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SPIN(4) ACTIONS ON 8-DIMENSIONAL MANIFOLDS (I)

PHILIPPE MAZAUD

ABSTRACT. We study smooth Spin(4) actions on closed, orientable 8-dimensional manifolds, where Spin(4) is isomorphic to the group SU(2)×SU(2). We examine the isotropy structures that can arise, and give an equivariant classification in the case where the set of exceptional orbits, stabilized by finite-cyclic goups, is empty.

0. Introduction

The object of this paper is to begin the study of smooth, effective Spin(4) actions on closed, orientable, 8-dimensional manifolds. Spin(4) is defined to be the (universal) double-cover of SO(4), and is isomorphic, as a Lie group, to $S^3 \times S^3$, where S^3 denotes the unit-quaternions.

One way topologists have constructed large, interesting families of manifolds is by considering spaces that fiber over a surface, allowing for the existence of exceptional or singular fibers, in some controlled way. The earliest and best known instance of this approach is perhaps Seifert's work on three-dimensional spaces fibered by circles [Sei]. Recently, Richard Scott [Sco] has shown that any closed, 3-connected, 8-dimensional manifold, with $H_4 \simeq \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$, smooth except possibly at a finite number of points, can be realized as $(S^3 \times S^3 \times P)/\lambda$, where P is an r-gon, and λ prescribes how the fibers over the boundary of P are to be collapsed (a complete set of algebraic invariants for 3-connected 8-manifolds manifolds was known from C.T.C. Wall's classification [Wal]).

As it turns-out, "relatively few" of our Spin(4) manifolds are 3-connected. In fact, even if one were to restrict to the simply-connected case, $\pi_2(M)$ is generally non-trivial, and one easily constructs examples where $\operatorname{rk}(H_2 \simeq \pi_2)$ is as large as one wishes. But asking for the existence of a Spin(4) action certainly imposes symmetry requirements on manifolds, yielding classes of spaces that enjoy a nice "stratification": essentially, they are parametrized by orbit data over a surface. We shall assume throughout that the *principal orbits are free*. Our viewpoint is decidedly concrete, the approach geometric with very explicit constructions, in the spirit of some of Orlik and Raymond's work concerning toral actions on 4-manifolds ([OR1] and [OR2]).

Equivariantly, one can distinguish between three general situations (cf. Section 1.2). (a) The action is principal, so all orbits are free over a *closed* surface. Any such principal Spin(4) bundle is trivial, and diffeomorphic to Spin(4) $\times M^2$; so that the interesting

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situations are the following two. (b) "Seifert-like manifolds" over closed surfaces. All orbits 13are 6-dimensional, but a finite number are stabilized by finite-cyclic subgroups of Spin(4): these non-principal orbits of maximal dimension are called "exceptional orbits" (and denoted E). The spaces that arise in this manner can be viewed as bundles with singularities. Finally (c) comprises all the cases where singular orbits are present, that is, where some points in M have stabilizers of positive dimension. This is the situation to which we devote much of our attention. The quotient space M^* is a surface *with boundary*. The singular orbits occur over this boundary; the interior of M^* consists entirely of free orbits, and possibly a finite number of E orbits.

The main result in this paper is an equivariant classification of Spin(4) actions in the case $E = \emptyset$. It may be stated as follows.

THEOREM A. *Suppose Spin*(4) *acts on M according to the conditions given above. Then, up to equivalence, the action is completely characterized by*:

(1) *The homeomorphism type of the orbit space* M^*.

(2) *The isotropy weights.*

(3) *An element* $o \in \mathbb{Z}_2{}^b$, *where b is the number of boundary components of* M^*; *given some fixed ordering of the components of the boundary, to the* j^{th} *boundary circle is associated an element in* $\mathbb{Z}_2$, *and this is the* j^{th} *coordinate of* o.

Put differently, two actions on M_0 and M_1 are essentially the same, if there is an isomorphism of the weighted orbit spaces M_0^* and M_1^*. That is, a homeomorphism h that takes any element in M_0^* with a given isotropy to one with the same isotropy in M_1^*; furthermore, assuming equivalent orderings of the boundary components so that the j^{th} component in M_0^* has the same isotropy as the j^{th} in M_1^*, we must have $o(M_0) = o(M_1)$.

The invariant o is directly interpreted as an obstruction to the "normalization", or "uniformization" of a global section to the action (we show a section always exists). An interesting question that arises naturally pertains to whether this invariant encodes topological information, or whether it is strictly equivariant, i.e., distinguishes between actions on the same space. We find examples where the former is true: Spin(4) acts on two topologically distinct manifolds, with homeomorphic orbit spaces, and the same orbit structure (so the only difference is in the value of o). On the other hand, we give conditions under which o is purely equivariant. We do not have a completely general answer to this question however.

The results of this paper are fairly specific. For instance, as far as we know, there isn't a precedent for the o invariant in the literature–though one might expect some uniformization invariant of this kind to arise in other settings involving codimension-2 actions of non-abelian, compact Lie groups. At the same time, Spin(4)-manifolds make up an extremely rich class of spaces, and the equivariant data provides a great deal of information on how these are built up. Thus a further investigation of the topology of these spaces is of interest, and a first step in that direction will be taken in a follow-up paper.

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1. Weighted orbit spaces

As we indicated in the introductory remarks, Spin(4) is the double cover of SO(4) $\simeq S^3 \rtimes \mathrm{SO}(3)$. Spin(4) $\simeq S^3 \times S^3$ (S^3 is the unit three-sphere), thus a compact, 6-dimensional Lie group. In turn S^3 is Spin(3) (the double-cover of SO(3)). It is easy to see that $S^3 \simeq \mathrm{SU}(2)$. We almost always write an element in Spin(4) as $g \times h$ (or $(g; h)$), where $g, h \in S^3$. We write g as (g_1, g_2), where $g_j \in \mathbb{C}$, and $\|g_1\|^2 + \|g_2\|^2 = 1$. With this notation, S^3 multiplication takes the form

$$gg' = (g_1, g_2)(g_1', g_2') = (g_1g_1' - g_2\overline{g_2'}, g_1g_2' + g_2\overline{g_1'}).$$

Of course, it is not abelian; the center of Spin(4) is $\mathbb{Z}_2 \times \mathbb{Z}_2$. Spin(4) is a semi-simple Lie group, as the product of the simple Lie group S^3 (the identity component of any proper normal subgroup in S^3 is trivial). The automorphism group of Spin(4) is $(\mathrm{Inn}(S^3) \times \mathrm{Inn}(S^3)) \rtimes \mathbb{Z}_2 \simeq (\mathrm{SO}(3) \times \mathrm{SO}(3)) \rtimes \mathbb{Z}_2$. Inn denotes the group of inner automorphisms; the $\mathbb{Z}_2$ term corresponds to the outer automorphism that permutes the factors in $S^3 \times S^3$.

1.1. *Closed subgroups.* We wish to determine the possible isotropy types, that is, conjugacy classes of Spin(4) subgroups occuring as stabilizers. These must always be closed subgroups. There are many distinct conjugacy classes of Spin(4) closed subgroups, but we shall see that relatively few do in fact arise as isotropy. In the following subsections, "subgroup" will always be taken to mean "closed subgroup", unless stated otherwise.

1.1.1. *Closed subgroups of S^3.* With the exception of the odd-order, finite cyclic groups, all the proper, non-trivial subgroups of S^3 arise as pull-backs of the closed subgroups of SO(3), which are well known. In other words, they are $\mathbb{Z}_2$ extensions of the SO(3) subgroups that they double-cover (see [Wol], for instance). To summarize, we have:

(i) S^1, covering SO(2). We will often refer to the "distinguished" circle subgroup $\mathcal{C} \doteqdot \{(e^{i\theta}, 0) \mid 0 \leq \theta < 2\pi\}$.

(ii) O^*, covering O(2). It normalizes S^1 in S^3. Like O(2), it is topologically the disjoint union of two circles: $O^* = N(\mathcal{C}) = \mathcal{C} \coprod \hat{\mathcal{C}}$, where $\hat{\mathcal{C}} = (0, 1)\mathcal{C} = \{(0, e^{i\theta})\}$. It is generated by $\mathcal{C}$, and $\mathbb{Z}_4 = \langle(0, 1)\rangle$. Since $\mathcal{C} \cap \mathbb{Z}_4 = \langle(-1, 0)\rangle = \mathbb{Z}_2$, and $\mathcal{C} \cdot \mathbb{Z}_4 = \mathbb{Z}_4 \cdot \mathcal{C} = \mathrm{N}(\mathcal{C})$, we can write $N(\mathcal{C}) = \mathcal{C} \rtimes_{\mathbb{Z}_2} \mathbb{Z}_4$. Of course, a generic O^* is conjugate to this. Note that it is *not* isomorphic to O(2): any element that is not in the component of the identity has order 4 (rather than 2 as in O(2)). Note also that $N(O^*) = O^*$.

(iii) Finite groups: cyclic, D^*_{4n}, T^*_{12}, O^*_{24}, I^*_{60}, the binary dihedral, binary tetrahedral, binary octahedral and binary icosahedral subgroups respectively. The only proper, non-trivial normal subgroup of S^3 is its center $\mathbb{Z}_2 = \pm 1$.

1.1.2. *Closed subgroups of Spin*(4). (a) *Finite subgroups.* We shall not list all such subgroups, for, as Lemma 1 will show, we need only be concerned with those that embed as finite subgroups of O(2).

(b) *One-dimensional subgroups.* The circle subgroups sit in the maximal tori T^2. We will refer to the group $\mathcal{C} \times \mathcal{C} = \{(e^{i\alpha}, 0) \times (e^{i\beta}, 0) \mid 0 \le \alpha, \beta < 2\pi\}$ as the "distinguished", or "standard torus" ($\mathcal{C}$ as in 1.1.1 (i)). $S(m, n)$ will denote subgroups conjugate to the circle in $\mathcal{C} \times \mathcal{C}$ defined by $m\alpha - n\beta = 0$, $(m, n) = 1$. In particular, the distinguished $S(1, 0) = \{1\} \times \mathcal{C} = \{(1, 0) \times (e^{i\beta}, 0) | 0 \le \beta < 2\pi\}$ (and similarly for $S(0, 1)$). $S(\pm m, \pm n)$ is conjugate to $S(m, n)$ (conjugating by $(0, 1)$ in one or both S^3 factors), and the outer automorphism of Spin(4) sends $S(m, n)$ to $S(n, m)$. We note that $S(m', n')$ is not the image of $S(m, n)$ under any automorphism, unless m, n, m', n' are related in one of these ways. We will see (Lemma 2) that no other 1-dimensional group can occur as isotropy. The other subgoups are subgroups of $O^* \times F$ (F finite), and the 1-dimensional subgroups of $O^* \times O^*$.

(c) *Two-dimensional subgroups.* We have the tori and their normalizers $N(T^2)$, and the normalizers $N(S(m, n))$ when $m, n \ne 0$. The latter are the conjugates of $\mathcal{C} \times \mathcal{C} \coprod \hat{\mathcal{C}} \times \hat{\mathcal{C}}$: clearly (up to conjugacy), $N(S(m, n)) \subseteq N(\mathcal{C}) \times N(\mathcal{C}) = O^* \times O^*$, and one checks directly that neither $\mathcal{C} \times \hat{\mathcal{C}}$ nor $\hat{\mathcal{C}} \times \mathcal{C}$ normalizes $S(m, n)$. Setting $\Delta\mathbb{Z}_4 = \langle (0, 1) \times (0, 1) \rangle$ (a diagonal $\mathbb{Z}_4$), we have $N(S(m, n)) = \mathcal{C} \times \mathcal{C} \cdot \Delta\mathbb{Z}_4 = \Delta\mathbb{Z}_4 \cdot \mathcal{C} \times \mathcal{C}$, and $\mathcal{C} \times \mathcal{C} \cap \Delta\mathbb{Z}_4 = \Delta\mathbb{Z}_2$, hence $N(S(m, n)) \simeq \mathcal{C} \times \mathcal{C} \rtimes_{\Delta\mathbb{Z}_2} \Delta\mathbb{Z}_4$.

If m or $n = 0$, we have a factor circle whose normalizer, obviously, is isomorphic to $O^* \times S^3$. We note that in all cases, $N(S(m, n))$ has two path-components: this fact will play a crucial role in later sections (see 2.2).

$N(T^2)$ are the conjugates of $N(\mathcal{C} \times \mathcal{C}) \simeq O^* \times O^*$. We note that these are the "largest" 2-dimensional subgroups of Spin(4) (with 4 path-components, each one a *topological* torus), and their own normalizers. Indeed, suppose $H \subset \mathrm{Spin}(4)$ is 2-dimensional. Let H_j be the image of H in S^3 under the standard j^{th} projection $(j = 1, 2)$ of Spin(4) onto S^3. Clearly, $H \subseteq H_1 \times H_2$, and since there are no 2-dimensional subgroups of S^3, $H \subseteq O^* \times O^*$.

(d) *Subgoups of dimension* ≥ 3. First, we have either the "factor" S^3's: $\{1\} \times S^3$ or $S^3 \times \{1\}$, or the "diagonal" S^3's: ΔS^3. The latter have the form $\{g_0 g g_0^{-1} \times g \mid g_0 \text{ fixed}, g_0, g \in S^3\}$. If $g_0 = \pm 1$, $\Delta S^3 = \{g \times g\}$, and we will often refer to this subgroup as the "strictly diagonal" S^3. Its normalizer is isomorphic to $S^3 \times \mathbb{Z}_2$, and consists of the elements $\{g \times g\} \cup \{g' \times -g'\}$. So in particular, there is no automorphism of Spin(4) that can can take a factor S^3 (which is normal) to a diagonal one.

Any S^3 subgroup must be either a factor or diagonal. To see this explicitly, suppose i is the inclusion of an S^3 subgroup into $S^3 \times S^3$. Composing i with projection on one factor gets a homomorphism of S^3 into S^3, whose kernel is either trivial, or S^3 itself (possibilities corresponding to the diagonal or factor cases), or else $\mathbb{Z}_2$; but $S^3/\mathbb{Z}_2$ is not a subgroup of S^3, excluding that case.

We claim, furthermore, that any subgroup H of dimension ≥ 3 must be isomorphic to $S^3 \times G$ ($G \subseteq S^3$). First, it is clear that there must be a surjective homomorphism $H \longrightarrow S^3$. For $H \subseteq H_1 \times H_2$, where H_j is the image of H in S^3 under the projection to the j^{th} S^3 factor, and least one of the H_j must be S^3 (there are no 2-dimensional subgroups of S^3). Now consider the sequence ($K \subseteq \operatorname{Spin}(4)$)

$$\{1\} \longrightarrow K \longrightarrow H \longrightarrow S^3 \longrightarrow \{1\}.$$

We need to examine the possibilities for K in dimensions 0, 1, or 2. In decreasing order we have the following.

Dimension 2. This case must be excluded, since no 2-dimensional subgoup can be normal in a subgroup of dimension > 2.

Dimension 1. If K is a circle, then it must be $S(0, 1)$ or $S(1, 0)$ (i.e., a factor), since no $S(m, n)$ with m or $n \neq 0$ can be normal in a subgoup of dimension > 2. Similarly O^* would have to be a factor. But then $H \simeq S^3 \times K$.

Dimension 0. Since $S^3 \simeq H/K$ is simply-connected, H must be, as a space, a product $K \times S^3$. We claim that as a group, it must also be a product. First, observe that the connected component of the identity must be S^3, so we have a copy of S^3 that is a normal subgroup of H. Thus we consider

$$\{1\} \longrightarrow S^3 \longrightarrow H \longrightarrow F \longrightarrow \{1\}$$

where $F \simeq H/S^3$ is some finite group ($\pi_0(H)$), not necessarily isomorphic to K (the finiteness actually is not needed in what follows).

In the diagonal case, since ΔS^3 is normal in H, the only possibility is $S^3 \times \mathbb{Z}_2$ (the normalizer itself). For a factor S^3, say $S^3 \times \{1\}$ for definiteness, suppose we have

$$\{1\} \longrightarrow S^3 \times \{1\} \longrightarrow H \longrightarrow F \longrightarrow \{1\}$$

so $F \simeq H/(S^3 \times \{1\})$. Consider $H \overset{i_1 \times i_2}{\hookrightarrow} S^3 \times S^3$, and write $\operatorname{Im}(i_j) = H_j$, ($j = 1, 2$). Then $H \subseteq H_1 \times H_2$. Now $\ker(i_2) = H_1 \cap H = S^3 \times \{1\} \lhd H$. Thus we have

$$\{1\} \longrightarrow H_1 \cap H \longrightarrow H \longrightarrow H_2 \longrightarrow \{1\}$$

so that $H_2 \simeq H/_{S^3 \times \{1\}} \simeq F$. Thus, F is some subgroup of S^3, and $H \subseteq S^3 \times F$, so that in fact, $H = S^3 \times F$.

1.2. *Admissible weighted orbit spaces.* What follows uses some terminology and fundamental results from the theory of compact transformation groups, concerning principal isotropy, the slice representation, and equivariant tubular neighborhoods. Standard references for this are [Bre] (in particular Chap. II, §4, 5, and Chap. VI, §2), or [Bor2] (Chap. VIII, §3). We fix some notation: G_x denotes the stabilizer of a point $x \in M$, $\mathcal{S}_x$ a slice to the action at x. M^* is the orbit space, and generally, for any $K \subset M$ or $x \in M$, K^* and x^* shall denote their image under the natural projection map $\pi\colon M \to M^*$. We will often write an element in a cartesien product $A \times B$ as $a \times b$, when it makes a formula clearer or easier to read. By a *weighted orbit space*, we mean an orbit space under a Spin(4) action, together with a specification of the isotropy type for each orbit; this will typically be given as a labeled diagram.

We now proceed to show that many of the Spin(4) subgroups do not actually arise as stabilizers: we will see shortly that this is forced upon us by the condition that M be orientable, and by our assumption that the principal isotropy is trivial. Smoothness and the slice theorem give an O(k) ($k = \dim(\mathcal{S}_x)$) representation of the G_x action on $\mathcal{S}_x$. Since the principal orbits are free, the representation must be faithful; so the subgroup must be embeddable in $O(k)$. Furthermore, it also follows from the freeness of principal orbits that *the slice action is effective.*

1.2.1. *Stabilizers for* Spin(4) *actions.*

CLAIM. *If* Spin(4) *acts smoothly on a closed, orientable,* 8*-manifold with free principal orbits, the only possible isotropy at singular and exceptional orbits are*: (i) Spin(4), (ii) S^3, (iii) S^1, (iv) T^2, (v) $S^1 \times S^3$, (vi) *finite cyclic subgroups.*

The lemmas that follow aim at eliminating any other case; our main tool throughout is the slice representation, which we examine in each dimension (from 0 to 6). Then, Theorem 1 in Section 1.2.2 will make explicit the orbit structures that do arise.

LEMMA 1. *Exceptional orbits are isolated, and $\mathbb{Z}_n$-stabilized.*

Proof. Suppose $G_x = F$, finite, then the orbit through x is 6-dimensional, so the transverse slice $\mathcal{S}_x \approx D^2$. That is, F must act effectively as an O(2) subgroup. Now, those elements in O(2) not contained in SO(2) all have order 2, and are non-central. The only order-two elements in Spin(4) are in the center. This leaves either the cyclic groups, or $D_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. The latter however, as well as $\mathbb{Z}_2$ acting by reflection, would yield a $\mathbb{Z}_2$ stabilized edge in M^* (two of these in the D_4 case): it is seen further below that this would contradict our orientability assumption (cf. the last part in the proof of Lemma 4, after Figure 1). So the only remaining possibilities are the cyclic subgroups, and the lemma follows immediately. □

A similar argument establishes:

LEMMA 2. *The only one-dimensional isotropy is S^1.*

Proof. For H one-dimensional, the slice at x must be a three-ball; so H must embed in O(3). Now, the closed, one-dimensional subgroups of O(3) are SO(2), O(2)

($\subset$ SO(3)), and their $\mathbb{Z}_2$ extensions. Except for SO(2), these contain an entire circle of (non-central) order-two elements. □

If the isotropy is 2-dimensional, then the orbit is 4-dimensional and so is the slice, and we have a representation in O(4).

LEMMA 3. *T^2 is the only possible 2-dimensional isotropy type.*

Proof. The slice is a 4-disc, and we consider the linear representation of the action on the S^3 boundary. First take $O^* \times O^*$. Recall this is the (4-component) normalizer of T^2; in fact $O^* \times O^*/T^2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Also, note that up to isomorphism, we have two $\mathbb{Z}_2$ extensions of T^2 (2-component subgroups of $O^* \times O^*$), namely $S^1 \times O^*$ ($\simeq C \times C \coprod \hat{C} \times \hat{C}$), and $N(S(m,n))$ ($\simeq C \times C \coprod \hat{C} \times C$), both sitting in $O^* \times O^*$. An effective T^2 action on S^3 gives a quotient I, with endpoints stabilized by $S(m,n)$ and $S(m',n')$ respectively (cf. 1.1.2 for definitions, and see the proof of Theorem 1, for further details on T^2). Composing

$$S^3 \xrightarrow{/T^2} I \xrightarrow{/\mathbb{Z}\times\mathbb{Z}_2} S^3/O^* \times O^*,$$

we see that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on I is at least $\mathbb{Z}_2$ ineffective. Therefore, the $O^* \times O^*$ action on S^3 couldn't possibly be effective if neither $S^1 \times O^*$ nor $N(S(m,n))$ can act *freely* on the $S^1 \times S^1$ fibers in S^3 that sit above the interior points in I. But this in turn is impossible, since the T^2 subgroup ($\simeq C \times C$) acts transitively there. Next, considering the groups $N(S(m,n))$ and $S^1 \times O^*$, we see that if they could be made to act effectively on S^3, that would require $\mathbb{Z}_2$ to act effectively on $I = S^3/T^2$, by reflection across an interior point. This would yield a $\mathbb{Z}_2$-stabilized edge in the quotient space (upon coning), forcing the total space to be non-orientable (again, we refer the reader to the proof of Lemma 4). Since there are no other 2-dimensional subgroups beside the tori, the lemma follows. □

We have seen that Spin(4) subgroups of dimension ≥ 3 must contain a copy of S^3, and that these have the form $S^3 \times H$ ($H \subseteq S^3$) (1.1.2.d).

LEMMA 4. *$S^3 \times F$ does not occur as isotropy.*

Proof. The slice here is a 5-ball, and $S^3 \times F$ must act effectively as a subgroup of SO(5); this can be viewed as coning the action on $S^4 = \mathrm{Bd}(B^5)$. By a theorem of Richardson [Ric1], up to equivalence there is only one (effective) S^3 action on S^4, the obvious suspension of S^3 operating by (left, say) multiplication. We then have

$$\begin{tikzcd} S^4 \arrow[r, "/F"] \arrow[d, "/S^3"'] & S^4/F \arrow[d, "/S^3"'] \\ I \arrow[r, "/F"] & S^4/(S^3 \times F). \end{tikzcd}$$

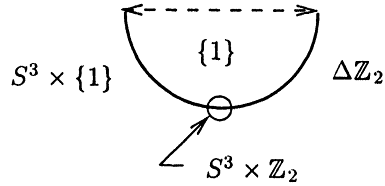

Figure 1a

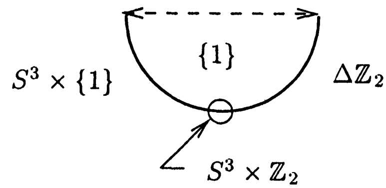

Figure 1b

Consider the bottom row. The only finite group that will act effectively on I is $F = \mathbb{Z}_2$. And F must act effectively in order for the total action to be effective. Indeed, suppose it did not. Then there exists an element $\xi \neq 1 \in F$, such that ξ does not move anything in $I = S^4/S^3$: therefore, ξ must be acting non-trivially in each S^3 fiber over I. Thus, we would have an effective $S^3 \times \langle \xi \rangle$ action induced on S^3; in fact, it would have to be free (otherwise we would not get free principal orbits upon coning). But the only positive-dimensional Lie groups that act freely on S^3 are known to be subgroups of S^3 (see for instance Theorem 8.5 in III of [Bre]).

The only possibility then is $F = \mathbb{Z}_2$, acting of course by reflecting across a fixed interior point of I. Now, $\mathbb{Z}_2$ might simultaneously act linearly on the S^3 fibers by the antipodal map; alternatively, it might act trivially on those fibers. In fact, these two $\mathbb{Z}_2$ actions are the only ones that commute with the S^3 action by multiplication. Indeed, given an involution ξ of S^3, $\xi(1) = w_0$, for some $w_0 \in S^3$. Then for all $z \in S^3, \xi(z) = \xi(z \cdot 1) = z\xi(1) = zw_0$. But $1 = \xi(\xi(1)) = w_0^2$, implying that $w_0 = \pm 1$.

If $\mathbb{Z}_2$ acts by the antipodal map of S^3, then it is not hard to check that we have the weighted orbit space shown on figure 1a. Alternatively, if $\mathbb{Z}_2$ does act trivially on the S^3 fibers, then the weighted orbit space is as shown in 1b.

In Figure 1b, all points correspond to free orbits, except for points along the semicircular arc: this arc is partitioned into two subarcs and a vertex, according to the isotropy. Over the $S^3 \times \mathbb{Z}_2$ vertex, we have a copy of $\mathbb{R}P(3)$, and what maps down to the given quotient is a tubular neighborhood. Its boundary sits over the arc α (dashed line in figure) joining the $\{1\} \times \mathbb{Z}_2$-stabilized edge to the $S^3 \times \{1\}$ one (this is an S^4 bundle over $\mathbb{R}P(3)$) and it is non-orientable. To see this, consider a subarc of α, joining an interior point to the $\mathbb{Z}_2$ stabilized end-point. Observe that above this arc, we have the mapping cylinder of the natural map $S^3 \times S^3 \to S^3 \times \mathbb{R}P(3)$, which is $S^3 \times \overset{\circ}{\mathbb{R}P}(4)$, where $\overset{\circ}{\mathbb{R}P}(4) \approx \mathbb{R}P(4) - \{\text{openball}\}$. Now, crossing this subspace with an interval yields an non-orientable, 8-dimensional submanifold of the tubular neighborhood. This implies that the neighborhood, and therefore M, cannot be orientable.

Finally, we a have a diffeomorphism from the space yielding the weighted quotient in Figure 1a to the one yielding Figure 1b. It is induced by a diffeomorphism of the boundary, itself descending from $h : S^3 \times S^3 \times I \to S^3 \times S^3 \times I$ defined by $(u, v, t) \mapsto (vu, v, t)$ $(u, v \in S^3, t \in I)$. □

LEMMA 5. *$S^3 \times O^*$ cannot occur.*

Proof. The slice is 6-dimensional with S^5 boundary. Again by a theorem of Richardson [Ric2], there is only one effective S^3 action on the 5-sphere (up to equivalence); view S^5 as the join $S^3 * S^1$, using S^3 multiplication on the S^3 term (equivalently, as the double suspension $\Sigma\Sigma S^3$ and multiplying in the obvious way). The weighted quotient is a 2-disk with a boundary of fixed points. We have

$$S^5 \xrightarrow{/S^3} D^2 \xrightarrow{/O^*} I.$$

The O^* action cannot be effective on D^2 (O^* does not embed in O(2)), and so effectiveness will fail unless $S^3 \times O^*$ can be made to act freely on the $S^3 \times S^1$ subspaces of S^5 (sitting over the concentric circles of D^2): but this is not possible. □

1.2.2. *Quotients and orbit structure.*

THEOREM 1. *Suppose* Spin(4) *acts effectively and smoothly on a closed, orientable* 8*-manifold M, with free principal orbits. Then*:

(1) *M^* is an orientable* 2*-surface, possibly with boundary.*

(2) *All singular orbits occur on the boundary of M^** (*which consists only of such*). *Locally, the orbit structure is determined according to the possibilities* (*a*)-(*d*) *described below* (*Figure* 2).

(3) *The interior of M^* consists entirely of free principal orbits, except possibly for a finite number of isolated exceptional orbits.*

Note. In Figure 2(a), $S_1 \neq S_2$, with $S_j = \{1\} \times S^3$, or $S^3 \times \{1\}$, or ΔS^3. In 2(c), we must have $\det \begin{pmatrix} m & n \\ m' & n' \end{pmatrix} = \pm 1$. In Figure 2(d), H equals $S(m, n)$ or an S^3.

Remarks. 1. Fixed points, $S^3 \times S^1$ and T^2-stabilized orbits are isolated (we will be using the above graphic conventions consistently in the sequel); S^3 and S^1-stabilized

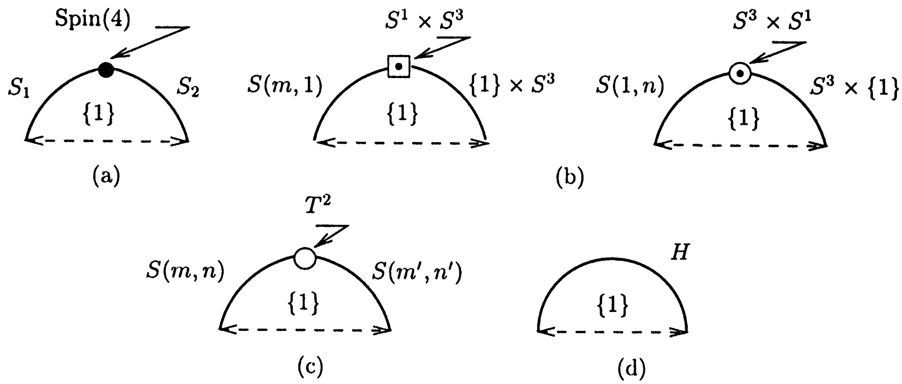

Figure 2

orbits form arcs. This determines a partition of each circle boundary-component of M^* into vertices and edges. Of course, a boundary circle can very well not have any vertices (in which case, it consists entirely of orbits with the same S^1 or S^3 isotropy type).

2. Each one of the weighted quotients pictured in Figures 2(a)–(c) is the image of a disc bundle over the singular orbit sitting above the vertex. In (a), of course, the singular orbit is just a point, and the neighborhood an 8-ball; we can directly see its boundary S^7 (above α) as the join $S^3 * S^3$. In (b), we have S^2 above the vertex, and the tubular neighborhood is the non-trivial, orientable B^6 bundle over S^2 (recall that such orientable k-disk bundles are in 1-1 correspondance with elements in $\pi_1(\mathrm{SO(k)}) \simeq \mathbb{Z}_2$, for $k > 2$). Above the vertex in (c) sits an $S^2 \times S^2$ subspace of M, and we have a D^4 bundle over $S^2 \times S^2$, with structure group T^2. For example, if $(m, n) = (0, 1)$ and $(m', n') = (1, 0)$, this is exactly a product of Hopf bundles, where the Hopf bundle is $\overline{\mathbb{C}P}(2)$ with an open 4-ball removed (equivalently, the closed disc bundle of the canonical complex-line bundle over $\mathbb{C}P(1)$).

Note that some of the above remarks make implicit use of the fact that there is a local section to the action (this is because there is one to the linear slice representation). We will be returning to this with greater care in Section 2, where the question of a global section is taken-up.

3. In Figure (d), it is not too hard to check that if $H \simeq S^3$, the pictured quotient corresponds to $S^3 \times D^5$ (it is easy to see that $\mathrm{Spin}(4)/S^3$ is a 3-sphere). If $H \simeq S(m, n)$, $\mathrm{Spin}(4)/H \approx S^3 \times S^2$. This is immediate if $(m, n) = (0, 1)$ or $(1, 0)$, as is the fact that the tubular neighborhood is diffeomorphic in this case to the product with S^3 of the non-trivial D^3 bundle over S^2; this D^3 bundle is isomorphic to $I \times$ Hopf. The statement regarding the topology of $\mathrm{Spin}(4)/S(m, n)$ is not completely obvious in the general case, and we now supply an argument, as follows.

Let $N = \mathrm{Spin}(4)/S(m,n)$ ((m,n) is a relatively prime pair). It follows from the homotopy sequence of the principal S^1 bundle: $S^1 \hookrightarrow S^3 \times S^3 \underset{/S(m,n)}{\longrightarrow} N$ that N is a simply connected 5-manifold, and that $\pi_2(N) \simeq \mathbb{Z} \simeq H_2(N)$. Then by Barden's classification [Bard], N is diffeomorphic to either $S^3 \times S^2$ or $S^3 \tilde{\times} S^2$. Showing the second Whitney class $w_2(N)$ to be zero will establish the claim.

Take $S(m', n')$ such that $\binom{m\ n}{m'\ n'} = \pm 1$. It acts freely on N with $S^2 \times S^2$ quotient. Thus N is a (principal) S^1 bundle: $S^1 \hookrightarrow N \underset{/S(m',n')}{\xrightarrow{p}} S^2 \times S^2$. Then, we have

$$\begin{aligned} w_2(N) &= w_2(TS^1 \oplus p^*(S^2 \times S^2)) \\ &= w_2(TS^1) + w_1(TS^1)w_1(p^*(S^2 \times S^2)) + w_2(p^*(S^2 \times S^2)) \\ &= 0. \end{aligned}$$

4. If Spin(4) acts smoothly on an 8-manifold, then we must get a weighted orbit space satisfying the local conditions of theorem 1. On the other hand, given some 2-dimensional complex, we can assign any orbit structure to it; certainly, there must be a Spin(4) action on some space X that could give rise to it. Indeed, we may take the cartesian product of the complex with Spin(4), and collapse each fiber to a (left) coset according to the isotropy: Spin(4) acts on the resulting space (by left multiplication on the cosets) with the desired orbit structure. If X^* is a 2-surface, we speak of an *admissibly weighted orbit space*, if the orbit structure satisfies the conditions of the theorem.

Two observations are in order now. First, if we are given an admissibly weighted space, then there exists a smooth Spin(4) action giving rise to it. Indeed an action is a certain map Ψ: $\mathrm{Spin}(4) \times X \longrightarrow X$, and we can assume that locally, this map is given (by the differentiable Slice Theorem) as a smooth map

$$\mathrm{Spin}(4) \times \mathcal{N}(O_x) \longrightarrow \mathcal{N}(O_x)$$

where $\mathcal{N}(O_x)$ is an equivariant tubular neighborhood of the orbit O_x for any $x \in X$, under the Spin(4) action. This neighborhood is parametrized by $\mathrm{Spin}(4) \times_H \mathcal{S}_x$, where H is the stabilizer of x, and $\mathcal{S}_x$ a slice at x on which H acts linearly (slice representation); Spin(4) acts by (left) multiplication on the first factor.

Second, the (admissibly) weighted orbit space basically describes a stratification of the total space X, but it is by no means obvious that X is topologically entirely specifed by it, or that the action is uniquely determined (up to equivalence) by the weighted quotient. This uniqueness question is in fact the object of Section 2.

Proof of Theorem 1. Part (1) follows from the local picture prescribed by (2), (3) and the Figure, as well as the orientability condition on M. Part (3) has been established in Lemma 1.

If x is a fixed point, $\mathcal{S}_x = B^8$, and the action is the coning of a linear action on S^7. The latter is a codimension 1 action. It has a section, and S^7 must decompose as the union of two mapping cylinders of p_j: $S^3 \times S^3 \longrightarrow S^3 \times S^3/H_j$ ($j = 1, 2$) where

H_j is a subgroup of Spin(4) ([Bre], IV, Theorem 8.2). It is easy to verify that we must have $H_j = S_j$ (as shown on Figure 2a). Taking $S^7 = S^3 * S^3$, we easily check that (i) $(g \times h) \cdot \langle u; v; t \rangle = \langle gu; hv; t \rangle$ or (ii) $\langle guh^{-1}; hv; t \rangle$, where $u, v \in S^3, t \in I$ (collapse the first factor at $t = 0$, the second at $t = 1$, or vice-versa) will yield the quotients of Figure 2a. Up to equivalence any Spin(4) action must have this form.

If G_x is an S^3, or $S(m, n)$, the slice representation is uniquely determined; $\mathcal{S}_x$ is a five or a three-ball, on which G_x acts by coning the linear action on ΣS^3, or on ΣS^1, respectively, and we get the weighted quotients in Figure 2(d).

In the torus case (Figure 2(c)), the action on the 4-ball slice viewed as $D^2 \times D^2$, given by

$$(e^{i\gamma} \times e^{i\lambda}) \cdot (\rho_1, e^{i\alpha} \times \rho_2, e^{i\beta}) = (\rho_1 e^{i(\alpha+m\gamma-n\lambda)} \times \rho_2 e^{i(\beta+m'\gamma-n'\lambda)}),$$

is effective iff $\det \binom{m\ n}{m'\ n'} = \pm 1$, and the local picture is given above.

Finally, consider the linear action of $S^1 \times S^3$ on the boundary S^5 of the slice $\mathcal{S}_x$. As in Lemma 5, we view S^5 as the join $S^1 * S^3$ (collapsing S^1 at $t = 0$, S^3 at $t = 1$), so a typical element may be written as $\langle s; u; t \rangle$, where $s \in S^1, u \in S^3, t \in I$. Let $\xi \times g \in S^1 \times S^3$. By [Ric1], S^3 acts by (left) multiplication on the S^3 factor, with D^2 quotient, on which S^1 acts in the standard (linear) way. We have

$$\begin{array}{ccc} \langle s; u; t \rangle & \overset{g\cdot}{\mapsto} & \langle s; gu; t \rangle \\ \downarrow \xi\cdot & & \downarrow \xi\cdot \\ \langle \xi^n s; u\xi^m; t \rangle & \overset{g\cdot}{\mapsto} & \langle \xi^n s; gu\xi^m; t \rangle. \end{array}$$

One checks immediately that at $t = 0$, the isotropy is $S(m, 1)$. Note, furthermore, that if $n \neq 1$, we either get $\mathbb{Z}_n$ principal isotropy, or S^1 if $n = 0$. Clearly, we have $\{1\} \times S^3$ isotropy at $t = 1$, and obtain the quotient pictured on figure 2(b).

As this slice representation suggests, there cannot be adjacent $S(m, n)$ and ΔS^3 edges. Notice in particular that these subgroups do not generate $S^1 \times S^3$ (whereas $S(1, n)$ and $S^3 \times \{1\}$ do). Indeed, first consider the case of H generated by $C \times \{1\}$ and ΔS^3. Simply notice that $(g \times g) \cdot (e^{i\theta} g^{-1} \times g^{-1}) = g e^{i\theta} g^{-1} \times \{1\}$. Since every element in S^3 can be obtained as the conjugate of some $e^{i\theta}$, this means $S^3 \times \{1\} \subset H$, and since $\Delta S^3 \subset H$, it follows that $H = S^3 \times S^3$. More generally, for any $(m, n) \neq (1, 1)$, again we have $H \doteqdot \langle S(m, n) \cdot \Delta S^3 \rangle = S^3 \times S^3$ (if $(m, n) = (1, 1)$, $S(1, 1) \subset \Delta S^3$). By the previous case, it suffices to show that $C \times \{1\} \subset H$. Just multiply $e^{in\theta} \times e^{im\theta} \in S(m, n)$ by $e^{-im\theta} \times e^{-im\theta} \in \Delta S^3$ to get $e^{i(n-m)\theta} \times \{1\}$: unless $n = m = 1$, we can obviously hit all elements in $C \times \{1\}$. Thus, the local orbit structure for X^*, if it did arise from a linear slice representation, would give a vertex corresponding to a *fixed point*, rather than S^2 (with a pair of issuing edges, weighted by $S(m, n)$ and ΔS^3): so the neighborhood of that fixed point would be a cone over what sits above an arc joining the edges, and that cannot be a 7-sphere (for instance, an easy Mayer-Vietoris sequence gives $H_5 \simeq \mathbb{Z}$). In other words: one could certainly construct a space X with a Spin(4) action yielding this weighted quotient (just take $D^2 \times \text{Spin}(4)$ and then collapse over the vertex and the edges by a subgroup in the appropriate isotropy type), but X could not possibly be a manifold. $\square$

2. An equivariant classification

Our primary goal in this section is to establish Theorem A stated in the introduction. $E = \emptyset$ is assumed in what follows.

Recall that two actions on M are equivalent if there exists an equivariant diffeomorphism Ψ of M, so $\Psi(g \cdot m) = g \odot \Psi(m)$, for all $g \in \text{Spin}(4)$, and $m \in M$ (where $\cdot$ and $\odot$ denote the actions).

Once an admissibly weighted orbit space is given, one might reasonably hope that it contains all the equivariant information regarding the action, up to equivalence; this is the case, for instance, in the context of toral actions on 4-manifolds, with singular isotropy and $E = \emptyset$. The key question is whether one can find a global section to the action. For Spin(4) actions, we do find such a section, but the discussion centers on a possible obstruction to "normalizing" it. We'll obtain the extra invariant o, which is made necessary, roughly speaking, by the fact that Spin(4) is not abelian.

First note that there is no obstruction to a global section over M_0^*, where M_0 is any subset of the union of principal orbits, including cases where $M_0^* = M^* = M^2$ a closed surface. For such an obstruction would lie in $H^2(M_0^*; \pi_1(\text{Spin}(4)) = \{0\}$ [Ste]. Generally, one finds no obstruction to extending a section over some $K^* \subset M_0^*$ to one over M_0^*, for the same reason. In particular, all principal Spin(4) bundles over a closed M^2 are trivial. So we consider henceforth the case where M^* *is a surface* M^2 *with boundary.*

In what follows, we explicitely construct a global section, but the question that needs to be addressed simultaneously has to do with how much control we have over the image of that section. Namely whether it can be made to pass through points that are stabilized by *specific* subgroups of Spin(4), which could remain fixed for orbits of the same type, rather than determined only "up to conjugation". If such a section can be obtained, we call it a normalized section. More precisely:

Definition 1. A *normalized section* (n-section for short) is a section such that orbits with $S^1 \times S^3$, $S^1 \times S^1$ and $S(m, n)$ isotropy are mapped to points stabilized by $C \times S^3$, $C \times C$, and $S(m, n) \subset C \times C$, respectively, and the ΔS^3 orbits to points stabilized by $\{g \times g | g \in S^3\}$ (strict diagonal).

We will see that the obstructions to this normalization lie, essentially, in $\pi_0(N(H))$, where $N(H)$ is the normalizer of H in Spin(4), $H = S(m, n)$, or ΔS^3. As we've indicated, once a section over an annular neighborhood for each boundary component of M^* is obtained, then there is no difficulty in extending the resulting section, whose domain is a disjoint union of annuli, to one that is globally defined on M^*. So we need to examine a generic (half-open, say) annular region $\mathcal{A}$. The point is this: if we can construct an n-section, the total space is obtained by taking the cartesian product $\mathcal{A} \times \text{Spin}(4)$, and then collapsing each singular orbit over the boundary by its isotropy in a *uniformly* specified way. Indeed, for each isotropy type that occurs on the boundary, we collapse by the corresponding distinguished subgroup H, to $\text{Spin}(4)/H$. Spin(4) then acts by (left) multiplication on the Spin(4) factor of the

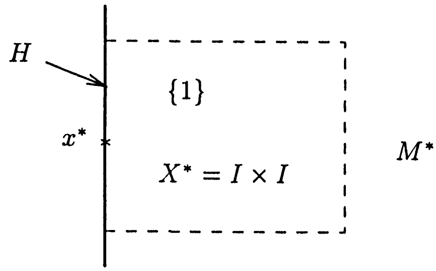

Figure 3

resulting quotient. Now, if the obstruction mentioned above does not vanish, this uniformity fails in the sense that we can no longer assign a *fixed* subgroup to each isotropy type. However, it does not fail badly, as the explicit construction will make apparent. For there still exists a section over $\mathcal{A}$. Furthermore, we can still fix the isotropy subgroup over all but one edge of $\mathrm{Bd}(\mathcal{A})$; and even there, we have control over the conjugates of the distinguished subgroup we collapse with. In other words, what we will show is that there are at most two distinct actions yielding a given $\mathcal{A}$ and isotropy weights.

We now proceed to establish theorem A, which we restate, in a slightly different form, in Section 2.3. The proof is constructive and occupies Sections 2.1, 2.2 and 2.3. The strategy, for a given annular neighborhood, is to get local n-sections (Section 2.1), and then attempt to fit them together suitably: the possibility of an obstruction becomes evident (2.2.1), and explicit models exhibit the actions where it does arise (2.2.2 and 2.2.3).

2.1. *Local n-sections near singular orbits.*

LEMMA 6. *Let X^* be any closed rectangular neighborhood of an $S^2 \times S^3$ orbit in M^* as shown on Figure 3. Then there is an n-section over X^*.*

Proof. Choose a point x in X stabilized by $H = S(m, n) \subset \mathcal{C} \times \mathcal{C}$, and mapping to a point $x^* \in \{0\} \times \mathrm{int}(I)$. The orbit is 5-dimensional so the slice $\mathcal{S}_x$ is a transverse 3-ball to that orbit, on which S^1 acts as SO(2). This action has a section, the image of which is in fact transverse to *all* the orbits in X that it intersects: thus, it can be viewed as the image of a section to the Spin(4) action itself. We now have a section σ_x over $\mathcal{S}_x^*$ which must be extended to X^*. We may take $\mathcal{S}_x^* \subseteq X^*$ to be closed.

Now consider a singular orbit $y^* \in \mathrm{Bd}(\mathcal{S}_x^*) \cap \{0\} \times I$ (see Figure 4). We can find similarly another section σ_y to the slice action about some y such that $\pi(y) = y^*$, and y is also stabilized by the standard $S(m, n)$. Next, we need to "fit" the two sections

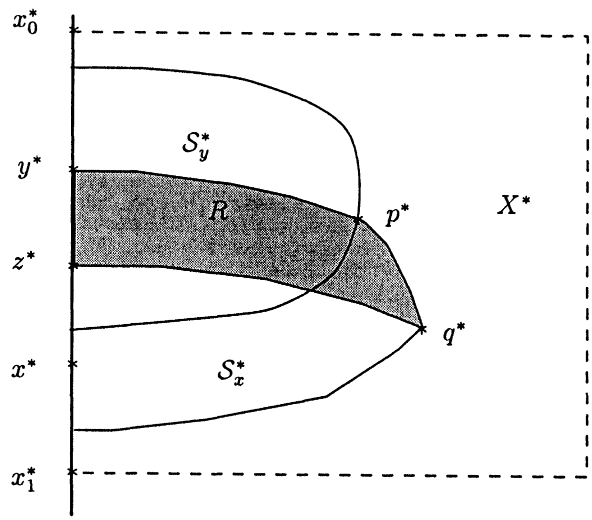

Figure 4

together over the arc $\alpha^* = \mathrm{Bd}(\mathcal{S}_x)^* \cap \mathcal{S}_y^*$, joining y^* to p^* (clearly, the images of α^* under σ_x and σ_y will in general not be equal). To do this, we will deform σ_x suitably. We may assume that we have a smooth path α: $I \to \mathrm{Im}(\sigma_x)$ over α^*; then for each $\alpha(t)$, there is a unique corresponding point in $\mathrm{Im}(\sigma_y)$, lying on the same Spin(4) orbit.

The image of α^* under σ_y is the path $\hat{\alpha}(t) = g(t) \cdot \alpha(t)$ in $\sigma_y(\mathcal{S}_y^*)$, with $g(t)$ a unique smooth path in Spin(4) (necessarily, $g(0) = 1$). Pick a point z^* lying between y^* and x^* on the $S(m, n)$-edge of $\mathcal{S}_x^*$, and form a region R parametrized by the square $(\gamma(s), \lambda(t))$, $(s, t) \in I \times I$; so $z^* = (\gamma(0), \lambda(0))$, $y^* = (\gamma(0), \lambda(1))$, $p^* = (\gamma(1), \lambda(1))$, $q^* = (\gamma(1), \lambda(0))$.

We now "match" the images of the n-sections σ_x, σ_y by

$$\sigma_x(\gamma(s), \lambda(t)) \mapsto g(st) \cdot \sigma_x(\gamma(s), \lambda(t)).$$

In particular, the arc joining z^* to q^* (corresponding to the bottom edge $t = 0$ of $I \times I$) is sent to its image under σ_x; the arc α joining y^* to p^* (corresponding to the upper edge $t = 1$) is sent to $g(s) \cdot \sigma_x(\gamma(s), \lambda(1)) = g(s) \cdot \alpha(s) = \hat{\alpha}(s)$, that is, its image under σ_y.

Note that to ensure smoothness, we can straighten the angle along the (smooth) arc $\sigma_x(\gamma(s), \lambda(0))$ in $\mathcal{S}_x$, as well as along $\hat{\alpha}$ in $\mathcal{S}_y$. Finally, it is clear by compactness that a finite number of such matched slice sections will result in an n-section over a subset of M^* that contains x_0^* and x_1^* and, therefore, contains X^*. $\square$

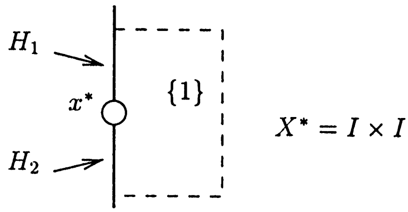

Figure 5

LEMMA 7. *Let X^* be any rectangular neighborhood of an S^3 orbit in M^*, as shown on Figure* 3. *Then there is an n-section over X^*.*

Proof. As in the previous lemma, we choose a point x stabilized by $H = S^3$ (strictly diagonal for ΔS^3 orbits), and mapping to x^* in $\{0\} \times \mathrm{int}(I)$. The slice representation gives an effective S^3 ($\hookrightarrow$ SO(5)) action on $\mathcal{S}_x = B^5$, which is the coning of the S^3 action on $S^4 = \mathrm{Bd}(\mathcal{S}_x)$. By Richardson' theorem [Ric1], any effective action of S^3 on S^4 is conjugate to the suspension of the standard linear action of S^3 on S^3. Thus the slice representation is equivalent to a multiplicative action on $B^5 = \mathrm{Cone}(\Sigma S^3)$, and there is no difficulty in finding a section. We obtain an n-section over $\mathcal{S}_x^*$, and we extend it to one over X^* proceeding as before. □

LEMMA 8. *Let X^* be any rectangular neighborhood of an $S^2 \times S^2$ orbit in M^* of the form given on Figure* 5. *Then there is an n-section over X^*.*

Proof. Choose $x \in S^2 \times S^2$ with stabilizer $C \times C$ generated by $H_1 = S(m, n)$ and $H_2 = S(m', n')$. The slice representation gives $S^1 \times S^1$ acting as SO(2) × SO(2) ⊂ SO(4) on $\mathcal{S}_x \approx B^4$. From a result in [OR1], there is a section to such an action , and in turn this is an n-section to the Spin(4) over $\mathcal{S}_x^*$. $\mathcal{S}_x^*$ may not coincide with X^*, but one extends appropriately as previously. □

LEMMA 9. *Let X^* be any rectangular neighborhood of an S^2 orbit in M^* (refer to Figure* 5). *Then there is an n-section over X^*.*

Again the slice representation at $\mathcal{S}_x$ is the coning of a linear action on $S^5 \approx S^1 * S^3$ which has a section. As before, we make an appropriate choice for x in X to get an n-section and extend over X^*.

LEMMA 10. *Let X^* be any rectangular neighborhood of a fixed point in M^*. Then there is an n-section over X^*.*

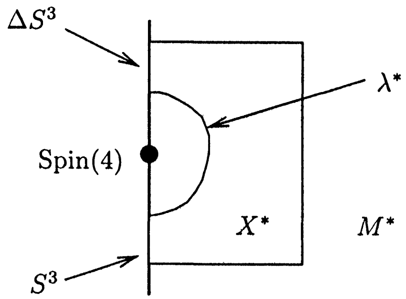

Figure 6

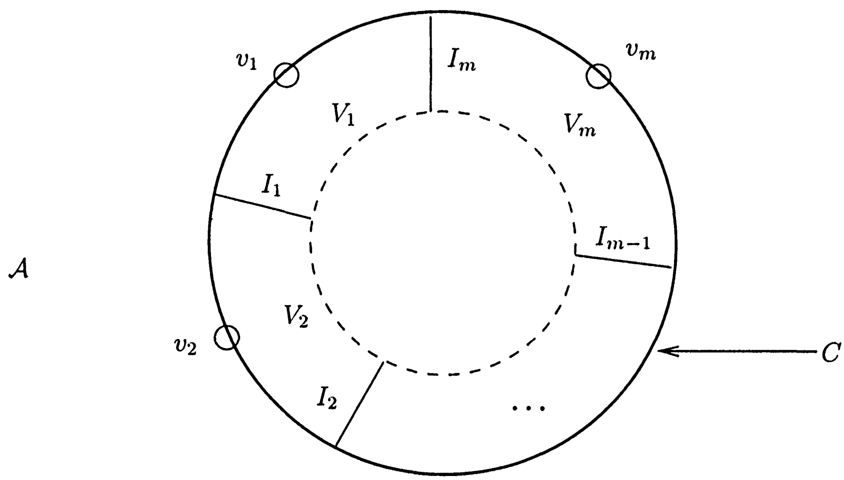

Figure 7

Proof. By Lemma 7 we can ensure that in the cases where one of the two incoming edges is ΔS^3 stabilized, the section sends that edge to strictly diagonal points. That is, we may assume that we have an n-section over the shaded area in Figure 6: we extend it by coning over the arc λ^*. $\square$

2.2. *Global sections and the normalization obstruction.* Given a boundary component C of M^* with vertices $v_1, \dots, v_m$ (ordered, say, counterclockwise), we now have m n-sections σ_j over closed neighborhoods V_j of v_j, which cover the (closed) annular neighborhood $\mathcal{A}$ of C, as shown (Figure 7). We let I_j denote the intersection $V_{[j]} \cap V_{[j+1](\mathrm{mod}\, m)}$.

Of course, these sections may not fit together. In particular, the images $x_0 = \sigma_j(x^*)$ and $x_1 = \sigma_{j+1}(x^*)$, $(x^* = I_j \cap C_k)$ may be distinct points. Generally, $x_0 = \overline{g} \cdot x_1$,

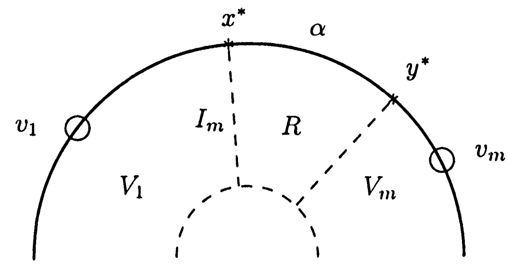

Figure 8

where $\overline{g} \in N(H_{x_i})$, the normalizer in Spin(4) of the isotropy subgroup of both points.

Starting at $x^* = I_1 \cap C_k$, if $\sigma_1(x^*) = x_1 = \overline{g} \cdot x_2$, ($x_2 = \sigma_2(x^*)$), we translate the entire image Im(σ_2) to $\overline{g} \cdot Im(\sigma_2)$. This evidently defines a new n-section (call it σ_2 again), which now agrees with σ_1 over I_1, possibly after an additional adjustment of the free orbit points in $\sigma_2(I_1)$, analogous to that made in the proof of Lemma 6. We proceed in this fashion, always modifying σ_{j+1} if it doesn't agree with σ_j, until we get to the final pair σ_m and σ_1. These need to agree over I_m, but here we can no longer repeat the same procedure.

The edge $[v_1, v_m]$ is stabilized either by (i) $S^3 \times \{1\}$, $\{1\} \times S^3$, or (ii) ΔS^3, or (iii) $S(m, n)$. We examine these in turn.

2.2.1. *The obstruction over an annular neighborhood.* Suppose we can set $[v_1, v_m]$ to have isotropy (i). Let $x^* = I_m \cap C_k$, $\sigma_1(x^*) = x_1 = \overline{g} \cdot x_m$, $x_m = \sigma_m(x^*)$, where $\overline{g} \in N(S^3 \times \{1\}) = N(\{1\} \times S^3) = \text{Spin}(4)$. Choose a path $\overline{g}(t)$ in Spin(4) joining $1 = \overline{g}(0)$ and $\overline{g} = \overline{g}(1)$. Pick a point y in Im($\sigma_m$) such that $y^* \in C_k$ between x^* and v_m. Let $\alpha(t)$ parametrize the arc α on C_k joining y^* to x^*, $\alpha(0) = y^*$, $\alpha(1) = x^*$ (see Figure 8).

Viewing the region R in V_m as $\alpha \times I_m = \{\alpha(t) \times I_m | t \in I\}$, we deform $\sigma_m(\alpha \times I_m)$ to $\{\overline{g}(t) \cdot \sigma_m(\alpha(t) \times I_m) | t \in I\}$. The resulting modified image of V_m in X is that of an n-section which agrees with σ_1 over I_m (possibly after some additional straightening). We are done in this case.

Thus, given a boundary circle of M^*, whenever there is at least one $\{1\} \times S^3$ or $S^3 \times \{1\}$-stabilized edge, we may take this edge to be the m^{th} or last edge in the ordering; so that the construction above applies to get the n-section. The crucial feature was that we had a path in the normalizer joining $\overline{g}$ to 1. But in cases (ii) and (iii), $N(H)$ is disconnected, in fact we have

$$\pi_0(N(S(m, n)) \simeq \mathbb{Z}_2 \simeq \pi_0(N(\Delta S^3))$$

Therefore, if $\overline{g}$ is not in the component of the identity, we can still get a global section, but it cannot be normalized: for the image in M of the arc α on C_k will have to contain points that are stabilized by distinct *conjugates* of the given isotropy group.

2.2.2. ΔS^3*-weighted boundaries.* We consider the ΔS^3 case: here, the entire boundary component of M^* must be ΔS^3 stabilized (otherwise we are in case (i)).

Suppose $p \in M$ is such that its stabilizer is strictly diagonal. If $\hat{p} \in \mathrm{Orbit}(p)$ and $\hat{p} \neq p$, then the stabilizer of $\hat{p}$ is strictly diagonal as well iff $\hat{p} = (g \times -g) \cdot p$, $g \in S^3$. Furthermore this $\hat{p}$ is unique, indeed $\forall g, \hat{g} \in S^3$,

$$p = (g^{-1}\hat{g} \times g^{-1}\hat{g}) \cdot p = (g^{-1} \times -g^{-1})(\hat{g} \times -\hat{g}) \cdot p$$

$$\iff (\hat{g} \times -\hat{g}) \cdot p = (g \times -g) \cdot p = \hat{p}.$$

So we may write $\hat{p} = (1 \times -1) \cdot p$. Now suppose that in attempting to construct a section σ_1 over the annular neighborhood of $\mathrm{Bd}(M^*)$, we end up back in $\mathrm{Orbit}(p)$ at $\hat{p}$.

Starting again at $\hat{p}$, we proceed as before to get a second section σ_2. Observe that the union of the images of $\mathrm{Bd}(M^*)$ under the two sections form a circle that double-covers $\mathrm{Bd}(M^*)$. When forming the slice at $\hat{p}$ (after having gone around $\mathrm{Bd}(M^*)$ once), we intersect orbits projecting down to points y^* in $\mathrm{Bd}(M^*)$ for which we had already found an image y under σ_1; obviously $\hat{y}$ $(= \sigma_2(y^*))$ must be distinct from y if $\hat{p} \neq p$, in fact $\hat{y} = (1 \times -1) \cdot y$.

We now give a model for such an action, as well as a model for the normalizable case. Both models describe the actions over $\mathrm{Bd}(M^*) = S^1$, that is, the $S^3 \times S^3$ actions on $S^3 \times S^1$ with the single diagonal S^3 isotropy type.

(a) *n-sectioned action.* For $(g \times h) \in \mathrm{Spin}(4)$ set

$$(g \times h) \cdot (v \times s) = gvh^{-1} \times s, (v \in S^3, s \in S^1).$$

It is clear the n-section is given by

$$s \mapsto 1 \times s \text{ (or } -1 \times s).$$

Slightly more generally, one could write the action as:

$$(g \times h) \cdot (v \times s) = (w_0 g w_0^{-1}) v h^{-1} \times s (\text{where } w_0 \in S^3 \text{ is fixed}).$$

We see that if $v = \pm w_0$, then $w_0 g w_0^{-1} v h^{-1} = \pm w_0 g h^{-1} = \pm w_0 = v$ iff $g = h$.

(b) *Non-n sectioned action.* We make w_0 dependent on $s = e^{i\theta}$, setting

$$(g \times h) \cdot (v \times e^{i\theta}) = e^{i\theta/2} g e^{-i\theta/2} v h^{-1} \times e^{i\theta}.$$

Note this is well defined: we vary the angular parameter θ from $\theta = 0$ to 2π, where we conjugate trivially by $e^{i2\pi/2} = -1$ in the center of S^3. We also see that $\pm e^{i\theta_0/2} \times e^{i\theta_0}$ is strict ΔS^3 stabilized (θ_0 fixed). In particular, starting at $\theta = 0$, and picking 1×1 in that orbit, we then move continuously to the neighboring fibers, passing through $e^{i\theta/2} \times e^{i\theta}$ as we vary θ, to wind back at $e^{i2\pi/2} \times e^{i2\pi} = -1 \times 1$.

Remark. The two actions are not equivalent.

Proof. Suppose ψ were a weakly equivariant map of $S^3 \times S^1$. Then in particular, letting σ be an n-section,

$$\psi(\sigma(x)) = \psi(\Delta g \cdot \sigma(x)) = \mathrm{aut}(\Delta g) \cdot \psi(\sigma(x)), \qquad (*)$$

where $\Delta g \in \Delta S^3$ is strictly diagonal, and aut is an automorphism of Spin(4). Now any automorphism will send ΔS^3 to $\{g_0 g g_0^{-1} \times g | g, g_0 \in S^3, g_0 \text{ fixed}\}$. For the non n-sectioned action, the points in $S^3 \times S^1$ that are stabilized by $\mathrm{aut}(\Delta S^3)$ form a single circle C. $(*)$ says that the image under ψ of the circle Im (σ) is this circle C. But C must also be the image of $(-1 \times 1) \cdot \mathrm{Im}(\sigma)$ $(\neq \mathrm{Im}(\sigma))$, so that ψ could not be a homeomorphism. □

2.2.3. *Boundaries with $S(m, n)$ weights.* We now examine the case where all edges in $\mathrm{Bd}(M^*)$ are stabilized by subgoups of the form $S(m_j, n_j)$. First we will consider the special case where there are no vertices; the general situation will then proceed with only minor modifications.

Suppose all points in $\mathrm{Bd}(M^*)$ are $S(m, n)$ orbits. In fact, we'll first assume that m or $n = 0$, and consider $S(0, 1)$ for definiteness; we will be using this particular case in Section 3. Recall that $S(0, 1) = \mathcal{C} \times \{1\}$ and conjugates. $N(S(0, 1))$ is a conjugate of $N(\mathcal{C}) \times S^3$, where $N(\mathcal{C})$ has two components, $\mathcal{C} = \{e^{i\theta} | \theta \in [0, 2\pi)\} \equiv \{(e^{i\theta}, 0)\}$ and $(0, 1)\mathcal{C} = \{(0, e^{i\theta})\}$.

Proceed exactly as in the ΔS^3 case, attempting to construct a section σ_1 over the annular neighborhood of the boundary component through $\mathcal{C}$ stabilized points. Note that there are two disjoint copies of S^3 of such points above each point in $\mathrm{Bd}(M^*)$. Starting at some p, we wind back to hit orbit(p) at $\hat{p}$, where $\hat{p} = (n \times g) \cdot p$, and $n \times g \in N(\mathcal{C}) \times S^3$. If $n \in \mathcal{C}$, we can alter σ_1 so as to get $\hat{p} = p$, and we are done. This can certainly occur, and it is easy to write out a model for such an action (a product action on $S^2 \times S^3 \times S^1$). We omit the details. But if we had $n = (0, e^{i\alpha})$, we would not be able to modify the section without hitting points that are stabilized by conjugates of $\mathcal{C}$. However, we could certainly arrange for $\hat{p} = ((0, 1) \times 1) \cdot p$, and then, constructing σ_2 starting at $\hat{p}$, similarly arrange $\hat{y} = ((0, 1) \times 1) \cdot y$, (for each $\hat{y} \in \mathrm{Im}(\sigma_2)$ in the same orbit as $y \in \mathrm{Im}(\sigma_1)$). We would then wind back at

$$((0, 1) \times 1)((0, 1) \times 1) \cdot p = (-1 \times 1) \cdot p = p.$$

Again, $\mathrm{Im}(\sigma_1) \cup \mathrm{Im}(\sigma_2)$ above $\mathrm{Bd}(M^*)$ traces out a circle double covering $\mathrm{Bd}(M^*)$. We need to give an explicit model for this action.

We begin with the non-trivial S^2 bundle over S^1 (it is non-orientable), viewing it as $(S^2 \times I)/\sim$, where we glue by the antipodal map a: $S^2 \times \{0\} \longrightarrow S^2 \times \{1\}$. Expressing S^2 as $S^3/\mathcal{C}$, the following assignment is well defined and equivalent to a:

$$[u] = u\mathcal{C} \mapsto u(0, 1)\mathcal{C} = [u(0, 1)].$$

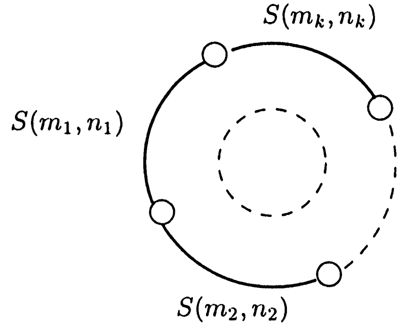

Figure 9a

(Note that $uC \sim ue^{i\alpha}C \mapsto ue^{i\alpha}(0,1)C = u(0,1)e^{-i\alpha}C \sim u(0,1)C$.) Next, define an S^3 action on the left in the obvious way; it is clear we have an $S^3 \times S^3$ action on $(S^2 \tilde{\times} S^1) \times S^3$ (using left S^3 multiplication on the $S^2 = S^3/C$ factor, and on the S^3 factor), which gives an S^1 quotient with $S(0,1)$ isotropy, and such that the strictly C stabilized points form a single circle $(\times S^3)$ double-covering the projection to S^1.

Now view $S^3 \times S^1$ as $(S^3 \times [0,1])/_\sim$ where we glue $S^3 \times \{0\}$ to $S^3 \times \{1\}$ by: $u \times 0 \mapsto u(0,1) \times 1$. Let $\mathrm{Cyl}(\phi)$ be the mapping cylinder of

$$\phi\colon S^3 \times S^1 \longrightarrow S^2 \tilde{\times} S^1$$

$$\langle u \times t\rangle \mapsto \langle [u] \times t\rangle.$$

ϕ is equivariant with respect to the left S^3 action (by multiplication). Thus, we have obtained an $S^3 \times S^3$ action on $\mathrm{Cyl}(\phi) \times S^3$; the weighted orbit space is an annulus with one boundary circle made up of $S(1,0)$ points (all other orbits principal), and the action has a non-normalizable section.

We now indicate briefly how to generalize the preceding discussion. First, if the entire boundary consists of $S(m,n)$ orbits. The standard $S(m,n) = \{e^{in\theta} \times e^{im\theta}\}$; recall also that for $m, n \neq 0$, $N(S(m,n)) = C \times C \coprod (0,1)C \times (0,1)C$. Take $(S^3 \times S^3/S(m,n)) \times [0,1]$, and then glue in one of two possible ways:

(i) $[u \times v] \times \{0\} \mapsto [u \times v] \times \{1\}$ (normalizable case)

(ii) $[u \times v] \times \{0\} \mapsto [u(0,1) \times v(0,1)] \times \{1\}$ (non-normalizable case)

The gluing maps are equivariant with respect to the usual left Spin(4) action by translation, and we easily extend this to a mapping cylinder as before.

Finally, suppose there are several distinct $S(m,n)$ (refer to Figure 9a).

Let $t_0 < t_1 < \cdots < t_k$ be $k+1$ real numbers. Form the product $S^3 \times S^3 \times [t_0, t_k]$. Next, over each interval $[t_{j-1}, t_j]$, collapse the $S^3 \times S^3$ factor by the appropriate $S(m_j, n_j)$, which we may take to be the circle $\{e^{in_j\theta} \times e^{im_j\theta}\}$ in the distinguished maximal torus. We collapse on the right, so over the interior points of $[t_{j-1}, t_j]$ we have cosets $(u \times v)S(m_j, n_j)$, i.e., classes $[u \times v] \times t \sim [ue^{in_j\theta}, ve^{im_j\theta}] \times t$. Over

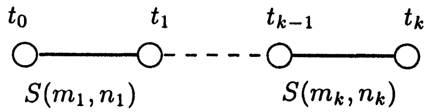

Figure 9b

each t_j, we have $S^3 \times S^3$ collapsed to $S^2 \times S^2 = S^3 \times S^3/T^2 = S^3 \times S^3/S(m_j, n_j) \cdot S(m_{j+1}, n_{j+1})$; we also collapse over the end points t_0 and t_k by $S(m_1, n_1)$ and $S(m_k, n_k)$.

Now, glue the $S^2 \times S^2$ fibers above t_0 and t_k in one of two ways:

(i) by the identity,

(ii) by the map $a \times a$, where a is the "antipodal" map as previously defined, that is, using multiplication on the right by (0, 1).

Clearly, the linear left Spin(4) action on the complexes makes sense; the gluing maps are equivariant with respect to the restriction of the action to the fibers above t_0 and t_k. The circle quotient has the same isotropy weights as the circle boundary of the orbit space on Figure 9. We will have our models after forming the mapping cylinders Cyl(Ψ) defined in the obvious way. Namely, letting $S^3 \times S^3 \times S^1$ be obtained by gluing the fibers over the end-points of $S^3 \times S^3 \times [t_0, t_k]$ together according to either (i) or (ii), for each $t \in [t_0, t_k]$, where $\Psi\ |_t$ is the natural map $u \times v \mapsto [u \times v]$. Spin(4) acts on the left, Ψ is well defined and Spin(4)-equivariant, and the induced action on Cyl(Ψ) gives the desired weighted annular region. (i) is the model for the n-sectioned action, and (ii) for the non-n-sectioned one.

2.3. *An equivariant theorem.* Let M be a smooth Spin(4)-manifold ($E = \emptyset$), and let ϵ be a specific orientation for the 2-manifold M^*. Let g denote its genus. Order the boundary components from 1 to b (b = number of components), writing them as $C_1, C_2, \ldots, C_b$. Let $\mathcal{O}_j(M)$ be the orbit structure on C_j; that is (given the sense determined by ϵ), an admissible sequence $\{H_{j_k}\}_{1 \le k \le m}$ of isotropy types, subject to the equivalence: $\{H_{j_k}\} \sim \{H'_{j_l}\}$, iff $H'_{j_i} = H_{j_1}, \ldots, H'_{j_m} = H_{j_{m-i+1}}, H'_{j_1} = H_{j_{m-i+2}}, \ldots, H'_{j_{i-1}} = H_{j_m}$, for some $i \in \{1, \ldots, m\}$.

Let $o(M) = (o_1(M), o_2(M), \ldots, o_b(M)) \in \mathbb{Z}_2{}^b$, where $o_j(M) = 0$ or 1, according to whether the action admits an n-section over an annular neighborhood $\mathcal{A}_j$ of C_j, or does not. Note that if a boundary component contains points that are stabilized by a factor S^3, the corresponding coordinate of o is necessarily 0.

THEOREM 2. *Suppose M and M' are closed, orientable 8-manifolds on which* Spin(4) *acts smoothly, with $E = \emptyset$. Then there is an equivariant orientation-preserving diffeomorphism between M and M' iff there exists an ordering of the boundary components of M^* and M'^* such that,*

$$(\epsilon, g, b;\ o(M)) = (\epsilon', g', b';\ o(M'))$$

and

$$\mathcal{O}_1(M) = \mathcal{O}_1(M'), \dots \mathcal{O}_b(M) = \mathcal{O}_b(M').$$

Proof. Let π denote the orbit map of M into M^*, and G_x denote the stabilizer subgroup of x ($x \in M$ or M'). If o vanishes everywhere, we have two smooth, normalized global sections σ and σ', and a (smooth) homeomorphism $h\colon M^* \longrightarrow M'^*$, such that

$$G_{\sigma(\pi(x))} = G_{\sigma'(h(\pi(x)))}.$$

If o does not vanish, notice that the identity is still satisfied. Indeed, for each of the distinguished subgroups H of type $S(m, n)$ or ΔS^3, we can once and for all fix the conjugates that stabilize points where the section fails to be normalized (which may happen over a small arc lying in the interior of an edge): this amounts to fixing a suitable (smooth) path in Spin(4), from the component of $N(H)$ not containing the identity, to the identity.

Now, identifying M^* and M'^* with their images under σ and σ', we extend h to $H\colon M \longrightarrow M'$ in the obvious way. Namely, set

$$H(x) = g \odot h(y)$$

where $x = g \cdot y$, for a unique $y \in M^*$ and some $g \in \mathrm{Spin}(4)$. It is not hard to check that H is well defined and a homeomorphism. For the sake of expliciteness, we show it is smooth at all $x \in M$.

If x lies in a free orbit, this is immmediate. Indeed there is a tube $D^2 \times \mathrm{Spin}(4)$ consisting only of free orbits, with $D^2 \times \{1\} \subset M^*$ a disk about y. In particular, $x = (y, g)$ for a unique g. Over the entire tube, H has the form $h \times \mathrm{Id}_{|\mathrm{Spin}(4)}$.

If x is in a singular orbit, then we have local coordinates $\mathcal{S}_x \times \mathcal{N}_1$, where $\mathcal{S}_x$ is a slice centered at x, and $\mathcal{N}_1$ is the image in Spin(4) of a smooth local section $\chi\colon U \subset \mathrm{Spin}(4)/G_x \longrightarrow \mathrm{Spin}(4)$, U an open ball about $1 \cdot G_x$, and $\chi(1 \cdot G_x) = 1$. In these local coordinates, $H = \tilde{h} \times \mathrm{Id}$, where $\tilde{h}$ is the obvious extension of h over $\mathcal{S}_x$; in particular, it is equivariant with respect to the linear G_x action. Clearly H is smooth if $\tilde{h}$ is. Now let $\gamma(t)$ be any smooth path in $\mathcal{S}_x$ through x ($\gamma(0) = x$). It may be written as $\gamma(t) = g(t) \cdot \alpha(t) = g_t(\alpha(t))$, where g_t is a smooth path in G_x through 1 ($g_0 = 1$), and α is a smooth path in $\mathcal{S}_x^* \subset \mathcal{S}_x$, through x. Then

$$\tilde{h}(\gamma(t)) = \tilde{h}(g_t(\alpha(t))) = g_t(\tilde{h}(\alpha(t))) = g_t(h(\alpha(t))).$$

Taking $\gamma(t)$ to represent a vector v in $T_x\mathcal{S}_x$ (α represents a vector w in $T_x\mathcal{S}_x^*$), we see that $\tilde{h}$ must be differentiable a x, and that, in fact,

$$d\tilde{h}_x(v) = \frac{d}{dt} g_t(h(\alpha(t)))_{|t=0} = \hat{g}_*(dh_x(w))$$

where $\hat{g}_* = \frac{d}{dt} g(t)_{|t=0}$ (we have an induced action of G_x on $T_{h(x)}\mathcal{S}_{(h(x))}$). This concludes the proof of the theorem. □

Remark. We wish to emphasize that this result will underlie many of our subsequent discussions. It is what makes it permissible to view, as we will, the total space as $\mathrm{Spin}(4) \times M^2$, with Spin(4) collapsed to cosets over the boundary, and the group acting only on the first factor. This is a *global statement*, not simply a local parametrization of the action. In particular, it enables us in some cases to directly "read off" from the weighted quotient the manifold which sits above it.

2.4. *Examples.* We identify a few simply connected Spin(4) manifolds. Recall that the orbit structure on a boundary circle partitions it into vertices (weighted by T^2, $S^1 \times S^3$ or Spin(4)) and edges (weighted by $S(m, n)$, or S^3), and satisfies the local conditions given in Theorem 1, Figure 2 (1.2.2). We will take M^* to be a disk, and since we will assume the presence of fixed points, o automatically vanishes.

1. Suppose there are two vertices, both of which are fixed points. Then $M = S^8$, the eight-sphere. Indeed, the invariant subspace in M that projects down to a straight arc in M^* joining an interior point in one edge to a second point in the interior the other edge, is either exactly the join $S^3 * S^3$, or diffeomorphic to it. We have a one-parameter ($t \in [0, 1]$) family of these, collapsed at the two fixed points ($t = 0, 1$): that is, the suspension of S^7 (alternatively, the Spin(4) -invariant 8-ball neighborhoods of the fixed-points are glued by the identity along their invariant S^7 boundaries).

2. Suppose we have three vertices, all of them fixed points. Moving, say clockwise, this forces the three edges to be weighted either by $S^3 \times \{1\}$, $\{1\} \times S^3$ and ΔS^3, or by $\{1\} \times S^3$, $S^3 \times \{1\}$ and ΔS^3, respectively. We see directly from the orbit data, that above one closed edge (here, this includes two fixed points on the ends) sits an invariant 4-sphere (suspension of S^3), i.e., HP(1), the quaternionic projective space of (quaternionic) dimension one. Then the total space is obtained by attaching an 8-cell (the interior of which is a neighborhood of the third fixed point) to this sphere by a Hopf map; thus M must be HP(2). It can be shown that the two distinct isotropy structures correspond to $\mathbb{H}P(2)$ and $\overline{\mathbb{H}P}(2)$ respectively, the quaternionic projective spaces with opposite orientations.

3. Given four vertices, all of which are again fixed points, there are two situations: either only two of the three S^3 isotropy occur, or all three do. In the former, $M = S^4 \times S^4$. Viewing S^4 as the suspension ΣS^3 and using S^3 multiplication, it is not hard to give the explicit forms that the action can take; these actions yield the desired weighted orbit space, and in turn, Theorem 2 guarantees that, up to equivalence, they are the only ones that do so. In the second situation, observe that we can always find a Spin(4)-invariant 7-sphere that does not bound an 8-ball: it sits over an arc joining two non-contiguous edges. So that, following Example 2, M is seen to be an equivariant connected sum $\mathbb{H}P(2)\#\ \overline{\mathbb{H}P}(2)$ (note in particular that $\pm(\mathbb{H}P(2)\#\ \mathbb{H}P(2))$ cannot occur).

4. The general case for all n vertices corresponding to fixed points proceeds similarly. It turns out that for $n = 5$ in particular, one can deduce directly from

consideration of the orbit spaces that

$$\mathbb{H}P(2) \# (S^4 \times S^4) = \overline{\mathbb{H}P}(2) \# (\overset{2}{\#}\ \mathbb{H}P(2)) \tag{1}$$

$$\overline{\mathbb{H}P}(2) \# (S^4 \times S^4) = (\overset{2}{\#}\ \overline{\mathbb{H}P}(2)) \# \mathbb{H}P(2) \tag{2}$$

paralleling analogous identities that exist in dimension 4, relating $S^2 \times S^2$, $\mathbb{C}P(2)$ and $\overline{\mathbb{C}P}(2)$. As a result, Spin(4) manifolds with M^* a disc, and all vertices ($n > 3$) fixed points, must be either connected sums of $S^4 \times S^4$ if only two of the three possible S^3 isotropy types occur, or connected sums of $\mathbb{H}P(2)$ and $\overline{\mathbb{H}P}(2)$ terms otherwise. We note here that these are the only 3-connected manifolds that arise in our context.

Remark. Other families of Spin(4) manifolds do not generally allow for such a straightforward topological identification.

3. The invariant $o \in \mathbb{Z}_2{}^b$

We begin by examining two instances where o encodes topological differences. First, we construct two actions with a disc quotient and ΔS^3 isotropy on the boundary, and show these actually correspond to actions on two topologically distinct manifolds (indeed, they are homotopy-inequivalent).

(i) Let $M = (D^2 \times \mathrm{Spin}(4))/_\sim$, where the second S^3 factor in Spin(4) is squashed to a point over $\mathrm{Bd}(D^2)$. Let $\langle e^{i\theta}, \rho\rangle$, $\rho \in [0, 1]$, denote an element in D^2, $u, v \in S^3$, and Spin(4) act by

$$(g \times h) \cdot (\langle e^{i\theta}, \rho\rangle \times u \times v) = \langle e^{i\theta}, \rho\rangle \times guh^{-1} \times hv$$

It is immediate that $M^* = D^2$ with ΔS^3 stabilized orbits forming the boundary, and that there is an n-section to the action. It is also not hard to see that $M \approx S^3 \times S^5$. Indeed, ignoring the first S^3 factor, we have $S^4 = \Sigma(S^3)$ above a diametral segment in D^2; taking the suspension again, and now crossing it with S^3, gives M.

(ii) The non-n-sectioned action on $S^1 \times S^3$ given earlier (cf. 2.2.2) induces one on $\mathcal{V} = D^4 \times S^3 \times S^1$, with $D^4 \approx ([1/2, 1] \times S^3)/_\sim$, collapsing S^3 to a point at $t = 1$:

$$(g \times h) \cdot [t, u] \times v \times e^{i\theta} = [t, gu] \times e^{i\theta/2} g e^{-i\theta/2} v h^{-1} \times e^{i\theta}.$$

$\mathcal{V}^*$ is an annulus, and we can then extend this action to one on M with $M^* = D^2$ as follows. Let $\mathcal{U} = D^2 \times S^3 \times S^3 = (S^1 \times [0, 1/2])/_\sim \times S^3 \times S^3$, and let Spin(4) act by

$$(g \times h) \cdot (\langle e^{i\theta}, t\rangle \times u \times v) = \langle e^{i\theta}, t\rangle \times gu \times vh^{-1}.$$

Next, glue $\mathrm{Bd}(\mathcal{U}) \to \mathrm{Bd}(\mathcal{V})$ by an equivariant diffeomorphism ϕ given by

$$e^{i\theta} \times u \times v \mapsto u \times e^{i\theta/2} u e^{-i\theta/2} v \times e^{i\theta}.$$

CLAIM. $M \approx SU(3)$.

First note that as a space, SU(3) is known to be distinct from $S^3 \times S^5$. In fact, their homotopy groups differ; for instance π_4 vanishes for SU(3) (cf. [Bor1], for instance), whereas it is $\mathbb{Z}_2$ for $S^3 \times S^5$. Thus, it suffices to show that there is a Spin(4) action on SU(3) with free principal orbits and disc quotient with boundary consisting of S^3 stabilized orbits. Call f the canonical isomorphism $S^3 \simeq \mathrm{SU}(2)$. Take the $S^3 \times \{1\}$ action to correspond to left multiplication by elements in $\begin{pmatrix} \mathrm{SU}(2) & 0 \\ 0 & 1 \end{pmatrix} \subset \mathrm{SU}(3)$. Thus, $(g \times 1) \cdot A = \begin{pmatrix} f(g) & 0 \\ 0 & 1 \end{pmatrix} A$, where $A \in \mathrm{SU}(3)$. Take the $\{1\} \times S^3$ action to correspond to right multiplication by elements in $\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SU}(2) \end{pmatrix}$, as follows: $(1 \times h) \cdot A = A \begin{pmatrix} 1 & 0 \\ 0 & f(h)^{-1} \end{pmatrix}$. This obviously defines a left Spin(4) action on SU(3). Observe that for $A = \mathrm{Id}$, $(g \times h) \cdot \mathrm{Id} = \mathrm{Id}$ iff $g \times h = 1 \times 1$. Therefore, the principal orbits are free. Now $\mathrm{SU}(3)/\mathrm{SU}(2) \approx S^5$, so we have $\mathrm{SU}(3)/(S^3 \times \{1\}) \approx S^5$. Further, $S^3 \simeq \{1\} \times S^3$ must be acting effectively on that quotient, and again by Richardson [Ric2] there is only one such action (up to equivalence) of S^3 on S^5, which has D^2 quotient with boundary consisting of S^3-stabilized orbits.

We have established:

PROPOSITION 1. *Suppose* Spin(4) *acts on a closed orientable M^8, with the quotient space a 2-disc whose boundary consists only of ΔS^3 orbits. Then $M \approx S^3 \times S^5$ if $o = 0$;* $M \approx \mathrm{SU}(3)$ if $o = 1$.

We can also show:

PROPOSITION 2. *Suppose* Spin(4) *acts on a closed orientable M^8, M^* a 2-disc with boundary consisting only of $S(0, 1)$ orbits. Then $M \approx S^3 \tilde{\times} S^2 \times S^3$ if $o = 0$, where $S^3 \tilde{\times} S^2$ denotes the orientable, non-trivial S^3 bundle over the two-sphere. $M \approx \mathrm{SU}(3)/\mathrm{SO}(3) \times S^3$ if $o = 1$.*

Proof. (i) Taking $D^2 \times S^3 \times S^3$ and collapsing by $/S(0, 1)$ over the boundary, and then letting Spin(4) act on the left in the obvious way, gives $M^* = D^2$ with boundary consisting only of $S(0, 1)$ orbits; obviously the action has an n-section. It is also clear that M is the product of a 5-manifold N with S^3. Further, $N^5 = S^3 \tilde{\times} S^2$. Indeed it is the double of $I \times H$, where $I = [0, 1]$ and H is the mapping cylinder of the Hopf fibration $S^3 \longrightarrow S^2$. We have

$$N = \mathcal{D}(I \times H) = \mathrm{Bd}(I \times I \times H) = \mathrm{Bd}(B^4 \tilde{\times} S^2) = S^3 \tilde{\times} S^2.$$

(Note that it turns out that in all cases where the boundary of the D^2 quotient consists of $S(m, n)$, the total space is homeomorphic to the manifolds above.)

(ii) Now suppose that the isotropy weights are as in (i), but the action does *not* have an n-section. First note that the total space still has the form $N \times S^3$. N is a

5-manifold with S^3 acting to give $N^* = D^2$ with a boundary of S^1-stabilized orbits. It is constructed following the prescription given in 2.2.3 for the subspace sitting above an annular neighborhood of $\mathrm{Bd}(N^*)$, and extending it as follows. View $\mathrm{Int}(N^*)$ as a family of circles indexed by $t \in [-1, 1)$ (the circle at -1 collapsed to a point), above each of which we have $(S_t^3 \times I)/\sim$. The gluing $S_t^3 \times \{0\} \longrightarrow S_t^3 \times \{1\}$ is as before for $t \in [0, 1)$; for $t \in [-1, 0]$, it is given by $u \times \{0\} \mapsto u\lambda(t) \times \{1\}$, where $\lambda\colon [-1, 0] \longrightarrow S^3$ is a smooth path with $\lambda(-1) = (1, 0)$ (the identity in S^3), and $\lambda(0) = (0, 1)$.

We can compute the homology of N using a Mayer-Vietoris sequence with $N = U \cup V$, $U \sim S^3$, and $V \sim S^2 \tilde{\times} S^1$, so U^* is a (closed) disk in M^*, V^* an annulus containing $\mathrm{Bd}(M^*)$. One obtains $\mathbb{Z}$ in dimensions 5 and 0, and $\mathbb{Z}_2$ in dimension 2. Thus, according to Barden's classification of simply-connected 5-manifolds, N is X_{-1} [Bar]. It turns out that $X_{-1} \approx \mathrm{SU}(3)/\mathrm{SO}(3)$. We can see this directly as follows.

Take $S^3 \simeq \mathrm{SU}(2) \equiv \begin{pmatrix} 1 & 0 \\ 0 & SU(2) \end{pmatrix} \hookrightarrow \mathrm{SU}(3)$. It acts by left multiplication on SU(3), hence on the coset space SU(3)/SO(3); here, we view SO(3) as those matrices in SU(3) with real entries. Clearly, $g \in S^3$ stabilizes $\langle A \rangle$, the equivalence class in SU(3)/SO(3) of the matrix $A \in \mathrm{SU}(3)$, iff $A^{-1}gA \in \mathrm{SO}(3)$.

Now take $A = \begin{pmatrix} c & d & 0 \\ -\bar{d} & \bar{c} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \begin{pmatrix} \mathrm{SU}(2) & 0 \\ 0 & 1 \end{pmatrix}$, and $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -\bar{b} & \bar{a} \end{pmatrix}$.

Then:

$$A^{-1}gA = \begin{pmatrix} |\bar{c}|^2 + a|d|^2 & \bar{c}d - a\bar{c}d & -db \\ \bar{d}c - ac\bar{d} & |d|^2 + a|c|^2 & bc \\ \overline{bd} & -\overline{bc} & \bar{a} \end{pmatrix}.$$

We can asssume $c, d \notin \mathbb{R}$. For the (1,1)-entry to be real, we would need $a \in \mathbb{R}$. But entry (1,2) is $(1-a)\bar{c}d$, so (for $a \neq 1$), we can certainly arrange $\bar{c}d \notin \mathbb{R}$, so that $g \neq \mathrm{Id}$ does not stabilize $\langle A \rangle$. Therefore, the principal isotropy for this action is trivial. Observe that $\langle \mathrm{Id} \rangle$ is S^1-stabilized. Thus $\langle \mathrm{Id} \rangle^*$ lies on the boundary of the orbit space which must be a disk since SU(3)/SO(3) is simply connected. The fact that, for S^3 actions on M^5, each boundary component must consist of only one singular orbit type concludes the argument. □

In both examples, o encoded topological differences between the spaces whose homeomorphic quotients had the same isotropy weights. Is this true generally? We now show that this is in fact *not* always the case. In particular:

PROPOSITION 3. *Let $M_j^* = D^2$, $(j = 0, 1)$, with only S^1 or T^2 isotropy over the boundary, and $o = j$. Suppose that there is an $S(1, 0)$ weighted edge contiguous to an $S(0, 1)$. Then $M_0 \approx M_1$.*

Proof. Consider annular neighborhoods of $\mathrm{Bd}(M_j^*)$. Refering to Figure 9, we may assume that $S(1, 0) = S(m_1, n_1)$ and $S(0, 1) = S(m_2, n_2)$. Set $t_0 = 0$, $t_1 = 1/2$ and $t_2 = 1$ for definiteness.

It is clear how M_0 is obtained, starting from the mapping cylinder that we defined. Write the cylinder parameter as ρ, ranging over the interval $[1/2, 1]$. We have another mapping cylinder for the map $S^3 \times S^3 \times S^1 \to S^3 \times S^3$ given by $(u \times v) \times t \mapsto (u \times v)$; write the cylinder parameter as ρ again, but ranging over the interval $[0, 1/2]$, with the target $S^3 \times S^3$ sitting at $\rho = 0$. The resulting two spaces are glued together by the identity along $S^3 \times S^3 \times S^1$, at $\rho = 1/2$ (so in the disc quotient, ρ is the radial coordinate). To obtain M_1, take the $S^3 \times S^3 \times S^1$ as defined earlier: recall that we attached by $a \times a$: $u \times v \times t_k \mapsto u(0, 1) \times v(0, 1) \times t_0$, so directly forming another mapping cylinder as in M_0 for $0 \leq \rho \leq 1/2$, would not be well defined. However it is clear that we can deform the way we attach by homotoping a to the identity: just pick a path $\gamma(\rho)$ in S^3 ($\rho \in [0, 1/2]$) joining $(0, 1)$ to $(1, 0)$, and attach by multiplying with $\gamma(1/2 - \rho) \times \gamma(1/2 - \rho)$.

Define a path λ: $[a, b] \to S^3$ parametrizing an arc joining $(0, -1)$ to $(1, 0)$. In fact, choose this arc to be $\{\gamma(\rho)^{-1}\}$, made up of the inverses (with respect to the S^3 product) of the points in $\mathrm{Im}(\gamma)$. Let $\lambda_1(t)$ be constantly $(0, -1)$ for $0 \leq t \leq 1/6$; over the interval $[a, b] = [1/6, 1/3]$, $\lambda_1 = \lambda$, and for $1/3 \leq t \leq t_k$, $\lambda_1 \equiv (1, 0)$. Define λ_2 in exactly the same way, but over the intervals $[0, 2/3]$, $[2/3, 5/6]$, and $[5/6, t_k]$ respectively. Let $\hat{M}_j \subset M_j$ be the mapping cylinders that map down to the annular neighborhoods of $\mathrm{Bd}(M_j^*)$ (corresponding to $\rho \in [1/2, 1]$). The following assignment defines a homeomorphism $\hat{M}_1 \approx \hat{M}_0$:

$$(u \times v) \times t \times \rho \mapsto (u\lambda_1(t) \times v\lambda_2(t)) \times t \times \rho$$

Indeed we see that

$$\begin{array}{ccc} u \times v \times t_k \times 1 & \mapsto & u \times v \times t_k \times 1 \\ ||| & & ||| \\ u(0,1) \times v(0,1) \times t_0 \times 1 & \mapsto & u(0,1)(0,-1) \times v(0,1)(0,-1) \times t_0 \times 1 = u \times v \times t_0 \times 1 \end{array}$$

It is not difficult now to extend this map to all of the M_j's. The assignment will have the same form, except that for each $0 \leq \rho \leq 1/2$, we take $\lambda^{(\rho)}$ to parametrize the portion of the arc $\mathrm{Im}(\lambda)$ that starts at $\gamma(1/2 - \rho)^{-1}$ (ending at $(1, 0)$); the λ_j^ρ 's are defined from $\lambda^{(\rho)}$ in the same way as λ_j from λ. At $\rho = 0$, we get the identity map of the $S^3 \times S^3$ fiber. $\square$

Example. Take $M = \mathbb{C}P(2)\sharp\overline{\mathbb{C}P}(2) \times \mathbb{C}P(2)\sharp\overline{\mathbb{C}P}(2)$, where each factor is obtained from $S^4 = \Sigma(S^3)$ by blowing-up the two suspension points. Then there is an obvious (left) Spin(4) action on M given by the S^3 actions on $\mathbb{C}P(2)\sharp\overline{\mathbb{C}P}(2)$ induced by the linear action on S^4 (using S^3 multiplication on the left). One checks that the resulting weighted quotient is a disc with two $S(1, 0)$ and two $S(0, 1)$ edges, and exactly four T^2 vertices. In this case, $o = 0$. But there must be an inequivalent action on M resulting in the same weighted disc, except with $o = 1$.

At present, we do not know whether the condition in the proposition that there be contiguous $S(1, 0)$ and $S(0, 1)$ edges is necessary; but our explicit construction would definitely fail to work using arbitrary weights. Let us note also, that if we have an orbit space with j^{th} boundary component weighted by a diagonal S^3 and there is a factor S^3 occuring on another component, then $o_j = 0$ or 1 does not affect the topology (this will become obvious in the follow-up study).

Remark. From the proof of Proposition 3, we see that generally, if M_0 and M_1 have homeomorphic orbit spaces (not necessarily discs), with the same weights everywhere, except possibly for o on boundary components satisfying the conditions of the proposition, then $M_0 \approx M_1$.

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Department of Mathematics, Idaho State University, Box 8085, Pocatello ID 83209
mazaphil@isu.edu