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ON THE L^1-BEHAVIOR OF THE MAXIMAL OPERATOR FOR THE CLASS OF MARTINGALES ADAPTED TO A GIVEN FILTRATION

ALEXANDRA BELLOW AND ULRICH KRENGEL

To the memory of Alberto Calderón

ABSTRACT. We study the boundedness in L^1 of the maximal operator of the class of martingales for a fixed increasing or decreasing filtration. We obtain necessary and sufficient conditions for several classes of martingales.

1. Introduction

The starting point of this investigation was the observation that in the discrete case, for the standard decreasing filtration on $\mathbb{N}$, the maximal (martingale) operator is bounded on L^1. It is well known that on any probability space, for any filtration (whether increasing or decreasing), the corresponding maximal (martingale) operator is always bounded in L^p with bound $p/(p-1)$ (for $1 < p < \infty$), and is weak-type 1-1 (the maximal inequality). Similar statements hold for the maximal ergodic operator and the Hardy–Littlewood maximal operator. Furthermore it is known that for the maximal ergodic operator (in the case when the underlying measure-preserving transformation is ergodic) and for the Hardy–Littlewood maximal operator we have $Mf \in L^1$ if and only if $f \in L\log^+ L$ (see the important papers [5] and [4], respectively). These considerations motivated our interest in the further study of the maximal (martingale) operator from the point of view of its boundedness on L^1.

In a certain sense, the problem considered in this paper can be regarded as a dual converse to the problem considered by Blackwell and Dubins. In their celebrated paper [1], Blackwell and Dubins start with a function $f \in L^1$, $f \notin L\log^+ L$ and look for a filtration such that for the corresponding maximal (martingale) operator, $Mf \notin L^1$. In many situations however, the filtration is given, and it is more natural to ask if there is $f \in L^1$ such that $Mf \notin L^1$.

Let us make the notation precise. Let $(\Omega, \mathcal{F}, P)$ be a probability space. We shall write $L^1 = L^1(\Omega, \mathcal{F}, P)$ and $L^1_+ = \{f \in L^1; f \geq 0\}$. All the σ–algebras considered below will be contained in $\mathcal{F}$.

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If $\mathcal{G}$ is a σ–algebra and $f \in L^1$, we write $E(f \mid \mathcal{G})$ for the conditional expection operator with respect to $\mathcal{G}$. By an increasing filtration on Ω we mean a sequence $(\mathcal{F}_n)_{n\geq 1}$ of σ–algebras such that

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \cdots .$$

By a decreasing filtration on Ω we mean a sequence $(\mathcal{F}_n)_{n\geq 1}$ of σ–algebras such that

$$\mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots \supset \mathcal{F}_n \supset \mathcal{F}_{n+1} \supset \cdots .$$

Let $(\mathcal{F}_n)_{n\geq 1}$ be an increasing (resp. decreasing) filtration on Ω. For $f \in L^1$ consider the (increasing) (resp. reversed) martingale $(E(f \mid \mathcal{F}_n))_{n\geq 1}$ and define the maximal operator

$$Mf = \sup_{n\geq 1} |E(f \mid \mathcal{F}_n)|.$$

If $(X_j)_{j\in J}$ is a family of real random variables on our probability space, we denote by

$$\sigma((X_j)_{j\in J})$$

the smallest σ-algebra making every X_j, $j \in J$ measurable.

In what follows we shall often consider the discrete probability spaces $(\mathbb{N}, \mathcal{F}, P)$, where

$\Omega = \mathbb{N} = \{1, 2, 3, \ldots, n, \ldots\}$,

$\mathcal{F}$ is the the collection of all subsets of $\mathbb{N}$,

$P = (p_1, p_2, \ldots, p_n, \ldots)$ is a probability on $\mathbb{N}$ with $p_n = P(\{n\}) > 0$ for all $n \in \mathbb{N}$.

Here is a brief outline of the paper. In Section 2 we consider discrete probability spaces and we show that, for the standard decreasing filtration, the maximal operator is bounded on L^1 (Proposition 1). In Section 3 we first establish two key lemmas for general probability spaces giving quantitive estimates for the maximal function corresponding to a finite decreasing filtration (Lemmas 1 and 2). As a first application we show that for an independent sequence of random variables, both in the case of the decreasing filtration and in the case of the increasing filtration associated with the sequence, the corresponding maximal operator is unbounded on L^1. Next we consider discrete probability spaces and for a decreasing filtration (respectively an increasing filtration) we give a simple criterion (necessary and sufficient condition) for the maximal operator to be unbounded on L^1; these are Theorems 1 and 2. In Section 4 we discuss a number of examples where the previous criteria apply. In Section 5 we consider non–atomic probability spaces. We show that for any decreasing non-atomic filtration for which the tail σ-algebra is trivial, the maximal operator is unbounded on L^1; this is Theorem 3.

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2. The discrete case; the standard decreasing filtration

The starting point of our investigation is the following observation.

PROPOSITION 1. *Let* $(\mathbb{N}, \mathcal{F}, P)$ *be a discrete probability space (no restriction on* P*). Let* $(\mathcal{F}_n)_{n\geq 1}$ *be the standard decreasing filtration on* $\mathbb{N}$, *that is* $\mathcal{F}_n$ *is the* σ*-algebra generated by the partition*

$$[1, n], \{n+1\}, \{n+2\}, \{n+3\}, \dots$$

(here $[1, n] = \{1, 2, \dots, n\}$*). Then the maximal operator*

$$Mf = \sup_{n\geq 1} |E(f \mid \mathcal{F}_n)|, \quad f \in L^1$$

satisfies

$$\|Mf\|_1 \leq \left(1 + \frac{1}{p_1}\right) \|f\|_1, \quad f \in L^1. \tag{2.1}$$

Hence $M : f_1 \longrightarrow Mf$ *is a bounded sublinear operator on* L^1 *with bound* $1+(1/p_1)$.

Proof. Let $P_k = p_1 + \cdots + p_k$ for $k \geq 1$. Let f_k denote the indicator function of $\{k\}$. We first calculate Mf_k. It is clear that $E(f_k \mid \mathcal{F}_n) = f_k$ for $n < k$. For $n \geq k$,

$$E(f_k \mid \mathcal{F}_n)(j) = \begin{cases} \frac{p_k}{P_n} & \text{for } j \leq n \\ 0 & \text{for } j > n. \end{cases}$$

As p_k / P_n decreases with n, we obtain

$$(Mf_k)(j) = \begin{cases} \frac{p_k}{P_k} & \text{for } j \leq k-1 \\ 1 & \text{for } j = k \\ \frac{p_k}{P_j} & \text{for } j > k. \end{cases}$$

Thus

$$\begin{aligned} E(Mf_k) &= \frac{p_k}{P_k} \cdot P_{k-1} + p_k + \sum_{j>k} \frac{p_k}{P_j} \cdot p_j \\ &= p_k \left[\frac{P_{k-1}}{P_k} + 1 + \sum_{j=k+1}^{\infty} \frac{p_j}{P_j} \right] \\ &\leq p_k \left[\frac{P_{k-1}}{P_k} + 1 + \frac{1}{P_k} \sum_{j=k+1}^{\infty} p_j \right] \\ &\leq p_k \left[1 + \frac{1}{P_k} \right] \\ &\leq p_k \left[1 + \frac{1}{p_1} \right] = \left(1 + \frac{1}{p_1}\right) E(f_k). \end{aligned}$$

Since the first factor in the right-hand side of the previous inequality does not depend on k, the subadditivity and homogeneity of M easily imply

$$\|Mf\|_1 \le \left(1 + \frac{1}{p_1}\right) \|f\|_1 \quad \text{for all} \quad f \in L^1_+.$$

For arbitrary $f \in L^1$, note that $0 \le Mf \le M(|f|)$. Hence (2.1) follows and the proof is complete.

3. Criterion for unboundedness of the maximal operator

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

LEMMA 1. *Let $\mathcal{G}_1 \supset \mathcal{G}_2 \supset \cdots \supset \mathcal{G}_l$ be a finite decreasing filtration on Ω. Let $D \in \mathcal{F}$ and assume that*

$$E(1_D \mid \mathcal{G}_j) = \alpha_j 1_{B_j}, \tag{3.1}$$

where $\alpha_j \in \mathbb{R}_+$ and $B_j = B_j(D) \in \mathcal{G}_j$ for $1 \le j \le l$. In addition assume that

$$(*) \qquad \frac{P(B_{j+1})}{P(B_j)} \ge 2 \quad \text{for} \quad 1 \le j \le l-1.$$

Then we have

$$\| \sup_{1 \le k \le l} E(1_D \mid \mathcal{G}_k)\|_1 \ge \frac{l+1}{2} \|1_D\|_1. \tag{3.2}$$

Proof. Clearly $\alpha_j = P(D)/P(B_j)$ and we may assume that $B_1 \subset B_2 \subset \cdots \subset B_l$. Thus the maximal function $\sup_{1 \le k \le l} E(1_D \mid \mathcal{G}_k)$ takes the value $P(D)/P(B_1)$ on B_1, the value $P(D)/P(B_k)$ on $B_k \backslash B_{k-1}$ for $k = 2, \ldots, l$. Since

$$\frac{P(B_k \backslash B_{k-1})}{P(B_k)} = 1 - \frac{P(B_{k-1})}{P(B_k)} \ge 1 - \frac{1}{2} = \frac{1}{2}$$

it follows that

$$\begin{aligned} \| \sup_{1 \le k \le l} E(1_D \mid \mathcal{G}_k)\|_1 &= \frac{P(D)}{P(B_1)} P(B_1) + \sum_{k=2}^{l} \frac{P(D)}{P(B_k)} P(B_k \backslash B_{k-1}) \\ &= P(D) \left(1 + \sum_{k=2}^{l} \frac{P(B_k \backslash B_{k-1})}{P(B_k)}\right) \\ &\ge P(D) \left(1 + (l-1)\frac{1}{2}\right) = \frac{l+1}{2} \|1_D\|_1 \end{aligned}$$

and the lemma is proved.

We also have the following result in the opposite direction.

LEMMA 2. *Let $\mathcal{G}_1 \supset \mathcal{G}_2 \supset \cdots \supset \mathcal{G}_m$ be a finite decreasing filtration on Ω. Let $D \in \mathcal{F}$ and assume that*

$$E(1_D \mid \mathcal{G}_j) = \alpha_j 1_{B_j} \tag{3.3}$$

where $\alpha_j \in \mathbb{R}_+$ and $B_j = B_j(D) \in \mathcal{G}_j$ for $1 \le j \le m$. In addition assume that for some $1 < l \le m/2$ we have

$$\| \sup_{1 \le k \le m} E(1_D \mid \mathcal{G}_k) \|_1 \ge 2l \| 1_D \|_1. \tag{3.4}$$

Then there exists a chain of length l, $1 \le k_1 < k_2 < \cdots < k_l \le m$ such that

$$(*) \qquad \frac{P(B_{k_{\nu+1}})}{P(B_{k_\nu})} \ge 2 \quad \textit{for} \quad 1 \le \nu \le l-1.$$

Proof. As before, $\alpha_j = P(D)/P(B_j)$, $B_1 \subset B_2 \subset \cdots \subset B_m$ and the maximal function $\sup_{1 \le k \le m} E(1_D \mid \mathcal{G}_k)$ takes the value $P(D)/P(B_1)$ on B_1, the value $P(D)/P(B_k)$ on $B_k \setminus B_{k-1}$ for $k = 2, \ldots, m$. Let $k_1 = 1$. If k_ν has been determined for some $\nu \le l-1$, set

$$k_{\nu+1} = \inf \left\{ k; k_\nu < k \le m \quad \text{and} \quad \frac{P(B_k)}{P(B_{k_\nu})} \ge 2 \right\}$$

if the set in the brackets is non-empty, and $k_{\nu+1} = +\infty$ otherwise. The proof is complete if we can show that $k_l \le m$. Suppose otherwise. Then there exists $1 \le r < l$ with $k_{r+1} = +\infty$. Now observe that we have the following estimate for the maximal function

$$\begin{aligned} \sup_{1 \le k \le m} E(1_D \mid \mathcal{G}_k) \le \sum_{\nu=1}^{r} E(1_D \mid \mathcal{G}_{k_\nu}) + \sum_{\nu=1}^{r-1} \sup_{k_\nu < j < k_{\nu+1}} (E(1_D \mid \mathcal{G}_j) - E(1_D \mid \mathcal{G}_{k_\nu}))^+ \\ + \sup_{k_r < j \le m} (E(1_D \mid \mathcal{G}_j) - E(1_D \mid \mathcal{G}_{k_j}))^+. \end{aligned} \tag{3.5}$$

The integral of each one of the first r summands is $P(D)$. Now look at a summand of the form

$$\sup_{k_\nu < j < k_{\nu+1}} (E(1_D \mid \mathcal{G}_j) - E(1_D \mid \mathcal{G}_{k_\nu}))^+. \tag{3.6}$$

The function $E(1_D \mid \mathcal{G}_{k_\nu})$ takes the value $P(D)/P(B_{k_\nu})$ on the set B_{k_ν}. For $k_\nu < j < k_{\nu+1}$ the function $E(1_D \mid \mathcal{G}_j)$ takes the value $P(D)/P(B_j) \le P(D)/P(B_{k_\nu})$ on B_j. Thus $(E(1_D \mid \mathcal{G}_j) - E(1_D \mid \mathcal{G}_{k_\nu}))^+$ is 0 on B_{k_ν} and $\le P(D)/P(B_{k_\nu})$ on $B_j \setminus B_{k_\nu} (k_\nu < j < k_{\nu+1})$. By construction, the union of the sets $B_j \setminus B_{k_\nu} (k_\nu < j < k_{\nu+1})$ has measure at most $P(B_{k_\nu})$. It follows that the integral of (3.6) is at most

$P(D)$. A similar argument applies to the last term on the RHS of (3.5). Hence we obtain

$$\| \sup_{1 \le k \le m} E(1_D \mid \mathcal{G}_k)\|_1 \le 2rP(D) = 2r\|1_D\|_1.$$

As $r < k$, this contradicts (3.4) and finishes the proof of Lemma 2.

Remark. There are analogous versions of Lemmas 1 and 2 for finite increasing filtrations (this is immediate by relabelling).

We say that M is bounded on L^1 if there exists a constant $C > 0$ such that $\|Mf\|_1 \le C\|f\|_1$ holds for all $f \in L^1$. It is easy to see that this is true iff M maps L^1 into L^1. In the opposite case, M is called unbounded on L^1.

The groundwork is now done to completely settle the boundedness of the maximal operator in the case of discrete probability spaces and also in the case of arbitrary probability spaces when the filtration is generated by independent random variables. We begin with the latter.

COROLLARY 1. *Let $(X_n)_{n\ge1}$ be an independent sequence of (non-constant) random variables on the probability space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_n)_{n\ge1}$ be the decreasing filtration on Ω given by*

$$\mathcal{F}_n = \sigma(X_n, X_{n+1}, \ldots).$$

Then the maximal operator M is unbounded on L^1.

Proof. It is enough to show that given any integer $l > 1$ we can find $g \in L^1_+$ such that

$$\|Mg\|_1 \ge \frac{l+1}{2}\|g\|_1.$$

Let $D_j = \{X_j \in E_j\}$, E_j some Borel set of $\mathbb{R}$, such that $0 < P(D_j) \le 1/2$ for $1 \le j \le l$ (such a D_j exists since X_j is non-constant). Let $D = D_1 \cap D_2 \cap \cdots \cap D_l$ and $g = 1_D$. Set $B_k = D_k \cap \cdots \cap D_l$ for $1 \le k \le l$; in particular $B_1 = D$. Note that

$$E(1_D \mid \mathcal{F}_1) = 1_D = 1_{B_1}$$

and by independence, for $2 \le k \le l$,

$$E(1_D \mid \mathcal{F}_k) = P(D_1 \cap \cdots \cap D_{k-1})1_{D_k \cap \cdots \cap D_l} = P(D_1 \cap \cdots \cap D_{k-1})1_{B_k}$$

where $B_k \in \mathcal{F}_k$ and $B_1 \subset B_2 \subset \cdots B_l$. Also note that

$$(*) \qquad \frac{P(B_{k+1})}{P(B_k)} = \frac{P(D_{k+1} \cap \cdots \cap D_l)}{P(D_k \cap \cdots \cap D_l)} = \frac{1}{P(D_k)} \ge 2$$

for $1 \leq k \leq l-1$. Taking $\mathcal{G}_k = \mathcal{F}_k$ for $k = 1, 2, \dots, l$, we see that the assumptions for Lemma 1 are satisfied and therefore

$$\| \sup_{1 \leq k \leq l} E(1_D \mid \mathcal{F}_k)\|_1 \geq \frac{l+1}{2} \|1_D\|_1.$$

COROLLARY 2. *Let $(X_n)_{n\geq 1}$ be an independent sequence of (non-constant) random variables on the probability space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_n)_{n\geq 1}$ be the increasing filtration on Ω given by*

$$\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n).$$

Then the maximal operator M is unbounded on L^1.

Proof. As before, given any integer $l > 1$ we show the existence of $g \in L^1_+$ such that

$$\|Mg\|_1 \geq \frac{l+1}{2} \|g\|_1.$$

Again, let $D_j = \{X_j \in E_j\}$, E_j some Borel set of $\mathbb{R}$, such that $0 < P(D_j) \leq 1/2$, for $1 \leq j \leq l$. Let $D = D_1 \cap D_2 \cap \cdots \cap D_l$ and $g = 1_D$. Set $B_k = D_1 \cap D_2 \cap \cdots \cap D_k$ for $1 \leq k \leq l$; in particular $B_l = D$. Note that by independence, for $1 \leq k \leq l-1$,

$$E(1_D \mid \mathcal{F}_k) = P(D_{k+1} \cap \cdots \cap D_l) 1_{D_1 \cap D_2 \cap \cdots \cap D_k} = P(D_{k+1} \cap \cdots \cap D_l) 1_{B_k}$$

and that

$$E(1_D \mid \mathcal{F}_l) = 1_D = 1_{B_l},$$

where

$$B_k \in F_k, \quad B_1 \supset B_2 \supset \cdots \supset B_l,$$

and

$$(*) \qquad \frac{P(B_k)}{P(B_{k+1})} = \frac{P(D_1 \cap \cdots \cap D_k)}{P(D_1 \cap \cdots \cap D_{k+1})} = \frac{1}{P(D_{k+1})} \geq 2.$$

The conclusion now follows from the analog of Lemma 1 for finite increasing filtrations.

We can now formulate the criteria for the unboundedness of the maximal operator in the case of discrete probability spaces.

THEOREM 1. *Let $(\mathbb{N}, \mathcal{F}, P)$ be a discrete probability space. Let $(\mathcal{F}_n)_{n\geq 1}$ be a decreasing filtration on $\mathbb{N}$ and for each $i \in \mathbb{N}$ and $n \geq 1$ let $A_n(i)$ be the (unique)* *atom of $\mathcal{F}_n$ containing i. Consider the maximal operator $Mf = \sup_{n\geq 1} |E(f \mid \mathcal{F}_n)|$, $f \in L^1$. Then the following assertions are equivalent:*

(1) *The maximal operator M is unbounded on L^1.*
(2) *For each integer $l > 1$ there exists an $i \in \mathbb{N}$ and a chain of length l, $1 \leq k_1 < k_2 < \cdots < k_l$ such that*
(*) $\frac{P(A_{k_{\nu+1}}(i))}{P(A_{k_\nu}(i))} \geq 2$ *for* $1 \leq \nu \leq l-1$.

Proof. (2) $\Rightarrow$ (1). Start with $C > 0$ and choose an integer $l > 1$ such that $(l+1)/2 > C$. Consider the finite decreasing filtration $\mathcal{G}_1 \supset \mathcal{G}_2 \supset \cdots \supset \mathcal{G}_l$, where $\mathcal{G}_1 = \mathcal{F}_{k_1}, \mathcal{G}_2 = \mathcal{F}_{k_2}, \ldots, \mathcal{G}_l = \mathcal{F}_{k_l}$. For the set $D = \{i\}$ with $B_j = A_{k_j}(i)$, $1 \leq j \leq l$, the assumptions of Lemma 1 are satisfied and hence by Lemma 1 (with $f_i = 1_{\{i\}}$), we have

$$\|\sup_{1\leq\nu\leq l} E(f_1 \mid \mathcal{F}_{k_\nu})\|_1 \geq \frac{l+1}{2}\|f_i\|_1 \geq C\|f_i\|_1.$$

Since

$$Mf_i \geq \sup_{1\leq\nu\leq l} E(f_i \mid \mathcal{F}_{k_\nu})$$

we deduce

$$\|Mf_i\|_1 \geq C\|f_i\|_1,$$

proving (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Let $l > 1$ and let $C = 2l$. Since we do not have unboundedness on L^1, the maximal operator M cannot be bounded on the set $\{f_i; i \in \mathbb{N}\}$. Hence there is $i \in \mathbb{N}$ and m large enough, $m \geq 2l$, such that

$$\|\sup_{1\leq k\leq m} E(f_i \mid \mathcal{F}_k)\|_1 \geq 2l\|f_i\|_1.$$

Consider the finite decreasing filtration $\mathcal{G}_1 \supset \mathcal{G}_2 \supset \cdots \supset \mathcal{G}_m$, where $\mathcal{G}_1 = \mathcal{F}_1, \mathcal{G}_2 = \mathcal{F}_2, \ldots, \mathcal{G}_m = \mathcal{F}_m$. For the set $D = \{i\}$ with $B_j = A_j(i)$, $1 \leq j \leq m$, the assumptions of Lemma 2 are satisfied, and hence by Lemma 2 we conclude that there exists a chain of length l, $1 \leq k_1 < k_2 < \cdots < k_l \leq m$ such that

$$(*) \qquad \frac{P(A_{k_{\nu+1}}(i))}{P(A_{k_\nu}(i))} \geq 2 \quad \text{for} \quad 1 \leq \nu \leq l-1,$$

proving (1) $\Rightarrow$ (2).

An entirely similar argument based on the analogs of Lemma 1 and 2 for finite increasing filtrations yields the next result.

THEOREM 2. *Let* $(\mathbb{N}, \mathcal{F}, P)$ *be a discrete probability space. Let* $(\mathcal{F}_n)_{n\geq 1}$ *be an increasing filtration on* $\mathbb{N}$ *and for each* $i \in \mathbb{N}$ *and* $n \geq 1$ *let* $A_n(i)$ *be the (unique) atom of* $\mathcal{F}_n$ *containing* i. *Consider the maximal operator* $Mf = \sup_{n\geq 1} |E(f \mid \mathcal{F}_n)|$, $f \in L^1$. *Then the following assertions are equivalent*:

(1) *The maximal operator* M *is unbounded on* L^1.
(2) *For each integer* $l > 1$ *there exists an* $i \in \mathbb{N}$ *and a chain of length* l, $1 \leq k_1 < k_2 < \cdots < k_l$ *such that*
(*) $\frac{P(A_{k_\nu}(i))}{P(A_{k_{\nu+1}}(i))} \geq 2$ *for* $1 \leq \nu \leq l-1$.

4. Examples

In this section we discuss several examples where the criteria of the previous section apply.

(1). Consider the discrete probability space $(\mathbb{N}, \mathcal{F}, P)$ *(no restriction on* P*) and the standard increasing filtration* $(\mathcal{F}_n)_{n\geq 1}$, that is $\mathcal{F}_1$ is the the trivial σ-algebra $\{\phi, \mathbb{N}\}$ and for $n > 1$, $\mathcal{F}_n$ is the σ-algebra generated by the partition

$$\{1\}, \{2\}, \ldots, \{n-1\}, [n, \infty)$$

(here of course $[n, \infty) = \{n, n+1, n+2, \ldots\}$). Let

$$R_k = \sum_{n=k}^{\infty} p_n = P([k, \infty)).$$

As $R_k \searrow 0$, it is clear that for each $l > 1$ there is a chain $1 \leq k_1 < k_2 < \cdots < k_l$ such that $R_{k_\nu} \geq 2R_{k_{\nu+1}}$ holds for $1 \leq \nu \leq l-1$. Take any $i \in [k_l, \infty)$. Then the geometric growth condition (*) of (2) in Theorem 2 is satisfied. Hence the maximal operator M is *unbounded on* L^1.

(2). We now give an example where the *unboundedness on* L^1 of the maximal operator *depends on the probability* P. Consider the discrete probability space $(\mathbb{N}, \mathcal{F}, P)$ and consider the blocks

$$B_1 = \{1\}, B_2 = \{2, 3\}, B_3 = \{4, 5, 6\}, \ldots.$$

In other words we split $\mathbb{N}$ into disjoint consecutive blocks B_n with card $(B_n) = n$. We have

$$B_n = \{a_n, a_n + 1, a_n + 2, \ldots, b_n\}$$

where

$$a_n = \frac{1}{2}(n-1)n + 1, \quad b_n = \frac{1}{2}n(n+1).$$

For $1 \leq k < n$ let $B_{n,k} = \{b_n - (n-k) + 1, \ldots, b_n\}$ be the subblock of B_n consisting of the $n-k$ largest numbers in B_n.

For $k \geq 1$ let $\mathcal{F}_k$ be the σ-algebra generated by all singletons $\{i\}$ with

$$i \leq b_k$$

and by those with

$$a_n \leq i \leq a_n + (k-1)(= b_n - (n-k)) \quad \text{for some} \quad n > k$$

and by the subblocks $B_{n,k}$ with $n > k$.

If P is given by a sequence $(p_1, p_2, \ldots, p_n, \ldots)$ which decreases fast enough (for instance $p_n = 1/2^n$), we have

$$\frac{P(B_{n,k-1})}{P(B_{n,k})} \geq 2 \quad \text{for} \quad 2 \leq k < n.$$

In this case the condition $P(A_{k-1}(j))/P(A_k(j)) \geq 2$ is met for $j = b_n$ and $2 \leq k < n$. Hence Theorem 2 applies and it follows that for such a probability measure P the maximal operator is *unbounded.*

On the other hand if the probability measure P has the property that $p_{b_n} \geq (2/3)P(B_n)$, then

$$\frac{P(B_{n,k})}{P(B_{n,l})} \leq \frac{P(B_n)}{(\frac{2}{3})P(B_n)} = \frac{3}{2}$$

for all $1 \leq k < l < n$. It follows that condition $(*)$ of (2) in Theorem 2 fails. Hence in this case M is *bounded.*

(3). For notational convenience we work here with $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ instead of $\mathbb{N} = \{1, 2, 3, \ldots\}$. Consider the discrete probability space $(\mathbb{N}_0, \mathcal{F}, P)$ (no restriction on P). Each $i \in \mathbb{N}_0$ admits a (unique) dyadic representation

$$i = \sum_{n=0}^{\infty} X_n(i) 2^n$$

where $X_n(i) \in \{0, 1\}$ and all but finitely many of the $X_n(i)$'s are equal to 0. For each $n \geq 0$ let

$$\mathcal{F}_n = \sigma(X_n, X_{n+1}, X_{n+2}, \ldots)$$

Clearly $\mathcal{F}_0 = \sigma(X_0, X_1, X_2, \ldots)$ contains every $\{i\}, i \in \mathbb{N}_0$ and thus $\mathcal{F}_0 = \mathcal{F}$. Consider the decreasing filtration $(\mathcal{F}_n)_{n\geq 1}$. The *atoms* of $\mathcal{F}_n$ are given by $\{X_n = x_n, X_{n+1} = x_{n+1}, X_{n+2} = x_{n+2}, \ldots\}$, where $x_\nu \in \{0, 1\}$ and all but finitely many x_ν's are equal to 0, i.e. they are the sets of the form

$$\{m \cdot 2^n, m \cdot 2^n + 1, m \cdot 2^n + 2, \ldots, (m+1) \cdot 2^n - 1\}, m \geq 1.$$

Thus $(\mathcal{F}_n)_{n\geq 1}$, is the dyadic filtration on $\mathbb{N}_0$.

As usual for $i \in \mathbb{N}_0$ and $k \geq 0$ let $A_k(i)$ denote the atom of $\mathcal{F}_k$ containing i. We show that statement (2) of Theorem 1 holds and thus the maximal operator M is *unbounded* on L^1. Let $l > 1$ and consider the atom

$$B_l = \{0, 1, 2, \ldots, 2^l - 1\} = \{X_l = X_{l+1} = X_{l+2} = \cdots = 0\} \quad \text{of} \quad \mathcal{F}_l.$$

We have

$$B_l = (B_l \cap \{X_{l-1} = 0\}) \cup (B_l \cap \{X_{l-1} = 1\})(\text{disjoint union}).$$

Thus there exists $x_{l-1} \in \{0, 1\}$ such that

$$P(B_l \cap \{X_{l-1} = x_{l-1}\}) \leq \frac{1}{2} P(B_l).$$

Set $B_{l-1} = B_l \cap \{X_{l-1} = x_{l-1}\}$ and continue the process. In this manner we find $x_{l-1}, x_{l-2}, \ldots, x_1, x_0$ such that

$$P(B_\nu) \leq \frac{1}{2} P(B_{\nu+1}),\ \nu = 0, 1, 2, \ldots, l-1$$

where $B_\nu = B_{\nu+1} \cap \{X_\nu = x_\nu\}$ is an atom of $\mathcal{F}_\nu$. Set

$$i = \sum_{\nu=0}^{l-1} x_\nu 2^\nu.$$

Then $X_0(i) = x_0, X_1(i) = x_1, \ldots, X_{l-1}(i) = x_{l-1}$ and $X_l(i) = X_{l+1}(i) = X_{l+2}(i) = \cdots = 0$. Thus $B_0 = \{i\} = A_0(i), B_1 = A_1(i), \ldots, B_{l-1} = A_{l-1}(i), B_l = A_l(i)$ and by construction the chain $A_1(i), \ldots, A_l(i)$ satisfies the geometric growth condition $(*)$ in statement (2) of Theorem 1.

(4). For *nonatomic* probability spaces $(\Omega, \mathcal{F}, P)$ one can also give examples of decreasing (respectively) increasing filtrations $(\mathcal{F}_n)_{n\in\mathbb{N}}$ on Ω such that for each $n \in \mathbb{N}$, $(\Omega, \mathcal{F}_n, P \mid \mathcal{F}_n)$ is nonatomic, but for which the corresponding maximal operator M is *bounded* on L^1.

5. Decreasing nonatomic filtrations

We now study the maximal operator for reversed martingales in nonatomic probability spaces. Our aim is to prove the following.

THEOREM 3. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_n)_{n\geq 1}$ a decreasing filtration on Ω. Assume that*

(i) *P restricted to $\mathcal{F}_n$ is nonatomic for each $n \geq 1$,*
(ii) *the tail σ-algebra $\mathcal{F}_\infty = \cap_{n\geq 1}\mathcal{F}_n$ is trivial.*

Then the maximal operator

$$Mf = \sup_{n \geq 1} |E(f \mid \mathcal{F}_n)|$$

is unbounded on L^1: There exists an integrable f for which Mf is not integrable.

Proof. It is enough to show that, for any integer $m \geq 1$, there exists $g_m \in L^1_+$ and an integer $N(m)$ for which

$$s_m = \sup_{1 \leq j \leq N(m)} |E(g_m \mid \mathcal{F}_j)|$$

satisfies

$$E(s_m) \geq \frac{m-1}{2} E(g_m). \tag{5.1}$$

Since $Mg_m \geq s_m$, this implies that there is a sequence of nonnegative numbers a_m such that $f = \sum a_m g_m$ is integrable, but Mf is not integrable.

To construct g_m we use an inductive argument. The reversed martingale theorem implies that, for any integrable h, the sequence $(E(h \mid \mathcal{F}_n))_{n \geq 1}$ converges a.e. and in L^1-norm to the constant function $\int h\,dP$. (Note that $\mathcal{F}_\infty$ is trivial.)

Let $k_1 = 1$ and let A_1 be an $\mathcal{F}_1$–measurable set with $P(A_1) = 1/2$. Given $\epsilon > 0$, there exists $k_2 > k_1$, with

$$\int |P(A_1) - E(1_{A_1} \mid \mathcal{F}_{k_2})|\,dP < \epsilon. \tag{5.2}$$

If $C_1 = A_1^c$, we also have

$$\int |P(C_1) - E(1_{C_1} \mid \mathcal{F}_{k_2})|\,dP < \epsilon. \tag{5.3}$$

If k_ν and $A_\nu \in \mathcal{F}_{k_\nu}$ have been determined for some $\nu \geq 1$, we can find $k_{\nu+1}$ such that

$$\int |P(E_{j,\nu}) - E(1_{E_{j,\nu}} \mid \mathcal{F}_{k_{\nu+1}})|\,dP < \epsilon \tag{5.4}$$

for all sets $E_{j,\nu}(1 \leq j \leq \nu)$ of the form $F_j \cap \cdots \cap F_\nu$, where each F_i is either equal to A_i or to $C_i = A_i^c$. Let $A_{\nu+1}$ be any $\mathcal{F}_{k_{\nu+1}}$-measurable set with $P(A_{\nu+1}) = 1/2$. The construction continues until we arrive at A_m and k_m. We set $N(m) = k_m$ and

$$g_m = 1_{A_1 \cap \cdots \cap A_m}.$$

We shall show that (5.1) is satisfied if $\epsilon = \epsilon(m) > 0$ is small enough. Consider the sets $B_1 \subset B_2 \subset \cdots \subset B_m$ where

$$\begin{aligned} B_1 &= A_1 \cap A_2 \cap \cdots \cap A_m \\ B_2 &= A_2 \cap \cdots \cap A_m \\ B_j &= A_j \cap \cdots \cap A_m \\ B_m &= A_m \end{aligned}$$

and the difference sets

$$\begin{aligned} D_1 &= B_1 = A_1 \cap \cdots \cap A_m \\ D_2 &= B_2 \backslash B_1 = C_1 \cap A_2 \cap \cdots \cap A_m \\ D_j &= B_j \backslash B_{j-1} = C_{j-1} \cap A_j \cap \cdots \cap A_m \end{aligned}$$

for $2 \le j \le m$. We have $g_m = 1_{D_1}$, hence s_m takes the value 1 on D_1. For $2 \le \nu \le m$ we have

$$s_m \ge E(g_m \mid \mathcal{F}_{k_\nu}) = 1_{A_\nu \cap \cdots \cap A_m} E(1_{A_1 \cap \cdots \cap A_{\nu-1}} \mid \mathcal{F}_{k_\nu}).$$

Thus

$$s_m \ge E(1_{A_1 \cap \cdots \cap A_{\nu-1}} \mid \mathcal{F}_{k_\nu}) \quad \text{on} \quad A_\nu \cap \cdots \cap A_m, \quad \text{hence on} \quad D_\nu; \tag{5.5}$$

since the sets $D_1, \ldots, D_m$ are disjoint, all we have to show is that (5.4) supplies enough "approximate independence" to give a convenient estimate for $P(D_\nu)$ and the integral of $E(1_{A_1 \cap \cdots \cap A_{\nu-1}} \mid \mathcal{F}_{k_\nu})$ on D_ν.

We first show inductively on ν that

$$|P(E_{j,\nu}) - 2^{-(\nu-j+1)}| \le \nu\epsilon \tag{5.6}$$

for any set $E_{j,\nu} (1 \le j \le \nu)$ of the form described in (5.4). The assertion is trivial for $\nu = 1$ since $P(A_1) = P(C_1) = 1/2$. Suppose the assertion has been verified for ν and consider $E_{j,\nu+1}$ with $1 \le j \le \nu + 1$. If $j = \nu + 1$, there is nothing to prove since $P(A_{\nu+1}) = P(C_{\nu+1}) = 1/2$. If $j \le \nu$ and $E_{j,\nu+1}$ is of the form $E_{j,\nu} \cap A_{\nu+1}$, by (5.4) we have

$$\begin{aligned} |P(E_{j,\nu+1}) - 2^{-(\nu+1-j+1)}| &= \left| \int_{A_{\nu+1}} \left[E(1_{E_{j,\nu}} \mid \mathcal{F}_{k_{\nu+1}}) - 2^{-(\nu-j+1)} \right] dP \right| \\ &\le \int_{A_{\nu+1}} \left| E(1_{E_{j,\nu}} \mid \mathcal{F}_{k_{\nu+1}}) - P(E_{j,\nu}) \right| dP \\ &\quad + \int_{A_{\nu+1}} \left| P(E_{j,\nu}) - 2^{-(\nu-j+1)} \right| dP \\ &\le \epsilon + \nu\epsilon = (\nu+1)\epsilon. \end{aligned}$$

The argument for the case $E_{j,\nu+1} = E_{j,\nu} \cap C_{\nu+1}$ is essentially the same.

Finally we can estimate the integral of s_m. We have

$$\int_{D_1} s_m \, dP = P(D_1) = E(g_m).$$

By (5.5) we know that $s_m \ge E(1_{A_1} \mid \mathcal{F}_{k_2})$ on D_2 and by (5.6), $|P(D_2) - 2^{-m}| \le m\epsilon$, $|P(D_1) - 2^{-m}| \le m\epsilon$, so that

$$\begin{aligned} \int_{D_2} s_m \, dP &\ge \int_{D_2} E(1_{A_1} \mid \mathcal{F}_{k_2}) \, dP \ge \int_{D_2} P(A_1) \, dP - \epsilon \\ &\ge \frac{1}{2} P(D_1) - (m+1)\epsilon = \frac{1}{2} E(g_m) - (m+1)\epsilon. \end{aligned}$$

For $3 \leq i \leq m$, note that by (5.6), (5.5) and (5.4),

$$\left|P(D_i) - 2^{-(m-i+2)}\right| \leq m\epsilon$$

and

$$\begin{aligned}
\int_{D_i} s_m \, dP &\geq \int_{D_i} E(1_{A_1 \cap \cdots \cap A_{i-1}} \mid \mathcal{F}_{k_i}) \, dP \\
&\geq \int_{D_i} P(A_1 \cap \cdots \cap A_{i-1}) \, dP - \epsilon \\
&\geq \int_{D_1} 2^{-(i-1)} \, dP - i\epsilon \geq 2^{-(i-1)} \left[2^{-(m-i+2)} - m\epsilon\right] - i\epsilon \\
&\geq 2^{-m-1} - 2m\epsilon \\
&\geq \frac{1}{2} P(D_1) - 3m\epsilon = \frac{1}{2} E(g_m) - 3m\epsilon
\end{aligned}$$

Summing all these estimates, we arrive at

$$\int s_m \, dP \geq E(g_m) \left[1 + \frac{m-1}{2}\right] - 3m^2\epsilon.$$

This yields the desired estimate (5.1) for small enough $\epsilon > 0$.

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Alexandra Bellow, Department of Mathematics, Northwestern University, Evanston, IL 60208

Ulrich Krengel, Institut für Mathematische Stochastik, Lotzestrasse 13, D-37083 Göttingen, Germany
krengel@math.uni-goettingen.de