\mathbf{O}_{\pi'}(GA) = 1$. Furthermore $Y \subseteq \mathbf{Z}(G)$ because $\mathbf{O}^\pi(GA) = GA$ and $GA/\mathbf{C}_G(Y)$ is abelian. In particular, if $Z = \mathbf{Z}(F)$, then Z is cyclic and $Z = \mathbf{Z}(G)$. Observing that GA is solvable when $|F|$ is even, we quote Theorem 1.9 of [MW] to conclude that F/Z is abelian and is a direct sum of irreducible GA/F—modules of even dimension (possibly of different characteristics). The same theorem shows also that $F = EZ$ for a normal subgroup E of GA and the Sylow subgroups of E are extra-special or of prime order. Furthermore, F/Z is a faithful G/F-module by a theorem of Gaschutz [MW, Theorem 1.12]. Since F is non-abelian, $F > Z$.

Now $e^2 = |F:Z|$ for an integer $e > 1$. Since V is a homogeneous G-module, the structure of F implies that $V_Z \cong teW$ for an irreducible Z-module W (e.g., see Lemma 2.4 of [MW]) and some positive integer t. Since W is a faithful Z-module for the cyclic group Z, we have that $|Z| \mid |W| - 1$. Every prime divisor of e divides $|Z|$ and $|W| - 1$. In particular, $|W| \geq 3$.

Theorem 1.2 yields $\mathbf{C}_V(A)| \leq |V|^{2/3}$ and furthermore $|\mathbf{C}_V(A)| \leq |V|^{1/2}$ unless A is a 2-group. If A does not centralize Z, we may pick z in Z with $A^z \neq A$ and then $< A, A^z >$ generates a subgroup of AZ that intersects Z non-trivially. Then $\mathbf{C}_V(< A, A^z >) = \{0\}$ and so $|\mathbf{C}_V(A)| \leq |V|^{1/2}$. So $|\mathbf{C}_V(A)| \leq |V|^{2/3}$, and furthermore $|\mathbf{C}_V(A)| \leq |V|^{1/2}$ unless A is a 2-group centralizing Z.

We first prove the conclusion is valid when $G = F$. In this case, $|G| = |F/Z||Z| = e^2|Z| < e^2|W|$. Recall $2.24 < \alpha < 2.25$. If A is a 2-group that centralizes Z, then $e > 2$ and

$$|G{:}C| \leq |G{:}Z| = e^2 \leq 3^{e\alpha/3}/1.5 \leq |W|^{e\alpha/3}/1.5 \leq |V|^{\alpha/3}/1.5 \leq |V{:}\mathbf{C}_V(A)|^{\alpha}/1.5,$$

as desired. By the last paragraph, we may assume that $|\mathbf{C}_V(A)| \leq |V|^{1/2}$. Since $e > 1$ we have

$$|G{:}Z| \leq e^2 \leq 3^{e\alpha/2}/1.5 \leq |W|^{e\alpha/2}/1.5 \leq |V|^{\alpha/2}/1.5 \leq |V{:}\mathbf{C}_V(A)|^{\alpha}/1.5.$$

So we assume that A does not centralize Z, $|Z| > 2$ and $|W| \geq 4$. Because $e > 1$, note that $e^2 \leq 4^{(\alpha e-2)/2}/1.5$. Then $|G{:}C| \leq |G| < e^2|W| \leq |W|4^{(\alpha e-2)/2}/1.5 \leq |W||W|^{(\alpha e-2)/2}/1.5 \leq |W|^{\alpha e/2}/1.5 \leq |V|^{\alpha/2}/1.5 \leq |V{:}\mathbf{C}_V(A)|^{\alpha}/1.5$. So the conclusion holds when $G = F$.

Now $F = \mathbf{F}(GA)$, $Z = \mathbf{Z}(F) = \mathbf{Z}(G)$ and $(|G|, |A|) = 1$. We have that F/Z is the direct sum $F_1/Z \oplus \cdots \oplus F_k/Z$ for irreducible GA/F-modules F_i/Z of order f_i^2 for prime powers f_i such that $f_1 \ldots f_k = e$. We may assume that G/F does not centralize F_1/Z, because $G/F \neq 1$ and G/F acts faithfully on F/Z. If $B = GA/\mathbf{C}_{GA}(F_1/Z)$, then B has a non-trivial normal Hall-π-subgroup $B_0 = G\mathbf{C}_{GA}(F_1/Z)/\mathbf{C}_{GA}(F_1/Z)$ because G does not centralize F_1/Z. In particular, $\mathbf{F}(B_0) \neq 1$ has order coprime to f_1 because F_1/Z is a faithful irreducible B-module. Also B_0 and F_1/Z are π-groups, while B/B_0 is a π'-group. Furthermore $\mathbf{O}^{\pi}(B) = B$ and so $B/B_0 \neq 1$. In particular, B is non-abelian and $|B|f_1$ is divisible by at least three distinct primes. If f_1 is 2 or 3, then $\mathrm{Aut}(F_1, Z)$ is a $\{2,3\}$-group and we have a contradiction. If $f_1 = 4$, then B/B_0 has odd order and F_1/Z is a faithful irreducible module of order 2^4 for the solvable group B, whence $B = B_0$ or B is abelian (e.g., see Corollary 2.15 of [MW]), again a contradiction. Thus f_1 cannot be 2, 3, or 4. Thus $e = f_1 \ldots f_k > 9$ or $e = f_1$ is 5, 7, 8, or 9.

A simple computation shows that $e^{2(\alpha+1)} < 3^{(2e\alpha-3)/3} \leq |W|^{(2e\alpha-3)/3}$ for $e > 9$. But because every prime divisor of e must divide $|W|-1$ and $|W|$ is a prime power, we see also that $e^{2(\alpha+1)} \leq |W|^{(2e\alpha-3)/3}$ when e is 5, 7, or 9. Indeed similar computations yield

$$e^{2(\alpha+1)} \leq |W|^{(2te\alpha-3)/3} \text{ except when } e = 8, t = 1 \text{ and } |W| = 3;$$

$$e^{2(\alpha+1)} \leq |W|^{(te\alpha-2)/2} \text{ except when } t = 1 \text{ and } (e, |W|) \in \{(8,3), (8,5), (9,4)\};$$

$$e^{2(\alpha+1)} \leq |W|^{te\alpha/3} \text{ for odd } e \text{ except when } t = 1 \text{ and } (e, |W|) \in \{(5,11), (7,8), (9,4), (9,7)\}.$$

Since G/F acts faithfully and completely reducibly on F/Z, Lemma 2.2 implies that $|G/F| \leq |F/Z|^{\alpha}/\lambda = e^{2\alpha}/\lambda$ and thus $|G/Z| = |G\colon F||F\colon Z| \leq e^{2(\alpha+1)}/1.5$. Furthermore $|Z|$ divides $|W|-1$ and so $|G| \leq |W|e^{2(\alpha+1)}/1.5$. Assume that $e > 9$ or that $t > 1$. We apply the last paragraph to conclude that $|G| \leq |W|^{te\alpha/2}/1.5$. Likewise, $|G\colon Z| \leq |W|^{te\alpha/3}/1.5$ for odd e. If A is a 2-group centralizing Z, then e is odd, $|\mathbf{C}_V(A)| \leq |V|^{2/3}$ and

$$|G\colon \mathbf{C}_G(A)| \leq |G\colon Z| \leq |W|^{te\alpha/3}/1.5 = |V|^{\alpha/3}/1.5 \leq |V\colon \mathbf{C}_V(A)|^{\alpha}/1.5.$$

Otherwise $|\mathbf{C}_V(A)| \leq |V|^{1/2}$ and

$$|G\colon \mathbf{C}_G(A)| \leq |G| \leq |W|^{te\alpha/2}/1.5 = |V|^{\alpha/2}/1.5 \leq |V\colon \mathbf{C}_V(A)|^{\alpha}/1.5.$$

The conclusion $|G\colon \mathbf{C}_G(A)| \leq |V\colon \mathbf{C}_V(A)|^{\alpha}/1.5$ of the theorem holds when $e > 9$ or $t > 1$. Thus we may assume that $t = 1$ and $e \in \{5, 7, 8, 9\}$. By the same argument, we may even assume that $(e, |W|) \in \{(5, 11), (7, 8), (8, 3), (8, 5), (9, 4), (9, 7)\}$.

Assume for this paragraph that A has even order. Because $(|A|, |G|) = 1$, we have that e is odd and $(e, |W|) \in \{(5, 11), (7, 8), (9, 4), (9, 7)\}$. Now $G/F \cong B_0$ has odd order and is a non-trivial normal subgroup of B, which acts irreducibly on F/Z. If s is a prime divisor of $\mathbf{F}(G/F)$, then s must divide $e^2 - 1$ since 1 is the centralizer in F/Z of the Sylow-s-subgroup of $\mathbf{F}(G/F)$. Thus routine arguments then show that $|G/F|$ is 3 or 5. Then

$$|G/Z| \leq 5e^2 \leq \min\{|W|^{te\alpha/3}/1.5, |W|^{(te\alpha-2)/2}/1.5\},$$

with the last part of the inequality is easily verified by inspection. If A centralizes Z, then

$$|G\colon \mathbf{C}_G(A)| \leq |G\colon Z| \leq |W|^{te\alpha/3}/1.5 = |V|^{\alpha/3}/1.5 \leq |V\colon \mathbf{C}_V(A)|^{\alpha}/1.5$$

Otherwise $|\mathbf{C}_V(A)| \leq |V|^{1/2}$ and

$$|G\colon \mathbf{C}_G(A)| \leq |G| \leq |W|^{te\alpha/2}/1.5 = |V|^{\alpha/2}/1.5 \leq |V\colon \mathbf{C}_V(A)|^{\alpha}/1.5.$$

Hence the theorem is valid when $|A|$ is even.

We now may assume that $|A|$ is odd. In particular, $|\mathbf{C}_V(A)| \leq |V|^{1/2}$. If e is 5 or 7, it follows from the preceding three paragraphs that

$$\begin{aligned} |G\colon C| &\leq |G| \leq |W|e^{2(\alpha+1)}/1.5 \leq |W||W|^{(te\alpha-2)/2}/1.5 \\ &= |W|^{e\alpha/2}/1.5 = |V|^{\alpha/2} \leq |V|\colon \mathbf{C}_V(A)|^{\alpha}/1.5, \end{aligned}$$

as desired. If e is 9, then the prime divisors of $|A|$ are larger than 3 and so $|\mathbf{C}_V(A)| \leq$

$|V|^{1/3}$ by Theorem 1.2. For $e = 9$, it follows from the preceding three paragraphs that

$$\begin{aligned}|G:C| &\leq |G| \leq |W|e^{2(\alpha+1)}/1.5 \leq |W||W|^{(2te\alpha-3)/3}/1.5\\ &= |W|^{2e\alpha/3}/1.5 = |V|^{2\alpha/3} \leq |V:\mathbf{C}_V(A)|^{\alpha}/1.5,\end{aligned}$$

as desired. Hence we may assume that e is 8.

With $e = 8$, we have that $|W|$ is 3 or 5 and that A is solvable of odd order. Thus Z is cyclic of order 2 or 4 and thus $Z \leq \mathbf{Z}(GA)$. If $A_0 = \mathbf{C}_A(F/Z)$, the A_0 centralizes G/F, F/Z and Z, whence A_0 centralizes G. Then $A_0 = 1$ because $\mathbf{O}_{\pi'}(GA) = 1$, and F/Z is a faithful irreducible GA/F-module. Since $Z \leq \mathbf{Z}(GA)$, GA/F even acts symplectically. Then $|GA/F|$ must divide $|\mathrm{Sp}(6,2)| = 3^4{*}5{*}7{*}2^9$. For $p > 2$, an abelian p-group of GA/F can have rank at most $\dim(F/Z)/2 = 3$ (see Lemma 12.5 of [MW]). If T is a Sylow-3-subgroup of $\mathbf{F}(GA/F) = \mathbf{F}(G/F)$, then T cannot be elementary abelian of order 3^4 and so $|\mathrm{Aut}(T)|$ is not divisible by 5 or 7. Now $\mathbf{F}(GA/F)$ must be a {3,7}-group with order coprime to $|A|$. If 3 divides $|\mathbf{F}(GA/F)|$, then T and $\mathbf{F}(GA/F)$ are centralized by A, a contradiction because the solvability of GA implies that $\mathbf{F}(GA/F)$ must contain its own centralizer in GA/F. So $|\mathbf{F}(GA/F)| = 7$. Since $\mathrm{Aut}(\mathbf{F}(GA/F))$ is abelian, it follows that $\mathbf{F}(GA/F) = GA/F$ has order 7 and GA/F is non-abelian of order 21. Now

$$\begin{aligned}|G:\mathbf{C}_G(A)| \leq |G:Z| &= 7{*}2^6 \leq 3^{4\alpha}/1.5 \leq |W|^{e\alpha/2}/1.5\\ &= |V|^{\alpha/2}/1.5 \leq |V:\mathbf{C}_V(A)|^{\alpha}/1.5,\end{aligned}$$

as desired to complete the proof. (Alternatively, one could derive a contradiction here, because a non-abelian group of order 21 cannot act irreducibly on a $GF(2)$-vector space of dimension 6). □

Applying Theorem 2.4 to the action of GA on $\mathbf{F}(G)$ now gives an affirmative answer to the question posed by Perez and Iranzo.

2.5 COROLLARY. *Assume Hypothesis CP with G solvable and $F = \mathbf{F}(G)$. Then* $|G:C| \leq |F:F \cap C|^{\alpha+1}$.

Proof. We will argue by induction on $|G|$. We may assume that $A \neq 1$ and that $\mathbf{O}_{\pi'}(GA) = 1$, i.e., that A acts faithfully on G. So $F = \mathbf{F}(GA)$. Because $(|A|,|G|) = 1$, $\mathbf{C}_{G/F}(A) = FC/F$.

Now $F/\Phi(G)$ is a completely reducible and faithful G/F-module (possibly of mixed characteristic) by a Theorem of Gaschutz (see Satz III.4.2(d) and III.4.5 of [Hu]). If $D/F = \mathbf{C}_{GA/F}(F/\Phi(G))$, then D/F is a π'-group that centralizes G/F and $F/\Phi(G)$. Since D/F is a π'-group and G is a π-group, D/F centralizes $G/\Phi(G)$. By Satz III.3.18 of [Hu] D/F centralizes G and hence $D/F = 1$. Thus $F/\Phi(G)$ is a faithful GA/F module. Furthermore it is a completely reducible GA/F-module because it is completely reducible as a G/F-module and $(|GA/G|, |F/\Phi(G)|) = 1$ (see Theorem VII.7.20 of [HB]). Since $A \neq 1$, Theorem 2.4 applied to the action of GA/F on $F/\Phi(G)$ shows that

$$\begin{aligned}|G: FC| &= |G/F: \mathbf{C}_{G/F}(A)| \leq |F/\Phi(G): \mathbf{C}_{F/\Phi(G)}(A)|^{\alpha} \\ &= |F: \Phi(G)(F \cap C)|^{\alpha} \leq |F: F \cap C|^{\alpha}.\end{aligned}$$

Then $|G: C| = |G: FC||F: F \cap C| \leq |F: F \cap C|^{\alpha+1}$. □

2.6 *Example.* There is an infinite family (W_i, H_i) where W_i is an elementary abelian 3-group and is a faithful and irreducible module for H_i, a {2,3}-group and such that $|H_i| = |W_i|^{\alpha}/(24)^{1/3}$. Indeed W_0 may be chosen to have order 3^2, H_0 to be $GL(2, 3)$, and H_i to be $H_{i-1} \wr S_4$ (see Example 3.8 of [MW] for details).

For the moment, fix i. Let V be the direct sum of 5 copies of W_i and let G be the direct sum of 5 copies of H_i. Let Γ be $H_i \wr Z_5$ so that G is a normal Hall-π-subgroup of Γ with $\pi = \{2, 3\}$. Let A be a Hall-π'-subgroup of Γ so that $|A| = 5$. Also V is a faithful irreducible Γ-module. Applications of Lemma 1.1 show that $|G: \mathbf{C}_G(A)| = |H_i|^4$ while $|V: \mathbf{C}_V(A)| = |W_i|^4$. Thus $|G: \mathbf{C}_G(A)| = |V: \mathbf{C}_V(A)|^{\alpha}/(24)^{4/3}$. By letting $i \to \infty$, $|W_i| \to \infty$ and also $|V: \mathbf{C}_V(A)| \to \infty$. Thus the bound in Theorem 2.4 cannot be improved with an exponent less than α.

Let Γ^* be the semi-direct product ΓV and $G^* = GV$, so that G^* is a solvable normal Hall-π-subgroup of Γ^* and A is a Hall-π'-subgroup of Γ^*, $|A| = 5$. Set $C = \mathbf{C}_{G^*}(A)$ and note that $C = \mathbf{C}_G(A)\mathbf{C}_V(A)$. Since V is a faithful irreducible Γ-module, it follows that $V = \mathbf{F}(G^*) = \mathbf{F}(\Gamma^*)$. Now

$$\begin{aligned}|G^*: C| &= |G: \mathbf{C}_G(A)||V: \mathbf{C}_V(A)| \\ &= |V: \mathbf{C}_V(A)|^{\alpha+1}/(24)^{4/3} = |\mathbf{F}(G^*): \mathbf{F}(G^*) \cap C|^{\alpha+1}/(24)^{4/3}.\end{aligned}$$

By letting $i \to \infty$, we see that $|\mathbf{F}(G^*): \mathbf{F}(G^*) \cap C| \to \infty$. In particular, the exponent in Corollary 2.5 cannot be lowered. □

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